Quantum cohomology and the Seidel representation

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Introduction

Suppose we have a symplectic manifold (M, ω). From differential topology, we have a cohomology ring $\mathrm{H}^*(M)$. However, this does not take into account the symplectic structure of (M, ω). Instead, we can deform the ring structure, by considering J-holomorphic spheres which pass through the given homology classes. This gives us a ring $\mathrm{QH}^*(M, \omega)$, called the *quantum cohomology ring*. In chapter I, we will define this ring, subject to some technical conditions which simplify the theory. In the process, we will introduce some basic pseudoholomorphic curve theory, such as Gromov's compactness theorem.

Next, in chapter II, we introduce very briefly *Hamiltonian Floer homology*. This is a variant of Morse homology, where the critical points correspond to 1-periodic orbits under a Hamiltonian flow. This has a natural ring structure, by considering *pairs of pants* which join 1-periodic orbits. To conclude the section, we show that the quantum cohomology ring is isomorphic to the Hamiltonian Floer homology ring.

The second half of the essay is about the *Seidel representation*, which is a group homomorphism

$$\pi_0(\widetilde{G}) \to \mathrm{QH}^*(M, \omega)^\times$$

where $\widetilde{G}$ is a covering space of the group of smooth based loops in $\mathrm{Ham}(M, \omega)$. In chapter III, we will define the Seidel representation. We do this by defining an action of $\widetilde{G}$ on Hamiltonian Floer homology, then using the isomorphism we defined in chapter II to define the Seidel representation.

We finish the essay with two worked examples. In chapter IV, we use the Seidel representation to show that an element in $\pi_1(\mathrm{Ham}(M, \omega))$ has infinite order. In chapter V, we will use the Seidel representation to find relations in the quantum cohomology ring of a toric manifold, and compute the quantum cohomology ring of some examples.

To conclude this introduction, we make some remarks about the analytic aspects of the theory. Defining these structures usually involves counting the number of solutions to an elliptic partial differential equation. In turn, this involves constructing a moduli space of solutions, perhaps up to a generic choice of perturbation. We will either show that the moduli space is zero-dimensional, and so we can count the number of solutions, or we will intersect the moduli space with a manifold of complementary dimension.

For the most part, we will not discuss the analytic details, and we will instead focus on the geometric aspects of the theory.

Notation

In general, the notation that is used in the essay is either standard, or introduced in the text. However, we will use the following:

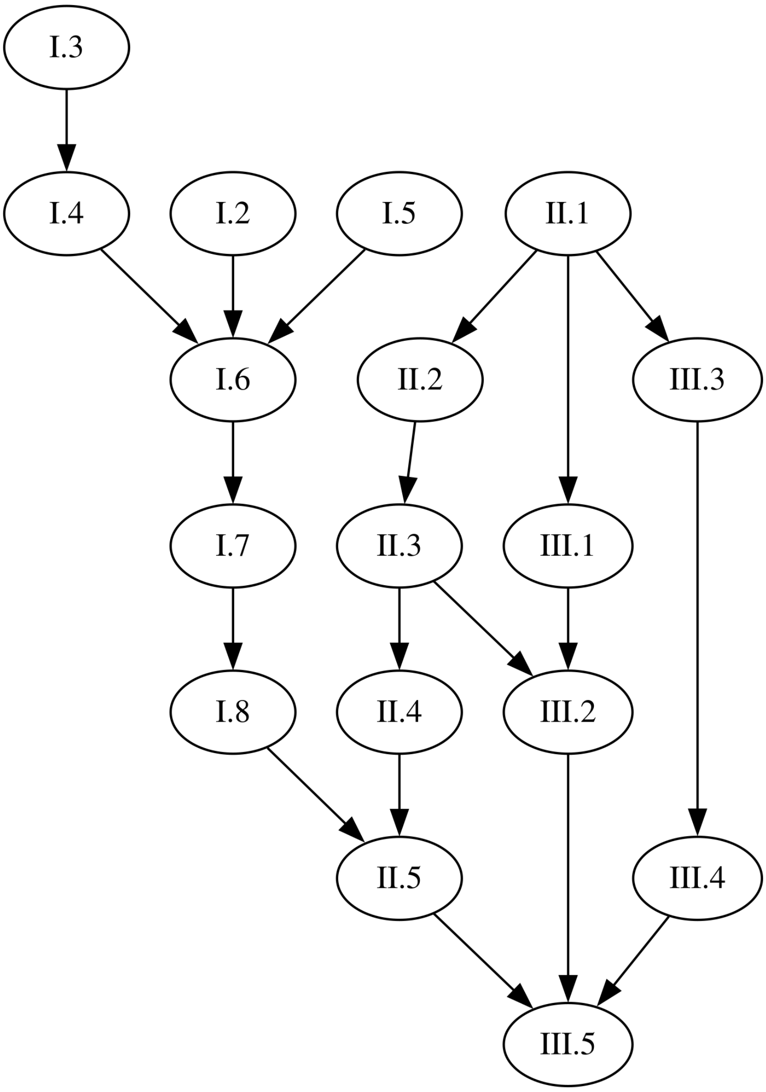
- a *manifold* will be smooth, and may or may not have boundary.
- If X is a compact oriented manifold (or more generally, has a fundamental class), and $f : X \to Y$ is a continuous map, we will write
$$[f] := f_*[X] \in \mathrm{H}_*(Y)$$
- For spaces where Poincaré duality hold, PD will be used to denote both maps in the isomorphism.
- $\mathbb{P}^n = \mathbb{CP}^n$ is complex projective space.
- we will omit the Hurewicz map, and consider $\pi_2(M)$ as a subgroup of $\mathrm{H}_2(M)$.

Note (June 25 2024) after submission:

There are several mistakes in this essay, which are not labelled or corrected. Read at your own peril...

Dependency

For the benefit of the reader[1], we include a dependency graph of the sections of this essay. This only includes the first three chapters, where we are developing the theory. The last two chapters are independent of eachother, and depend on the first three chapters.



[1] and mainly the author

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Chapter I

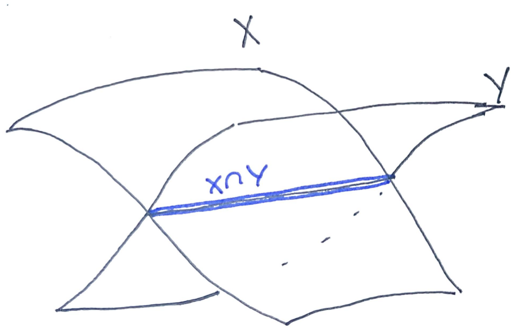
Quantum cohomology

I.1 Introduction

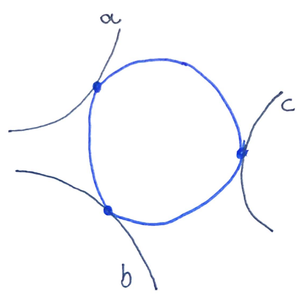
Let M be a compact oriented manifold, and let $\mathrm{H}^*(M)$ be its cohomology ring. Let $X, Y \subseteq M$ be compact oriented submanifolds, intersecting transversely. Under Poincaré duality,

$$\mathrm{PD}[X] \smile \mathrm{PD}[Y] = \mathrm{PD}[X \cap Y]$$

That is, the ring structure on $\mathrm{H}^*(M)$ is given by the intersection of submanifolds.



Now let (M, J) be an almost complex manifold. We would like to define a 'deformed' ring structure on M. Let $a, b, c \in \mathrm{H}_*(M)$ be cycles. We will define the coefficient of c in the product of a and b to be the number of J-holomorphic spheres passing through a, b, c.



The associated ring is called the *quantum cohomology* ring $\mathrm{QH}^*(M)$. More precisely,

Definition I.1.1. A *J-holomorphic curve u* is a smooth map $\mathbb{P}^1 \to M$, such that

$$J \circ \mathrm{d}u = \mathrm{d}u \circ j$$

where j is the complex structure on $\mathbb{P}^1$.

Since $\mathrm{PSL}(2,\mathbb{C})$ acts triply-transitively on $\mathbb{P}^1$ by Möbius transformations, it is equivalent to count the number of J-holomorphic curves u, with

$$u(0) \in a, u(1) \in b \text{ and } u(\infty) \in c.$$

For $A \in \mathrm{H}_2(M)$, define the moduli space

$$\mathcal{M}(A, J) = \{u : \mathbb{P}^1 \to M \ J\text{-holomorphic curve with } [u] = A\}$$

and we have an associated evaluation map

$$\mathrm{ev}(u) = (u(0), u(1), u(\infty)).$$

We would like to consider the intersection of $\mathrm{ev}(\mathcal{M}(A, J))$ with $a \times b \times c$, and we can use the intersection number to define the quantum cup product.

To define the homology class, we would like $\mathcal{M}(A, J)$ to be a compact oriented manifold. In this case, $\mathcal{M}(A, J)$ has a fundamental class, which we can push-forward using ev. For a generic choice of J, $\mathcal{M}(A, J)$ is a finite-dimensional oriented manifold. On the other hand, in general it is not compact.

Understanding the non-compactness of the moduli space will lead us to Gromov's compactness theorem, where we see that a sequence of J-holomorphic curves can degenerate via *bubbling*. Once we have this, we will see that the points which we need to add to $\mathrm{ev}(\mathcal{M}(A, J))$ to make it compact has codimension at least 2, assuming a technical condition called *monotonicity*. Once we have this, we will have an associated homology class.

Throughout, we will follow [MS94].

I.2 Pseudocycles

First of all, we will need to define a generalisation of cycles in homology, called *pseudocycles*. The main purpose is to allow for non-compactness, as long as the strata which we need to add to make the space compact has codimension at least 2.

I.2.1 Definitions

Let M be a compact manifold, with $\dim(M) = n$.

Definition I.2.1. A subset $B \subseteq M$ has *dimension at most* k if there exists a manifold V^k, and a smooth map $g : V \to M$, with $B \subseteq g(V)$. We will denote this by $\dim(B) \leq k$.

For a smooth map $f : V \to M$, let

$$\Omega_f = \bigcap_{K \subseteq V \text{ compact}} \overline{f(V \setminus K)}$$

be the set of all limit points of sequences $f(x_k)$, where $(x_k) \subseteq V$ is a sequence with no convergent subsequence. Intuitively, we can think of Ω_f as the boundary "at infinity" of $f(V)$.

Definition I.2.2. Let V be a k-dimensional oriented manifold. A smooth map $f : V \to M$ is called a *pseudocycle* if $\dim(\Omega_f) \leq k - 2$.

Definition I.2.3. Let $f_0 : V_0 \to M, f_1 : V_1 \to M$ be pseudocycles. They are *bordant* if there exists a $(k+1)$-dimensional manifold W, with $\partial W = V_1 - V_0$, and a smooth map $F : W \to M$, such that

- $F|_{V_0} = f_0$
- $F|_{V_1} = f_1$
- $\dim(\Omega_F) \leq k - 1$

I.2.2 Intersections

Definition I.2.4. Pseudocycles $e : U \to M, f : V \to M$ are *transverse* if:

- $\Omega_e \cap \overline{f(V)} = \overline{e(U)} \cap \Omega_f = \varnothing$,
- if $e(u) = f(v) = x$, then $\mathrm{T}_x M = \mathrm{im}(\mathrm{d}e(u)) + \mathrm{im}(\mathrm{d}f(v))$.

As with the intersection of submanifolds, if e and f are transverse, then

$$\{(u, v) \mid e(u) = f(v)\} \subseteq U \times V$$

is a compact manifold of dimension $\dim(U) + \dim(V) - \dim(M)$.

Proposition I.2.5 ([MS94, Lemma 7.1.2])**.** *Let $e : U \to M, f : V \to M$ be pseudocycle, with* $\dim(U) + \dim(V) = \dim(M)$*. Then*

(i) There exists a non-meagre subset $\mathcal{D} \subseteq \mathrm{Diff}(M)$, such that for all $\phi \in \mathcal{D}$, e is transverse to $\phi \circ f$,

(ii) If e and f are transverse, then $\{e(u) = f(v)\}$ is a finite set. Define

$$e \cdot f = \sum_{e(u)=f(v)} \nu(u, v)$$

where $\nu(u, v)$ is the intersection number of $e(U)$ and $f(V)$ at $e(u) = f(v)$.

(iii) $e \cdot f$ only depends on the bordism classes of e and f.

This is a generalisation of standard results in differential topology. For (ii), we have that if X, Y are compact oriented submanifolds, of complementary dimensions and intersecting transversely, then $X \cap Y$ is a compact zero-dimensional manifold, and so it is a finite set. For (iii), if we have $\partial Z = Z_1 - Z_0$, with $\dim(Z) + \dim(X) = \dim(M) + 1$, and assuming the intersection is transverse, we get that $Z \cap X$ is a 1-manifold with boundary $\partial Z \cap X = Z_1 \cap X \cup Z_0 \cap X$. Thus, when we count the boundary points with orientation, we get that the intersection numbers are the same.

I.2.3 Representing homology classes

Let $\beta \in \mathrm{H}_d(M; \mathbb{Z})$ be an integral homology class. We can represent this by a map $f : P \to M$, where P is a finite oriented simplicial complex without boundary, and so P has a fundamental class. In this case, $\beta = [f]$. Now consider V to be the union of the d and $d - 1$ dimensional faces of P. V is a smooth manifold, and let $f' : P \to M$ be a continuous approximation of f, which is smooth on V. In this case,

$$\Omega_{f'} = f(P \setminus V)$$

and so f' defines a pseudocycle.

Note that any two choices of f' are bordant, and so if $e : U \to M$ is an $n - d$ dimensional pseudocycle, we have a well defined homomorphism

$$\begin{aligned} \Phi_e : \mathrm{H}_d(M; \mathbb{Z}) &\to \mathbb{Z} \\ \Phi_e(\beta) &= e \cdot f' \end{aligned}$$

With all of this in mind our goal will be to define the moduli space $\mathcal{M}(A, J)$ and the evaluation map ev, and to show that $\mathrm{ev}(\mathcal{M}(A, J))$ is a pseudocycle.

I.3 Moduli space of curves

Let (M^{2n}, J) be an almost complex manifold, and let $A \in \mathrm{H}_2(M)$ be a homology class.

Definition I.3.1. A map $u : \mathbb{P}^1 \to M$ is *simple* if it does not factor as $u = v \circ \phi$, where $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree $d > 1$ map, and $v : \mathbb{P}^1 \to M$ is a smooth map.

Definition I.3.2. The *moduli space of curves* is

$$\mathcal{M}(A, J) = \{u : \mathbb{P}^1 \to M \mid u \text{ is } J\text{-holomorphic and simple, with } [u] = A\}$$

Remark I.3.3. We require the maps to be simple, as we will want to perturb the almost complex structure J locally. If the map was not simple, then we will need to perturb J at several places at the same time. We may not be able to assume transversality in this case.

The requirement for a smooth map $u : \mathbb{P}^1 \to M$ to be J-holomorphic is a first order linear elliptic PDE given by the Cauchy-Riemann equations. This can be represented as a section of an appropriate vector bundle, which we will now define.

Definition I.3.4. A smooth map $u : \mathbb{P}^1 \to M$ is *somewhere injective* if there exists $z_0 \in \mathbb{P}^1$ such that

$$\mathrm{d}u(z_0) \neq 0 \quad \text{and} \quad u^{-1}(u(z_0)) = \{z_0\}.$$

Remark I.3.5. If a J-holomorphic curve is somewhere injective, then it is simple.

Let $\mathcal{X}$ denote the set of all smooth maps $u : \mathbb{P}^1 \to M$, which satisfy $[u] = A$ and are somewhere injective. We can view $\mathcal{X}$ as an infinite-dimensional manifold, with tangent space

$$\mathrm{T}_u\mathcal{X} = \Gamma(u^*\mathrm{T}M) = \{\xi(z) \in \mathrm{T}_{u(z)}M\}$$

given by smooth vector fields along u. Now consider the vector bundle $\mathcal{E} \to \mathcal{X}$, with fibres

$$\mathcal{E}_u = \Omega^{0,1}(u^*\mathrm{T}M),$$

given by J anti-linear 1-forms on $\mathbb{P}^1$ with value in $u^*\mathrm{T}M$.

Definition I.3.6. For a smooth map $u : \mathbb{P}^1 \to M$, define

$$\bar{\partial}_J u = \frac{1}{2}\left(\mathrm{d}u + J \circ \mathrm{d}u \circ j\right) \in \Omega^{0,1}(u^*\mathrm{T}M) = \mathcal{E}_u$$

for the complex anti-linear part of $\mathrm{d}u$.

Note that u is J-holomorphic if and only if $\bar{\partial}_J u = 0$. The differential operator $\bar{\partial}_J$ defines a section of $\mathcal{E}$, and we have that $\mathcal{M}(A, J) = \bar{\partial}_J^{-1}(0)$. Thus, to show that $\mathcal{M}(A, J)$ is a smooth manifold, we will need to show that we have a transverse intersection, and then we can appeal to the implicit function theorem.

The first step is to linearise the operator. Let $\pi_u : \mathrm{T}_{(u,0)}\mathcal{E} = \mathrm{T}_u\mathcal{X} \oplus \mathcal{E}_u \to \mathcal{E}_u$ be the projection map, and define

$$\mathrm{D}_u : \Gamma(u^*\mathrm{T}M) \to \Omega^{0,1}(u^*\mathrm{T}M)$$

via

$$\begin{tikzcd} \mathrm{T}_u\mathcal{X} \arrow[rr, "\mathrm{D}_u"] \arrow[dr, "\mathrm{d}\bar{\partial}_J(u)"'] & & \mathcal{E}_u \\ & \mathrm{T}_{(u,0)}\mathcal{E} \arrow[ur, "\pi_u"'] & \end{tikzcd}$$

Transversality is equivalent to requiring D_u to be surjective for all $u \in \mathcal{M}(A, J)$. Computing, we find that D_u is a first order elliptic differential operator. In particular, it is Fredholm[1], and from Riemann-Roch, it has index

$$\mathrm{index}(\mathrm{D}u) = 2n + 2c_1(u^*\mathrm{T}M).$$

For a specific choice of J, $\mathcal{M}(A, J)$ need not be a smooth manifold. Suppose (M, ω) is a symplectic manifold. Let $\mathcal{J} = \mathcal{J}(M, \omega)$ be the space of ω-compatible almost complex structures, equipped with the C^∞ topology. We will let J vary in $\mathcal{J}$. For a generic choice of J, $\mathcal{M}(A, J)$ will be a smooth manifold.

Definition I.3.7. A pair (u, J) is *regular* if D_u is surjective. For $A \in \mathrm{H}_2(M)$, let $\mathcal{J}_{\mathrm{reg}}(A)$ be the set of $J \in \mathcal{J}$ such that (u, J) is regular for all $u \in \mathcal{M}(A, J)$.

[1]A bounded linear map $T : X \to Y$ between Banach spaces is *Fredholm* if $\ker(T)$ and $\mathrm{coker}(T)$ are finite dimensional, and $T(X)$ is closed. The *index* of T is

$$\mathrm{index}(T) = \dim(\ker(T)) - \dim(\mathrm{coker}(T))$$

Regular in this context can be considered to be a version of regular values from differential topology. In the finite dimensional case, we have Sard's theorem, which states that the complement of the set of regular values has measure zero. However, measure zero does not (a priori) make sense on an infinite dimensional manifold, and thus we will need to consider a generalisation, called the Sard-Smale theorem, which states that the set of regular values is non-meagre, in the sense of Baire category.

Once we have a regular value, in finite dimensions we have an implicit function theorem, which shows that (for example) the zero set is a submanifold. There is a natural generalisation to the case when the differential is a Fredholm operator. Combining these two statements, we obtain the following theorem:

Theorem I.3.8 ([MS94, Theorem 3.1.2]). *$\mathcal{J}_{\text{reg}}(A) \subseteq \mathcal{J}$ is non-meagre. For $J \in \mathcal{J}_{\text{reg}}(A)$, $\mathcal{M}(A, J)$ is an* oriented *smooth manifold of dimension $2n + 2c_1(A)$.*

Example I.3.9. Consider the case $M = \mathbb{P}^2$. Let $L \in \mathrm{H}_2(\mathbb{P}^2)$ be the class of a line. In this case, $c_1(L) = 3$. Note that by the degree-genus formula, the only curves which are biholomorphic to $\mathbb{P}^1$ are lines and conics. We then have that

$$\dim(\mathcal{M}(L, J)) = 4 + 6 = 10$$
$$\dim(\mathcal{M}(2L, J)) = 4 + 12 = 16.$$

These are larger than what we would expect. The moduli space of lines in $\mathbb{P}^2$ is $\mathbb{P}^2$, and the moduli space of conics in $\mathbb{P}^2$ is $\mathbb{P}^5$. Thus, we would expect the real dimensions to be 4 and 10 respectively.

The reason for the difference is as follows: The moduli spaces we constructed from algebraic geometry are for *unparametrised curves*, but the moduli space $\mathcal{M}(A, J)$ is for *parametrised curves*. In particular, $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PSL}(2, \mathbb{C})$ acts on $\mathcal{M}(A, J)$ by reparametrisation. One can show that $\dim_{\mathbb{C}}(\mathrm{PSL}(2, \mathbb{C})) = 3$. Thus, when we quotient out by the action, we get the expected dimensions.

In the introduction, we chose three *marked points* $(0, 1, \infty)$ on $\mathbb{P}^1$, and since three points determine a Möbius transformation, this is another way of quotienting out by the action of $\mathrm{PSL}(2, \mathbb{C})$.

Next, we will show that the resulting space does not depend on the choice of $J \in \mathcal{J}_{\text{reg}}(A)$ in an essential way. Let $J_0, J_1 \in \mathcal{J}_{\text{reg}}(A)$, and let $(J_t)_{t\in[0,1]} \subseteq \mathcal{J}$ be a smooth path connecting J_0 and J_1. Define

$$\mathcal{M}(A, (J_t)) = \bigsqcup_{t\in[0,1]} \mathcal{M}(A, J_t)$$

In general, $\mathcal{M}(A, J_t)$ can be singular.

Theorem I.3.10 ([MS94, Theorem 3.1.3]). *For a generic choice of (J_t), $\mathcal{M}(A, (J_t))$ is an oriented smooth manifold of dimension $2n + 2c_1(A) + 1$, with boundary*

$$\partial\mathcal{M}(A, (J_t)) = \mathcal{M}(A, J_1) - \mathcal{M}(A, J_0)$$

Thus, we have a cobordism between $\mathcal{M}(A, J_0)$ and $\mathcal{M}(A, J_1)$. However, for this to be of use, we will need to establish some form of compactness, which we will do in the next section.

I.4 Bubbling

I.4.1 Energy

Let $J \in \mathcal{J}(M, \omega)$. Then we have an associated Riemannian metric

$$g(v, w) = \omega(v, Jw)$$

on M.

Definition I.4.1. Let $u : \Sigma \to M$ be a J-holomorphic curve. The *energy* of u is

$$\mathrm{E}(u) = \frac{1}{2}\int_{\mathbb{P}^1} |\mathrm{d}u|^2 \mathrm{d}A,$$

where $\mathrm{d}A = \omega_{\mathrm{FS}}$ is the area form on $\mathbb{P}^1$.

Lemma I.4.2 ([MS12, Lemma 2.2.1])**.**

$$\mathrm{E}(u) = \int_{\mathbb{P}^1} u^*\omega = \omega([u])$$

In particular, it depends only on the homology class of u.

Proof. We will compute the integrand locally. Let $z = x + iy$ be the complex coordinate on an open subset of $\mathbb{C} \subseteq \mathbb{P}^1$. Then

$$\begin{aligned} \frac{1}{2}|\mathrm{d}u|^2\mathrm{d}A &= \frac{1}{2}\left(|u_x|^2 + |u_y|^2\right)\mathrm{d}x \wedge \mathrm{d}y \\ &= \frac{1}{2}\left|u_x + Ju_y\right|^2\mathrm{d}x \wedge \mathrm{d}y - g(u_x, Ju_y)\mathrm{d}x \wedge \mathrm{d}y \\ &= \left|\bar{\partial}u\right|^2\mathrm{d}A + \frac{1}{2}\left(\omega(u_x, u_y) + \omega(Ju_x, Ju_y)\right)\mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

The first term vanishes as u is J-holomorphic, and the second term is just $u^*\omega$. □

An important result in the theory of Sobolev spaces is *Morrey's inequality* [Eva10, Section 5.6.2, Theorem 4], which states that for $n < p < \infty$ and $u \in C_c^\infty(\mathbb{R}^n)$, we have

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

where $\gamma = 1 - n/p$. This implies that we have a compact embedding $W^{1,p}(\mathbb{R}^n) \to C^{0,\gamma}(\mathbb{R}^n)$. In our case, $n = 2$, and translating the results to maps on $\mathbb{P}^1$, we get the following:

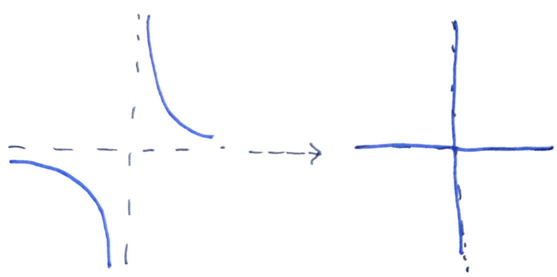
Theorem I.4.3 ([MS94, Theorem 4.1.3])**.** *Let $p > 2$. Suppose (u_k) is a sequence of J-holomorphic curves, with $\|u\|_{W^{1,p}(K)}$ bounded for all $K \subseteq \mathbb{P}^1$ compact. Then there exists a subsequence which converges uniformly with all derivatives on compact subsets of $\mathbb{P}^1$.*

However, when we only have control of the homology class, we have a $W^{1,2}$ bound. This is the critical case of the Sobolev embedding, and the theorem does not hold for $p = 2$. What we instead have is *bubbling*.

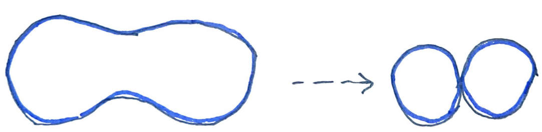
Example I.4.4. For $\alpha \in \mathbb{C}$ fixed, consider the curve C_α in $\mathbb{P}^2$, given by the equation $xy = \alpha z^2$.

C_α is a degree 2 curve, and so if $L \in \mathrm{H}_2(\mathbb{P}^2)$ is the homology class of a line, then $[C_\alpha] = 2L$. Thus, the energy is constant. When α is non-zero, C_α is a smooth conic, and so it is homeomorphic to $\mathbb{P}^1$. On the other hand, when $\alpha = 0$, we get $xy = 0$, which topologically is $\mathbb{P}^1 \vee \mathbb{P}^1$.

The "complex picture" is



and the "real picture" is



In this example. when we take the limit $\alpha \to 0$, we obtained a 'bubble' $\mathbb{P}^1$. On the other hand, if we have $\alpha_n \to \alpha$ with $\alpha_n \neq 0$, then we have a sequence of smooth curves converging to a $\mathbb{P}^1$, and no bubble forms.

I.4.2 Cusp curves

Definition I.4.5. A *cusp curve* $u = (u^1, \dots, u^N)$ consists of J-holomorphic curves u^j, such that if $C^j = u^j(\mathbb{CP}^1)$, then for all k,

$$C^1 \cup \dots \cup C^k$$

is connected.

Lemma I.4.6 ([MS94, Remark 4.4.2])**.** *If u is a cusp curve, then there exists a smooth, but not J-holomorphic map $v : \mathbb{P}^1 \to M$ parametrising it.*

Definition I.4.7. A sequence of J-holomorphic curves $u_k : \mathbb{P}^1 \to M$ *converges weakly* to a cusp curve $u = (u^1, \dots, u^N)$ if

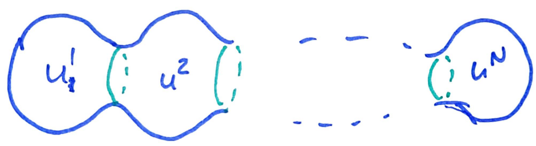
(i) for each j, there exists $\phi_k^j \in \mathrm{PSL}(2, \mathbb{C})$, $X^j \subseteq \mathbb{P}^1$ finite, such that $u_k \circ \phi_k^j$ converges to u^j uniformly with all derivatives on compact subsets of $\mathbb{P}^1 \setminus X^j$,

(ii) there exists $f_k \in \mathrm{Diff}^+(\mathbb{P}^1)$ such that $u_k \circ f_k$ converges in the C^0-topology to a parametrisation v of u.

Suppose u_k converges weakly to u. Then for k sufficiently large, u_k is homotopic to

$$u^1 \# \cdots \# u^N : \mathbb{P}^1 \to M$$

which is defined by



In particular,

$$\omega([u_k]) = \sum_j \omega([u^j]) \quad \text{and} \quad c_1([u_k]) = \sum_j c_1([u^j]).$$

From the first statement, we see that

$$\mathrm{E}(u_k) = \sum_j \mathrm{E}(u^j)$$

and we can see this as energy "bubbling off". This phenomenon was first observed for minimal surfaces in [SU81], and it is related to the conformal invariance of energy.

Theorem I.4.8 (Gromov's compactness, [MS94, Theorem 4.4.3])**.** *Suppose $J \in \mathcal{J}(M, \omega)$. Let $u_k : \mathbb{P}^1 \to M$ be a sequence of J-holomorphic curves, with*

$$\sup_k \mathrm{E}(u_k) < \infty$$

Then (u_k) has a subsequence with converges weakly to a cusp curve u.

Corollary I.4.9. *Any sequence $(u_k) \subseteq \mathcal{M}(A, J)$ has a sequence which converges weakly to a cusp curve.*

Example I.4.10. We continue with example I.4.4. We can parametrise the curve by

$$[s : t] \mapsto [s^2 : \alpha t^2 : st].$$

In affine coordinates $\zeta = s/t$, we have that

$$u(\zeta) = \left(\frac{1}{\alpha}\zeta^2, \frac{1}{\alpha}\zeta\right)$$

and

$$u'(\zeta) = \frac{1}{\alpha}(2\zeta, 1).$$

In particular, as $\alpha \to 0$, $|u'(\zeta)| \to \infty$ near 0. In the limit, we see that this energy bubbles off, and we get a cusp-curve in the limit.

We can also view this as "pinching", as illustrated in the following example:

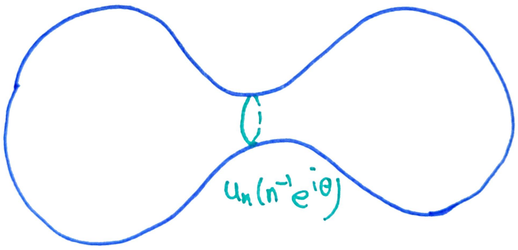
Example I.4.11. Define

$$u_n(z) = (z, n^{-2}z^{-1}) \in \mathbb{P}^1 \times \mathbb{P}^1.$$

Then consider the loop

$$u_n(n^{-1}e^{i\theta}) = (n^{-1}e^{i\theta}, n^{-1}e^{-i\theta}).$$

As $n \to \infty$, the loop shrinks to a point $(0, 0)$, which is where the two components of the cusp curve meet.



Another corollary of Gromov's compactness theorem is the following:

Corollary I.4.12. *Suppose $J \in \mathcal{J}(M, \omega)$. For fixed $K > 0$, there are only finitely many classes $A \in \mathrm{H}_2(M)$ with J-holomorphic representatives, and with $\omega(A) \leq K$.*

Proof. Suppose not. Then we have an infinite sequence of curves (u_n), each in a different homology class with bounded energy. But by Gromov's compactness theorem, $[u_n] \in \mathrm{H}_2(M; \mathbb{Z})$ is eventually constant. Contradiction. □

Corollary I.4.13. *There exists a constant $\hbar > 0$, such that any non-constant J-holomorphic curve has $E(u) \geq \hbar$.*

Proof. Fix K sufficiently large such that there exists a J-holomorphic curve u with $\mathrm{E}(u) \leq K$. By the previous corollary, we have finitely many homology classes $A_0, \ldots, A_k$, such that $\omega(A_i) \leq K$ and A_i has a J-holomorphic representative u_i. If $\omega(A_i) = 0$, then $\mathrm{E}(u_i) = 0$, and u_i is constant. In this case, $A_i = [u_i] = 0$. Without loss of generality, we may assume that $A_0 = 0$. Thus, we can take

$$\hbar = \min\{\omega(A_1), \ldots, \omega(A_k)\}.$$

□

I.4.3 Compactness of the moduli space

First of all, the non-compact group $G = \mathrm{PSL}(2, \mathbb{C})$ acts on $\mathcal{M}(A, J)$ by reparametrisation, and so $\mathcal{M}(A, J)$ cannot be compact. Let $\mathcal{C}(A, J) = \mathcal{M}(A, J)/G$ be the quotient space. We can use our understanding of bubbling to see when $\mathcal{C}(A, J)$ is compact.

Definition I.4.14. A homology class $A \in \pi_2(M)^2$ is *indecomposable* if it cannot be written as $A = A_1 + \cdots + A_k$ $(k \geq 2)$, where $A_i \in \pi_2(M)$, with $\omega(A_i) > 0$.

Theorem I.4.15 ([MS94, Theorem 4.3.4])**.** *Suppose A is indecomposable. Then $\mathcal{C}(A, J)$ is compact for all $J \in \mathcal{J}(M, \omega)$.*

Proof. We will show that it is sequentially compact. Let (u_k) be a sequence in $\mathcal{C}(A, J)$. By Gromov's compactness theorem, there exists a subsequence which converges to a cusp curve $u = (u^1, \ldots, u^N)$. In this case, we have that $A = [u^1] + \cdots + [u^N]$, where each $[u^j] \in \pi_2(M)$ has positive energy.

But we assumed A was indecomposable, and so we must have $N = 1$, and so $u = u^1 \in \mathcal{C}(A, J)$. □

Example I.4.16. Let $M = \mathbb{P}^n$, with the Fubini-Study form. In this case, $\mathrm{H}_2(M) = \mathbb{Z} \cdot L$, where L is the class of a line. In particular, a degree d curve in $\mathbb{P}^n$ has homology class dL. The above theorem implies that $\mathcal{C}(L, J)$ is compact. On the other hand, example I.4.4 shows that in $\mathcal{C}(2L, J)$ is not compact.

[2] Recall that we identify $\pi_2(M)$ as a subgroup of $\mathrm{H}_2(M)$ through the Hurewicz map.

I.4.4 Aside: Higher genus curves

In the definition of J-holomorphic curves, we have only considered the case of genus 0-curves. However, a lot of the theory still works for higher genus curves. The following example is inspired by the discussion of the genus of a plane curve found in [Ara12, Section 1.5].

Example I.4.17. Let $C = \mathbb{V}(f)$ be a smooth plane quartic, and let $g = (x^3 + y^3 + z^3)z$. Now consider the pencil $g_t = tf + (1-t)g$. Let $C_t = \mathbb{V}(g_t)$, so $C_1 = C$. As a curve, C_0 has two components: a smooth cubic and a line. By Bézout, the cubic intersects the line in 3 points.

On the other hand, C_1 has genus 3. What we can see is that as $t \to 0$, three circles on C_1 shrink to a point, and so we get a $\mathbb{P}^1$-bubbling off.

In general, the bubbles will all have genus 0, i.e. are $\mathbb{P}^1$s. We can relate this to the genus of the curve as follows:

Corollary I.4.18. *Let $C_d = \mathbb{V}(f_d)$ be a smooth plane curve of degree d. Then the genus of C_d is*

$$g_d = \frac{(d-1)(d-2)}{2}.$$

Sketch. Consider the pencil

$$f_{d+1,t} = tf_{d+1} + (1-t)f_d z$$

where f_d defines a smooth plane curve of degree d. Let $C_{d+1,t} = \mathbb{V}(f_{d+1,t})$. In the limit as $t \to 0$, the smooth plane curves converges to a cusp curve with two components, a smooth plane curve C_d of degree d and a line L. Now $L \cap C_d$ is d-points. By a version of Gromov's compactness for higher genus curves, C_{d+1} is homotopic to the result when we connect L and C_d. Thus, the genus of C_{d+1} is

$$g_{d+1} = g_d + d.$$

The result then follows by induction. □

I.5 Monotone symplectic manifolds

Definition I.5.1. A symplectic manifold (M, ω) is *monotone* if there exists $\lambda > 0$, such that

$$\omega(A) = \lambda c_1(A)$$

for all $A \in \pi_2(M)$.

This is a form of positivity condition, and although this won't appear explicitly in any of the statements, it is required for the "limiting" strata to be of lower dimension, and so the evaluation map defines a pseudocycle.

Definition I.5.2. The *minimal Chern number* N of M is defined by

$$\langle c_1, \pi_2(M) \rangle = N\mathbb{Z}.$$

Example I.5.3. Let $M = \mathbb{P}^n$. The anticanonical bundle is $\mathcal{O}(n+1)$, and so the minimal Chern number is $n+1$. Moreover, $c_1 = (n+1)[\omega]$, and so $\mathbb{P}^n$ is monotone.

I.6 Evaluation map

In this section, we will define a homomorphism

$$\Psi_{A,p} : \mathrm{H}_d(M^p, \mathbb{Z}) \to \mathbb{Z}$$

for $p = 3, 4$, and $d = 2n(p-1) - 2c_1(A)$

Intuitively, fix a tuple $z = (z_1, \ldots, z_p) \in (\mathbb{P}^1)^p$ and cycles $Z_i \in \mathrm{H}_{d_i}(M; \mathbb{Z})$, with $d_1 + \cdots + d_p = d$. We would like to define

$$\Psi_{A,p}(Z_1, \ldots, Z_p) = \left|\{u \in \mathcal{M}(A, J) \mid u(z_j) \in Z_j\}\right|$$

d is chosen such that for a generic choice of J and z, the right hand side is a finite set.

To make this precise, let $J \in \mathcal{J}_{\text{reg}}(A)$, and so $\mathcal{M}(A, J)$ is a manifold of dimension $2n + 2c_1(A)$. Consider the evaluation map

$$\begin{aligned} \mathrm{ev}_{A,J,z} : \mathcal{M}(A, J) &\to M^p \\ u &\mapsto (u(z_1), \ldots, u(z_p)). \end{aligned}$$

For a generic choice of J, $\mathrm{ev}_{A,J,z}$ is a pseudocycle. The limit points correspond exactly to cusp curves from Gromov's compactness theorem, and these form the lower-dimensional strata of $\mathrm{ev}_{A,J,z}(\mathcal{M}(A, J))$. With this, we have an associated homomorphism

$$\widetilde{\Psi} : \mathrm{H}_d(M^p; \mathbb{Z}) \to \mathbb{Z}$$

as in section I.2.3, by choosing a smooth pseudocycle representing a homology class. Thus, we can define

$$\Psi_{A,J,z}(Z_1, \ldots, Z_p) = \widetilde{\Psi}(Z_1 \times \cdots \times Z_d)$$

where $\times$ denotes the cross product in the Künneth theorem.

Theorem I.6.1 ([MS94, Lemma 7.4.1]). *$\Psi_{A,J,z}$ is independent of the choice of $J \in \mathcal{J}_{\text{reg}}(A)$ and $z \in (\mathbb{P}^1)^p$.*

The usual arguments show that if $J_0, J_1 \in \mathcal{J}_{\text{reg}}(A)$ and $z_0, z_1 \in (\mathbb{P}^1)^n$, then ev_{A,J_0,z_0} and ev_{A,J_1,z_1} determine bordant pseudocycles, and so the associated homomorphisms are equal. With this, $\Psi_{A,p}$ is then well defined.

Example I.6.2. Consider the case when $M = \mathbb{P}^n$. Let L be the homology class of a line, H be the homology class of a hyperplane. Then $c_1(A) = n + 1$, and so when $p = 3$, we get

$$d = 2n(2) - 2(n + 1) = 4n - 2n - 2 = 2n - 2.$$

Thus, $\Psi_L(\mathrm{pt}, \mathrm{pt}, H)$ is well defined. This counts the number of lines through two generic points, intersecting a generic hyperplane. One can verify that the standard complex structure on $\mathbb{P}^n$ satisfies the transversality requirements, and that $\Psi_L(\mathrm{pt}, \mathrm{pt}, H) = 1$ as expected.

Example I.6.3. For example, in $\mathbb{P}^2$, if L is the class of a line, we have that

$$\Psi_{2L}(\mathrm{pt}, \mathrm{pt}, \mathrm{pt}, \mathrm{pt}) = 1.$$

In algebraic geometry, five points determine a conic, but in the above, we only have four. To see this, the pair $(\mathbb{P}^1, z)$ is a *marked* curve. $\mathrm{PSL}(2, \mathbb{C})$ acts triply-transitively on $\mathbb{P}^1$, and so when $p = 3$, they are all equivalent. When $p = 4$, markings $z, z' \in (\mathbb{P}^1)^4$ are equivalent if and only if they have the same cross-ratio.

In particular, the space of all conics through four generic points $q_1, \ldots, q_4$ has complex dimension 1, and there exists a unique element in this space which sends z_i to q_i.

Remark I.6.4. It is also possible to define $\Psi_{A,p}$ for different values of p. However, as we will only need $p = 3$ to define the quantum cohomology ring, and $p = 4$ to prove that the quantum product is associative, we will not consider the general case.

I.7 Quantum cohomology

Assume that (M, ω) is a $2n$-dimensional compact monotone symplectic manifold, with minimal Chern number $N \geq 2$. By rescaling ω, we will assume $[\omega]$ defines an integral homology class, with $\langle \omega, \pi_2(M) \rangle = 1$. That is, $\lambda = 1/N$.

Definition I.7.1. The *quantum cohomology ring* is the tensor product of graded rings

$$\mathrm{QH}^*(M) = \mathrm{H}^*(M) \otimes \mathbb{Z}[q, q^{-1}]$$

where $\deg(q) = 2N$. Thus, $\mathrm{QH}^k(M)$ consists of elements of the form

$$a = \sum_{i \in \mathbb{Z}} a_i q^i$$

where $a_i \in \mathrm{H}^{k-2Ni}(M)$.

Remark I.7.2. Multiplication by q defines an isomorphism $\mathrm{QH}^k(M) \cong \mathrm{QH}^{k+2N}(M)$, and we have a $\mathbb{Z}/2N$-grading on $\mathrm{QH}^*(M)$. Equivalently, we can define

$$\mathrm{QH}^k(M) = \bigoplus_{j \equiv k \pmod{2N}} \mathrm{H}^j(M)$$

which gives us a $\mathbb{Z}/2N$-graded abelian group. The Novikov variable q allows us to get a $\mathbb{Z}$-grading.

I.7.1 Quantum product

Definition I.7.3. For $a \in \mathrm{H}^k(M), b \in \mathrm{H}^\ell(M)$, define

$$a * b = \sum_A (a * b)_A q^{c_1(A)/N} \in \mathrm{QH}^{k+\ell}(M).$$

This extends to a product on $\mathrm{QH}^*(M)$. Here, $(a * b)_A \in \mathrm{H}^{k+\ell-2c_1(A)}(M)$ is defined as follows: for $\gamma \in \mathrm{H}_{k+\ell-2c_1(A)}(M)$,

$$\int_\gamma (a * b)_A = \Psi_A(\mathrm{PD}(a), \mathrm{PD}(b), \gamma).$$

We note that

$$\deg(\mathrm{PD}(a)) = 2n - k \quad \text{and} \quad \deg(\mathrm{PD}(b)) = 2n - \ell$$

and

$$(2n - k) + (2n - \ell) + (k + \ell - 2c_1(A)) = 4n - 2c_1(A)$$

and so the right hand side is well defined.

In the introduction, we claimed that the quantum product is a 'deformation' of the usual cup product. The constant term is the case when $A = 0$. Since J is ω-compatible, we have that for any $u \in \mathcal{M}(A, J)$,

$$\mathrm{E}(u) = \omega(A) = 0$$

Thus, u is constant. In particular, $\Psi_0(Z_1, Z_2, Z_3)$ is just the triple intersection number $Z_1 \cdot Z_2 \cdot Z_3$. From this, we see that $(a * b)_0 = ab$.

I.8 Associativity

Next, we will show that the quantum product is associative. Again, it suffices to consider the quantum product of ordinary cohomology classes. Fix $a \in \mathrm{H}^j(M), b \in \mathrm{H}^k(M), c \in \mathrm{H}^\ell(M)$. For $A \in \mathrm{H}_2(M)$, define

$$\xi_A = \mathrm{PD}((a * b)_A) \in \mathrm{H}_{2n+2c_1(A)-j-k}(M)$$

With this, we have that

$$\int_\delta ((a * b) * c)_A = \sum_B \Psi_B(\xi_{A-B}, \mathrm{PD}(c), \delta)$$

First of all, we would like to find a pseudocycle with represents the homology class ξ_A. Fix $z_1, z_2, z_3 \in \mathbb{P}^1$, and consider the manifold

$$V = \{u \in \mathcal{M}(A, J) \mid u(z_1) \in \mathrm{PD}(a), u(z_2) \in \mathrm{PD}(b)\}$$

with the map

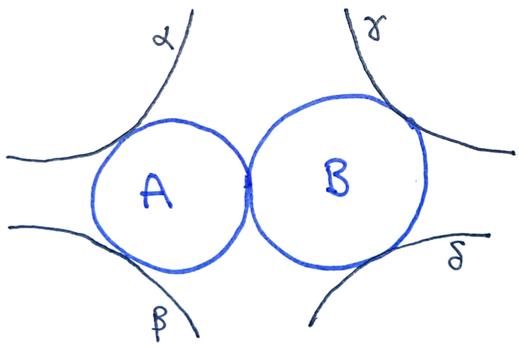
$$\begin{aligned} f : V &\to M \\ u &\mapsto u(z_3). \end{aligned}$$

This defines a pseudocycle representing ξ_A. Geometrically, ξ_A is the union of all curves in $\mathcal{M}(A, J)$, which pass through $\mathrm{PD}(a)$ and $\mathrm{PD}(b)$. Thus, $\Psi_B(\xi_A, \gamma, \delta)$ counts the number of curves in $\mathcal{M}(B, J)$, which passes through ξ_A, γ, δ. But by definition, every curve which meets ξ_A meets a curve in $\mathcal{M}(A, J)$, which passes through $\mathrm{PD}(a)$ and $\mathrm{PD}(b)$. Thus, $\Psi_B(\xi_A, \gamma, \delta)$ counts the number of cusp curves $u = (u^1, u^2)$, with $[u^1] = A, [u^2] = B$, such that u^1 passes through $\mathrm{PD}(a), \mathrm{PD}(b)$, and u^2 passes through γ, δ.

Let

$$\Psi_{A,B}(\alpha, \beta; \gamma, \delta) = \Psi_B(\xi_A, \gamma, \delta)$$

where in the right hand side, to define ξ_A, we take $a = \mathrm{PD}(\alpha), b = \mathrm{PD}(\beta)$. Geometrically, $\Psi_{A,B}$ represents



We then have that

$$\int_{\delta} ((a * b) * c)_A = \sum_B \Psi_{B,A-B}(\mathrm{PD}(a), \mathrm{PD}(b); \mathrm{PD}(c), \delta).$$

Analogous to our definition of the quantum cup product, we can also define a triple product, via

$$P(a, b, c) = \sum_A (a * b * c)_A q^{c_1(A)/N}$$

where

$$\int_{\delta} P(a, b, c)_A = \Psi_A(\mathrm{PD}(a), \mathrm{PD}(b), \mathrm{PD}(c), \delta)$$

The triple product is clearly skew-symmetric, and so it suffices to show the following:

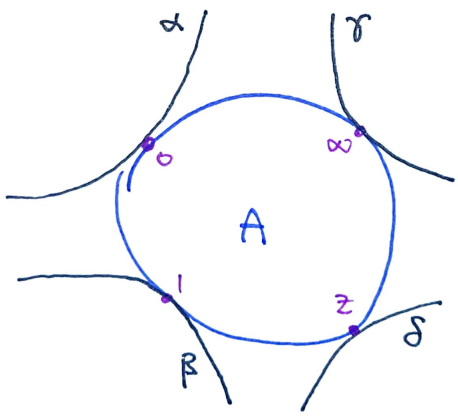
Proposition I.8.1 ([MS94, Lemma 8.2.4])**.**

$$\Psi_A(\alpha, \beta, \gamma, \delta) = \sum_B \Psi_{B,A-B}(\alpha, \beta; \gamma, \delta)$$

Sketch proof. Consider the evaluation map

$$\begin{aligned} \mathrm{ev}_z : \mathcal{M}(A, J) &\to M^4 \\ \mathrm{ev}_z(u) &= (u(0), u(1), u(\infty), u(z)) \end{aligned}$$

where $z \in \mathbb{C} \setminus \{0, 1\}$. $\Psi_A(\alpha, \beta, \gamma, \delta)$ counts the intersection between ev_z and $\alpha \times \beta \times \gamma \times \delta$. Diagrammatically, this is



Consider a sequence (u_k) in $\mathcal{M}(A, J)$, and a sequence (z_k) in $\mathbb{C}$ converging to 0, such that

$$u_k(0) \in \alpha, u_k(1) \in \beta, u_k(\infty) \in \gamma \text{ and } u_k(z_k) \in \delta.$$

We have two cases

(i) The sequences converge to different points.

(ii) The sequences $u_k(z_k)$ and $u_k(0)$ converge to the same point in $\gamma \cap \delta$.

In the first case, the derivative of w_k must blow up near 0, and we obtain a bubble which meets γ, δ and lies in a class B. In this case we get a limiting cusp curve corresponding to

$$\Psi_{A-B,B}(\alpha, \beta; \gamma, \delta)$$

In the second case, the limit is a curve which contributes to

$$\Psi_A(\alpha, \beta, \gamma \cap \delta) = \Psi_{A,0}(\alpha, \beta; \gamma, \delta)$$

That it, we can think of this as bubbling off a curve with zero energy. Thus, if z is sufficiently close to 0, every curve we count in $\Psi_A(\alpha, \beta, \gamma, \delta)$ is counted in $\sum_B \Psi_{B,A-B}(\alpha, \beta; \gamma, \delta)$. $\square$

I.9 Example - Projective space

To illustrate this theory, we will now compute the quantum cohomology of $\mathbb{P}^n$. We will revisit this in section V.5, where we will use the machinery developed in this essay to compute it using a combinatorial description. For now, we will compute it from the definitions.

Recall that

$$\mathrm{H}^*(\mathbb{P}^n) = \frac{\mathbb{Z}[p]}{\langle x^{p+1} \rangle}$$

where $p \in \mathrm{H}^2(\mathbb{P}^n)$ is Poincaré dual to a hyperplane. Let $L \in \mathrm{H}_2(\mathbb{P}^n)$ be the class of a line. The first Chern class in this case is $c_1(L) = n + 1$, and so $N = n + 1$.

As a graded abelian group,

$$\mathrm{QH}^k(\mathbb{P}^n) = \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}.$$

Next, to define the quantum product, recall that we defined a homomorphism

$$\Psi_{A,3} : \mathrm{H}_{4n-2c_1(A)}(M^3; \mathbb{Z}) \to \mathbb{Z}.$$

Thus, when $A = mL$, we have

$$\Psi_{mL,3} : \mathrm{H}_{4n-2m(n+1)}(M^3; \mathbb{Z}) \to \mathbb{Z}.$$

For this to be non-zero, we will need $0 \leq 4n - 2m(n+1) \leq 6n$. Rearranging to get a constraint on m, we get that

$$0 \leq m \leq \frac{2n}{n+1} < 2$$

and so $m = 0, 1$ are the only possibilities. Thus, for $a \in \mathrm{H}^k(M)$, $b \in \mathrm{H}^\ell(M)$, we have that

$$a * b = (a * b)_0 + (a * b)_L q = ab + (a * b)_L q$$

as the constant term is the usual cup product. By definition, $(a*b)_L \in \mathrm{H}^{k+\ell-2(n+1)}(M)$, and so if $k+\ell < 2(n+1)$, then $(a * b)_L = 0$, and so in this case, $a * b = ab$.

Let $p = \mathrm{PD}(\mathbb{P}^{n-1}) \in \mathrm{H}^2(M)$. We claim that

$$p * p^n = q \in \mathrm{QH}^{2n+2}(M).$$

In this case, $(p * p^n)_0 = 0$ for degree reasons, and so we just need to show that $(p * p^n)_L = 1 \in \mathrm{H}^0(\mathbb{P}^n)$. $\mathrm{H}_0(\mathbb{P}^n)$ is generated by the class of a point, and so all we need to show is that

$$\int_{\mathrm{pt}} (p * p^n)_L = 1.$$

But we have that

$$\int_{\mathrm{pt}} (p * p^n)_L = \Psi_L(\mathrm{PD}(p), \mathrm{PD}(p^n), \mathrm{pt}) = \Psi_L([\mathbb{P}^{n-1}], \mathrm{pt}, \mathrm{pt}).$$

and this is 1, as computed in example I.6.2. In general, one can show that for $k + \ell \geq n + 1$,

$$p^k * p^\ell = p^{k+\ell-(n+1)}.$$

To conclude, we have that

$$\mathrm{QH}^*(\mathbb{P}^n) = \frac{\mathbb{Z}[p, q^{\pm 1}]}{\langle p^{n+1} = q \rangle}$$

which we can view as a deformation of the cohomology ring of $\mathbb{P}^n$, since if we set $q = 1$, then we get

$$\frac{\mathbb{Z}[p]}{\langle p^{n+1} = 1 \rangle}.$$

Chapter II

Hamiltonian Floer homology

In this chapter, we will discuss the construction of Hamiltonian Floer homology. The starting point for Hamiltonian Floer homology is the study of 1-periodic orbits of a Hamiltonian vector field, inspired by the Arnol'd conjecture. The description in this chapter is a mix of the material from [Sei97] and [MS94]. In particular, in [MS94, Chapter 10], they define Hamiltonian Floer *cohomology*. In this case, the difference is in the grading, and won't affect any of the results.

Let (M, ω) be a compact monotone symplectic manifold of dimension $2n$.

Definition II.0.1. The *free loop space* of M is

$$\Lambda M = C^\infty(S^1, M)$$

and $\mathcal{L}M \subseteq \Lambda M$ is the space of all contractible loops.

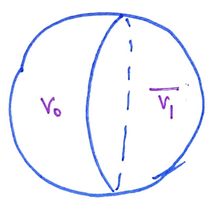
Hamiltonian Floer homology can be considered as an infinite-dimensional version of Morse homology, performed on $\mathcal{L}M$. A reference for Morse homology in the finite-dimensional setting is [Nic11].

II.1 Covering space

Consider pairs $(v, x) \in C^\infty(D^2, M) \times \mathcal{L}M$, such that $x = v|_{\partial D^2}$. We can define an equivalence relation $\sim$, where $(v_0, x_0) \sim (v_1, x_1)$ if $x_0 = x_1$, and

$$\omega(v_0 \# \overline{v_1}) = c_1(v_0 \# \overline{v_1}) = 0$$

Here, $v_0 \# \overline{v_1} : S^2 \to M$ is defined by



Let $\widetilde{\mathcal{L}M}$ be the set of equivalence classes under $\sim$. The projection map $p[v, x] = x$ defines a covering map $\widetilde{\mathcal{L}M} \to \mathcal{L}M$.

Let

$$\Gamma = \frac{\pi_2(M)}{\omega = c_1 = 0}.$$

For $a \in \pi_2(M)$, $\omega(a)$ and $c_1(a)$ only depends on its equivalence class in Γ. Thus, for $\gamma \in \Gamma$, we can define $\omega(\gamma)$ and $c_1(\gamma)$. In particular, Γ is the deck group for the covering map p.

II.2 Floer chain complex

Let $H \in C^\infty(M \times S^1, \mathbb{R})$ be a 1-periodic Hamiltonian[1]. Let $X_H(t, x)$ be the associated Hamiltonian vector field. Then we have an *action* one-form on $\mathcal{L}M$, given by

$$\alpha_H(x)(\xi) = \int_{S^1} \omega(\dot{x}(t) - X_H(t, x(t)), \xi(t))\mathrm{d}t.$$

The zero set of α_H consists of the 1-periodic solutions to

$$\dot{x}(t) = X_H(t, x(t)).$$

The pullback $p^*\alpha_H$ is exact. That is, $p^*\alpha_H = \mathrm{d}a_H$, where

$$a_H[v, x] = -\int_{D^2} v^*\omega + \int_{S^1} H(t, x(t))\mathrm{d}t.$$

Let $Z = \{\alpha_H = 0\} \subseteq \mathcal{L}M$. Then $\mathrm{Crit}(a_H) = p^{-1}(Z)$ is the set of critical points of a_H. We will assume all critical points are non-degenerate, in the sense of Morse theory. Thus, for $c \in \mathrm{Crit}(a_H)$, we have an associated index $\mu_H(c) \in \mathbb{Z}$[MS94, Section 10.1, p. 155]. Let $\mathrm{Crit}_k(a_H)$ denote the set of critical points of index k.

Definition II.2.1. The *Floer chain complex* $\mathrm{CF}_k(H)$ is the group of formal sums

$$\sum_{c \in \mathrm{Crit}_k(a_H)} m_c \chi^c$$

with $\{c \in \mathrm{Crit}_k(a_H) \mid m_c \neq 0, a_H(c) \geq C\}$ finite for all $C \in \mathbb{R}$, and χ^c is a formal variable. To avoid issues with orientation, we will use $\mathbb{Z}/2$ coefficients in this chapter. That is, $m_c \in \mathbb{Z}/2$.

As in Morse homology, the boundary maps will be given by counting flow lines between critical points. Let $\mathbf{J} = (J_t)_{t \in S^1}$ be a family of almost complex structure on M, with $J_t \in \mathcal{J}(M, \omega)$ for all t. This induces a Riemannian metric on $\mathcal{L}M$, given by

$$\langle \xi, \eta \rangle_{\mathbf{J}} = \int_{S^1} \omega(\xi(t), J_t \eta(t))\mathrm{d}t$$

Pulling this back to $\widetilde{\mathcal{L}M}$ and computing, we find that $\widetilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ is a flow line of $-\nabla a_H$ if and only if $u = p \circ \widetilde{u}$ is a solution to the *Floer equation*

$$\begin{cases} \frac{\partial u}{\partial s} + J_t(u(s,t))\left(\frac{\partial u}{\partial t} - X_H(t, u(s,t))\right) & = 0 \\ (s,t) \in \mathbb{R} \times S^1 \end{cases}$$

Remark II.2.2. The Floer equation can be viewed as a perturbed Cauchy-Riemann equation. When $H = 0$, the Floer equation is the Cauchy-Riemann equation. In particular, it is a first-order elliptic PDE.

Analogous to the finite-dimensional setting, if u is a solution to the Floer equation, with

$$\mathrm{E}(u) = \int \left|\frac{\partial u}{\partial s}\right|^2 < \infty$$

then there exists $c_-, c_+ \in \mathrm{Crit}(a_H)$, such that

$$\lim_{s \to \pm\infty} \widetilde{u}(s) = c_\pm.$$

For $c_-, c_+ \in \mathrm{Crit}(a_H)$, let $\mathcal{M}(c_-, c_+; H, \mathbf{J})$ be the space of solutions with have a lift $\widetilde{u}$, with limits c_-, c_+. $\mathbb{R}$ acts on $\mathcal{M}(c_-, c_+; H, \mathbf{J})$ by translation in the s-direction. We then define for $c_- \in \mathrm{Crit}_k(a_H)$,

$$\partial(\chi^{c_-}) = \sum_{c_+ \in \mathrm{Crit}_{k-1}(a_H)} |\mathcal{M}(c_-, c_+; H, \mathbf{J})/\mathbb{R}| \cdot \chi^{c_+}.$$

Definition II.2.3. The *Hamiltonian Floer homology* of (M, ω) is the homology of the Floer chain complex, denoted $\mathrm{HF}_*(M, \omega)$.

This exists for a generic choice of H and $\mathbf{J}$, and is indepenent of the choice.

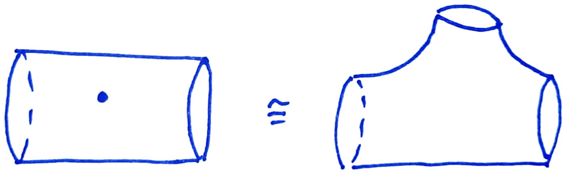
[1]We will identify $S^1 = \mathbb{R}/\mathbb{Z}$ throughout.

II.3 Pair-of-pants product

Consider the punctured surface $\Sigma = (\mathbb{R} \times S^1) \setminus (0;0)$. We can think of this as a pair-of-pants, via

$$\begin{aligned} e : \mathbb{R}_{<0} \times S^1 &\to \Sigma \\ (s,t) &\mapsto \left(\frac{1}{4} e^{2\pi s} \cos(2\pi t), \frac{1}{4} e^{2\pi s} \sin(2\pi t) \right) \end{aligned}$$

This defines a cylindrical structure near the puncture, as shown in the diagram below.



For $u : \Sigma \to M$, we will write $u^e = u \circ e$. Suppose $u \in C^\infty(\Sigma, M)$, $v_0 \in C^\infty(D^2, M)$ are such that

$$\lim_{s \to -\infty} u^e(s, \cdot) = v_0|_{\partial D^2}$$

then these two glue to a smooth map $u \# v_0 : \widehat{\Sigma} \to M$. Identifying $\widehat{\Sigma} = \mathbb{R} \times S^1$, we get a path $\mathbb{R} \to \Lambda M$. On the two ends of $\widehat{\Sigma}$, $u \# v_0$ has the same limits as u. Geometrically, we can attach a disc to one of the ends of Σ, which gives a cylinder.

For $c_-, c_0, c_+ \in \widetilde{\mathcal{L}M}$, we say that $u \in C^\infty(\Sigma, M)$ *converges to* (c_-, c_0, c_+) *if*

- $$\lim_{s \to \pm\infty} u(s, \cdot) = p(c_\pm)$$
- $$\lim_{s \to -\infty} u^e(s, \cdot) = p(c_0)$$
- for $c_0 = [v_0, x_0]$, $u \# v_0$ has a lift $\widetilde{u \# v_0} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ with limits c_-, c_+.

The appropriate version of the Floer equation in this case are

$$\frac{\partial u}{\partial s}(s,t) + J_{s,t}(u(s,t)) \left(\frac{\partial u}{\partial t}(s,t) - X_H(s,t,u(s,t)) \right) = 0 \quad \text{for} \quad (s,t) \in \Sigma \setminus e((-\infty, -1] \times S^1)$$

and

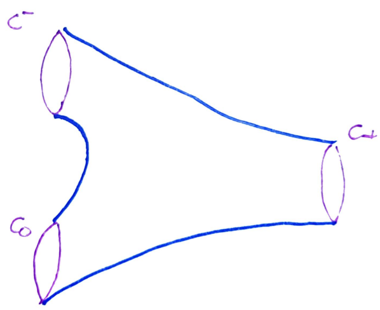
$$\frac{\partial u^e}{\partial s}(s,t) + J_{e(s,t)}(u^e(s,t)) \left(\frac{\partial u^e}{\partial t}(s,t) - X_H(e(s,t), u^e(s,t)) \right) = 0 \quad \text{for} \quad (s,t) \in \mathbb{R}_{<0} \times S^1.$$

For c_-, c_0, c_+, let $\mathcal{M}_{\mathrm{PP}}(c_-, c_0, c_+; H, \mathbf{J})$ be the space of solutions to the above, which converge to (c_-, c_0, c_+). Again, we need a generic choice of H and $\mathbf{J}$.

Definition II.3.1. Define

$$\mathrm{PP}(\chi^{c_-}, \chi^{c_0}) = \sum_{c_+} |\mathcal{M}_{\mathrm{PP}}(c_-, c_0, c_+; H, \mathbf{J})| \cdot \chi^{c_+}.$$

This induces a map $*_{\mathrm{PP}} : \mathrm{HF}_*(M, \omega) \otimes \mathrm{HF}_*(M, \omega) \to \mathrm{HF}_*(M, \omega)$, called the *pair-of-pants product*.



II.4 Novikov ring

In the next section, we will define an isomorphism between Floer homology and quantum cohomology. To do this, we will need to generalise the coefficients which we consider in quantum cohomology. Let R be a ring, which throughout will be one of $\mathbb{Z}, \mathbb{Z}/2, \mathbb{R}$.

Definition II.4.1. The *Novikov ring* Λ is the set of sums

$$\lambda = \sum_{A \in \Gamma} \lambda_A t^A$$

subject to the finiteness condition that for all $C > 0$, there are only finitely many A with $\lambda_A \neq 0$ and $\omega(A) \leq C$. Multiplication is defined by

$$\lambda * \mu = \sum_{A,B} \lambda_A \mu_B t^{A+B}.$$

We can then define the quantum cohomology ring as

$$\mathrm{QH}^*(M) = \mathrm{H}^*(M) \otimes \Lambda.$$

Λ is naturally a $\mathbb{Z}$-graded ring, where we define $\deg(t^A) = 2c_1(A)$. This gives a grading on the quantum cohomology ring as follows.

$$\mathrm{QH}^k(M) = \bigoplus_{j=0}^{2n} \mathrm{H}_j(M) \otimes \Lambda_{k-j}$$

Thus, a generic element of $\mathrm{QH}^k(M)$ is of the form

$$a = \sum_A a_A t^A$$

where $a_A \in \mathrm{H}^{k-2c_1(A)}(M)$, and the finiteness condition implies that for all $C > 0$, there are only finitely many A with $a_A \neq 0$ and $\omega(A) \leq C$. The same formula

$$\int_\gamma (a * b)_A = \Psi_A(\mathrm{PD}(a), \mathrm{PD}(b), \gamma)$$

defines the quantum cup product on $\mathrm{QH}^*(M)$, which makes it into a graded ring.

II.5 Isomorphism with quantum cohomology

For the next chapter on the Seidel representation, we will need an isomorphism between the Floer homology ring and the quantum cohomology ring. We will sketch the construction due to Piunikhin, Salamon, and Schwarz here, as we will not need the details. This isomorphism was originally constructed in [PSS96]. The exposition here is based on [PSS96, Example 3.3] and [MS12].

First of all, for $c \in \mathrm{Crit}(a_H)$, consider the manifold $\widetilde{M}(c)$ of maps $u : \mathbb{C} \to M$, where the map $(s,t) \mapsto u(e^{2\pi(s+it)})$ satisfies the Floer equation, with limit c.

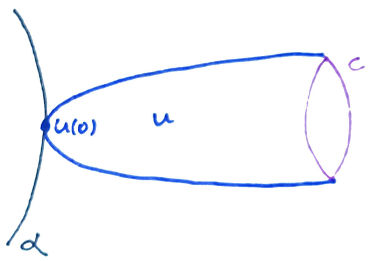
Let α be a cycle, with $\deg(\alpha) + \dim(\widetilde{M}(c)) = 2n$. Then we can define $n(\alpha, c)$ to be the number of $u \in \widetilde{M}(c)$ such that $u(0) \in \alpha$, counted with appropriate signs. Then we can define[2]

$$\begin{aligned} \phi : \mathrm{QH}^*(M, \omega) &\to \mathrm{CF}_*(H) \\ \alpha = \sum_{A,c} \alpha_A t^A &\mapsto n(\alpha_A, (-A)\#c)\chi^C \end{aligned}$$

where we sum over all $A \in \Gamma$, and all c such that $n(\alpha_A, (-A)\#c)$ makes sense. $\phi(\alpha)$ defines a cycle, and the resulting homology class is independent of choices. Thus, we have an induced map $\Phi : \mathrm{QH}^*(M, \omega) \to \mathrm{HF}_*(M, \omega)$.

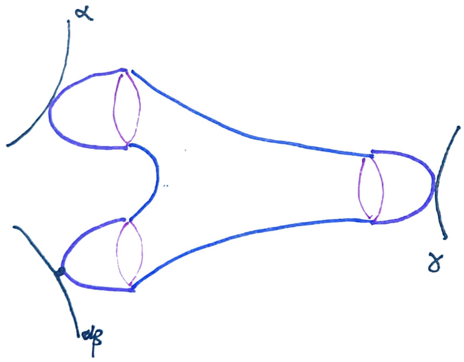
Diagrammatically, Φ is defined by counting *spiked discs* as follows:

[2] with implicit dependence on H, J and representatives for the homology classes.



The inverse map is given by reversing the diagram.

The fact that it is a ring homomorphism follows from the following diagram:



We can see that if we add three "caps" to the pair-of-pants, we get a sphere, and this relates the pair-of-pants product to the quantum cup product.

Chapter III

Seidel representation

In this chapter, we will define the Seidel representation. First, we will define the covering space $\widetilde{G}$ of $\Lambda\,\mathrm{Ham}(M, \omega)$, and construct the action on loops in M. This action then lifts to an action of the covering space $\widetilde{\mathcal{L}M}$ as constructed in the previous chapter, which then induces an action on Hamiltonian Floer homology.

Next, we will define *Hamiltonian fibre bundles*, which are symplectic fibre bundles, with a globally defined closed 2-form which restrict to the symplectic form on the fibres. For each element of $\widetilde{G}$, we associate a Hamiltonian fibre bundle. Counting pseudoholomorphic sections of this bundle will define an element in quantum cohomology.

Finally, we will use the isomorphism defined in section II.5 to show that these two constructions are equivalent, and use this to deduce properties of the Seidel representation.

Throughout, we will follow [Sei97]. One change is that in [Sei97], the map is defined to take value in *quantum homology*, rather than quantum cohomology. In this case, we have a natural Poincaré duality pairing between quantum homology and quantum cohomology, and so we have translated the results. See [MT06, Section 2.2] for more details about the Poincaré duality pairing.

III.1 Group action on loops

Let $\mathrm{Ham}(M, \omega)$ denote the group of Hamiltonian diffeomorphisms of (M, ω), and G the group of smooth based loops in $\mathrm{Ham}(M, \omega)$. We will use the C^∞ topology on $\mathrm{Ham}(M, \omega)$ and on G.

G acts on ΛM by

$$(g \cdot x)(t) = g_t(x(t)).$$

Lemma III.1.1 ([Sei97, Lemma 2.2])**.** *This action restricts to an action on $\mathcal{L}M$, the space of contractible loops.*

Let H be a generic periodic Hamiltonian on M. Recall that the Floer chain complex $\mathrm{CF}_k(H)$ is generated by critical points of

$$a_H[v, x] = -\int_{D^2} v^*\omega + \int_{S^1} H(t, x(t))\mathrm{d}t.$$

We say that a Hamiltonian K_g *generates* $g \in G$ if

$$\frac{\partial g_t}{\partial t}(y) = X_{K_g}(t, g_t(y)).$$

If H is another Hamiltonian, we define

$$H^g(t, y) = H(t, g_t(y)) - K_g(t, g_t(y))$$

By a direct computation, we have that $g^*\alpha_H = \alpha_{H^g}$.

Proof. Suppose $g(\mathcal{L}M)$ is a connected component of ΛM which is distinct from $\mathcal{L}M$. If H is small, then α_H has no zeroes in $g(\mathcal{L}M)$ since for sufficiently small H, 1-periodic orbits of H will be contractible. Thus, α_{H^g} has no zeroes on $\mathcal{L}M$.

But this contradicts the Arnol'd conjecture, which implies that for every Hamiltonian there exists a contractible 1-periodic orbit. The Arnol'd conjecture was proven in the monotone case by Floer[Flo89]. ☐

In turn, this action lifts to a homeomorphism of $\widetilde{\mathcal{L}M}$. Let $\widetilde{G} \subseteq G \times \mathrm{Homeo}(\widetilde{\mathcal{L}M})$ be the subgroup of pairs $\mathbf{g} = (g, \widetilde{g})$, where $\widetilde{g}$ is a lift of g. We have an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{G} \xrightarrow{\mathrm{pr}_1} G \longrightarrow 1 \tag{III.1}$$

of topological groups, where Γ is the deck group of the covering map $p : \widetilde{\mathcal{L}M} \to M$ constructed in the previous chapter.

For $c = [v, x] \in \widetilde{\mathcal{L}M}$, we have an associated trivialisation of $x^*\mathrm{T}M$, given by $\tau_c : x^*\mathrm{T}M \to S^1 \times \mathbb{R}^{2n}$, which extends over $v^*\mathrm{T}M$. This is independent of the choice of v and x. Next, for $\mathbf{g} \in \widetilde{G}$, we have a loop

$$\ell(t) = \tau_{\widetilde{g}(c)}(t) \mathrm{d}g_t(x(t)) \tau_c(t)^{-1} \in \mathrm{Sp}(2n, \mathbb{R}).$$

Up to homotopy, this does not depend on the choice of c or the trivialisations. We define

$$I(\mathbf{g}) = \deg(\ell)$$

where $\deg : \mathrm{H}_1(\mathrm{Sp}(2n, \mathbb{R})) \to \mathbb{Z}$ is the isomorphism induced by the determinant map on $\mathrm{U}(n) \subseteq \mathrm{Sp}(2n, \mathbb{R})$.

Lemma III.1.2 ([Sei97, Lemma 2.6])**.** *$I(\mathbf{g})$ depends only $[\mathbf{g}] \in \pi_0(\widetilde{G})$, and $I : \pi_0(\widetilde{G}) \to \mathbb{Z}$ is a group homomorphism. For $\gamma \in \Gamma$, $I(1, \gamma) = c_1(\gamma)$.*

III.2 Action on Floer homology

Define

$$\begin{aligned} \mathrm{CF}(\mathbf{g}) : \mathrm{CF}_k(H^g) &\to \mathrm{CF}_{k-2I(\mathbf{g})}(H) \\ \chi^c &\mapsto \chi^{\widetilde{g}(c)} \end{aligned}$$

This defines an isomorphism of chain complexes, and so we have an induced isomorphism on homology

$$\mathrm{HF}(\mathbf{g}) : \mathrm{HF}_k(H^g) \to \mathrm{HF}_{k-2I(\mathbf{g})}(H)$$

Now, by the usual arguments, if $(H, \mathbf{J})$ is 'regular', then so is $(H^g, \mathbf{J}^g)$, where $J_t^g = g_t^* J_t$. With this in mind, we have a well defined map

$$\mathrm{HF}(\mathbf{g}) : \mathrm{HF}_*(M, \omega) \to \mathrm{HF}_*(M, \omega).$$

In particular, from the definitions, we have that

Proposition III.2.1 ([Sei97, Proposition 4.9])**.** *$HF(1) = \mathrm{id}$ and $\mathrm{HF}(\mathbf{g}_1\mathbf{g}_2) = \mathrm{HF}(\mathbf{g}_1)\mathrm{HF}(\mathbf{g}_2)$.*

Moreover, this action is compatible with the pair-of-pants product, in the following sense.

Proposition III.2.2 ([Sei97, Proposition 6.3])**.** *For all $\mathbf{g} \in \widetilde{G}$, $a, b \in \mathrm{HF}_*(M, \omega)$,*

$$\mathrm{HF}_*(\mathbf{g})(a *_{\mathrm{PP}} b) = \mathrm{HF}(\mathbf{g})(a) *_{\mathrm{PP}} b.$$

Thus, if u is the multiplicative unit of $\mathrm{HF}_*(M, \omega)$, then

$$\mathrm{HF}_*(\mathbf{g})(a) = \mathrm{HF}_*(\mathbf{g})(u) *_{\mathrm{PP}} a.$$

III.3 Symplectic fibre bundles

Definition III.3.1. A *symplectic fibre bundle* is a fibre bundle $\pi : E \to B$, with a smooth family $\Omega = (\Omega_b)_{b \in B}$ of symplectic forms on the fibres.

Let $\mathrm{V}E = \ker(\mathrm{d}\pi) \leq \mathrm{T}E$ denote the *vertical bundle*. Then Ω defines a symplectic structure on the vector bundle $\mathrm{V}E$.

We will focus on the case when $B = S^2$. We can think of $S^2 = D^+ \cup_{S^1} D^-$, where D^+, D^- are the closed upper and lower hemispheres. Fix a point $z_0 \in D^-$.

Definition III.3.2. A *symplectic fibre bundle* (E, Ω) *over* S^2 *with fibre* (M, ω) is a symplectic fibre bundle, along with a fixed symplectomorphism $i : (M, \omega) \to (E_{z_0}, \Omega_{z_0})$.

In particular, we will be interested in the following:

Definition III.3.3. A *Hamiltonian fibre bundle* is a symplectic fibre bundle $(E, \Omega) \to B$, along with a closed 2-form $\widetilde{\Omega}$ on E, such that $\widetilde{\Omega}|_{E_b} = \Omega_b$ for all $b \in B$.

III.3.1 Sections

Definition III.3.4. Let s_0, s_1 be continuous sections of a Hamiltonian fibre bundle $(E, \widetilde{\Omega})$ over S^2. We say that s_0, s_1 are *Γ-equivalent* if $\widetilde{\Omega}(s_0) = \widetilde{\Omega}(s_1)$, and $c_1(VE, \Omega)(s_0) = c_1(VE, \Omega)(s_1)$. If S is the Γ-equivalence class of s, we will write $\widetilde{\Omega}(S) = \widetilde{\Omega}(s)$ and $c_1(VE, \Omega)(S) = c_1(VE, \Omega)(s)$.

Lemma III.3.5 ([Sei97, Lemma 2.10])**.** *Let $(E, \widetilde{\Omega})$ be a Hamiltonian fibre bundle over S^2, with fibre (M, ω). Then*

(i) E has a continuous section,

(ii) for Γ-equivalence classes S_0, S_1, there exists a unique $\gamma \in \Gamma$ such that

$$\widetilde{\Omega}(S_1) = \widetilde{\Omega}(S_0) + \omega(\gamma)$$
$$c_1(VE, \Omega)(S_1) = c_1(VE, \Omega)(S_0) + c_1(\gamma)$$

Conversely, given a Γ-equivalence class S_0 and $\gamma \in \Gamma$, there exists a unique Γ-equivalence class S_1 such that the above holds.

In the setting of (ii), we will write $S_1 = S_0 + \gamma$.

Definition III.3.6. A *normalised Hamiltonian fibre bundle* $(E, \widetilde{\Omega}, S)$ is a Hamiltonian fibre bundle with a Γ-equivalence class S.

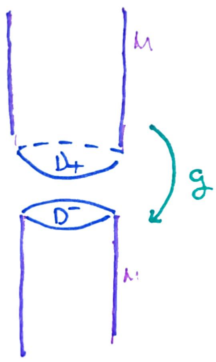
III.3.2 Clutching construction

Let g be a smooth loop in $\mathrm{Ham}(M, \omega)$. We will now construct a Hamiltonian fibre bundle associated to g, using the *clutching construction*. That is, we will take the trivial bundle over the upper and lower hemispheres of S^2, and glue them together along the boundary S^1 using g. A description in the case of vector bundles can be found in [Hat17].

More precisely, we glue the trivial bundles $D^\pm \times (M, \omega)$ together via

$$\begin{aligned} \phi_g : \partial D^+ \times M &\to \partial D^- \times M \\ (t, y) &\mapsto (t, g_t(y)) \end{aligned}$$

This gives a symplectic fibre bundle (E_g, Ω_g), with fibre (M, ω). Moreover, this is in fact Hamiltonian.



Next, for $\mathbf{g} \in \widetilde{G}$, choose a point $c \in \widetilde{\mathcal{L}M}$, and representatives $c = [v, x]$ and $\widetilde{g}(c) = [v', x']$. Then the maps

$$\begin{aligned} s^+_{\widetilde{g}} : D^+ &\to D^+ \times M \\ z &\mapsto (z, v(z)) \end{aligned}$$

and

$$\begin{aligned} s^-_{\widetilde{g}} : D^- &\to D^- \times M \\ z &\mapsto (z, v'(z)) \end{aligned}$$

glue together to form a continuous section $s_{\widetilde{g}}$ of E. The Γ-equivalence class is independent of the choices which we have made, and we obtain a normalised Hamiltonian fibre bundle $(E_g, \widetilde{\Omega}_g, S_{\widetilde{g}})$.

In fact, all normalised Hamiltonian fibre bundles are of this form, and all Hamiltonian fibre bundles are of the form $(E_g, \widetilde{\Omega}_g)$ for some $g \in G$.

III.4 Pseudoholomorphic sections

Let (E, Ω) be a symplectic fibre bundle over S^2, with fibre (M, ω). Let $\pi : E \to S^2$ be the projection map, $i : (M, \omega) \to (E_{z_0}, \Omega_{z_0})$ the fixed isomorphism.

Let $\mathcal{J}(E, \Omega)$ be the space of $\mathbf{J} = (J_z)_{z \in S^2}$, such that J_z is an almost complex structure on E_z compatible with Ω_z.

Definition III.4.1. Let j be the usual complex structure on $S^2 = \mathbb{P}^1$. We say that an almost complex structure $\widehat{J}$ on E is *compatible with j and* $\mathbf{J}$ if

1. π is $(\widehat{J}, j)$-holomorphic, i.e.
$$j \circ \mathrm{d}\pi = \mathrm{d}\pi \circ \widehat{J}$$
2. $\widehat{J}|_{E_z} = J_z$ for all $z \in S^2$.

Let $\widehat{\mathcal{J}}(j, \mathbf{J})$ be the space of all such almost complex structures.

Definition III.4.2. For $\widehat{J} \in \widehat{J}(j, \mathbf{J})$, a smooth section $s : S^2 \to E$ is $(j, \widehat{J})$-holomorphic if

$$\mathrm{d}s(z) \circ j = \widehat{J}(s(z)) \circ \mathrm{d}s(z)$$

Let $\mathcal{S}(j, \widehat{J})$ be the space of all such sections.

Locally, a holomorphic section looks like a J-holomorphic map $u : \mathbb{P}^1 \to M$, except we allow for the the complex structure to vary on each fibre. However, globally it can be very different, and the action of $\widetilde{G}$ means that the maps will be "twisted".

For a Γ-equivalence class S, we will write $\mathcal{S}(j, \widehat{J}, S)$ for the space of $(j, \widehat{J})$-holomorphic sections in S. Using the standard techniques, we obtain that

Proposition III.4.3. *For generic choices of $j, \mathbf{J}$ and $\widehat{J} \in \widehat{\mathcal{J}}(j, \mathbf{J})$, the map*

$$\begin{aligned} \mathrm{ev} : \mathcal{S}(j, \widehat{J}, S) &\to M \\ s &\mapsto i^{-1}(s(z_0)) \end{aligned}$$

is a pseudocycle of dimension $2n + 2c_1(\mathrm{V}E, \Omega)(S)$.

With this, we then have an associated element

$$Q(E, \widetilde{\Omega}, S) = \sum_{\gamma \in \Gamma} \left[\mathrm{ev}_{z_0} \left(\mathcal{S}(j, \widehat{J}, S + \gamma) \right) \right] \cdot t^{\gamma}$$

As before, different choices of $\mathbf{J}, \widehat{J}$ will define bordant pseudocycles, which will define the same homology class, and so the right hand side is independent of choices. Geometrically, this is counting the number of pseudoholomorphic sections which intersect cycles in the fibre above z_0.

III.5 Seidel representation

Definition III.5.1. The *Seidel representation* is the map $q : \widetilde{G} \to \mathrm{QH}^*(M, \omega)$ defined by

$$q(\mathbf{g}) = Q(E_g, \widetilde{\Omega}_g, S_{\widetilde{g}}) \in \mathrm{QH}^*(M, \omega)$$

where $(E_g, \widetilde{\Omega}_g, S_{\widetilde{g}})$ is the normalised Hamiltonian fibre bundle associated to $\mathbf{g}$.

Let $\Phi : \mathrm{QH}^*(M) \to \mathrm{HF}_*(M, \omega)$ be the isomorphism defined in section II.5. The relation between q and the $\widetilde{G}$-action on Floer homology is given by

Theorem III.5.2 ([Sei97, Theorem 8.2])**.** *$q(\mathbf{g}) = \Phi^{-1}\mathrm{HF}(\mathbf{g})\Phi(1)$ where $1 \in \mathrm{QH}^*(M)$ is the multiplicative unit.*

Let $u = \Phi(1)$ denote the multiplicative unit of $\mathrm{HF}_*(M, \omega)$. Then we have the following properties of the Seidel representation, which follow almost immediately from what we have shown.

Corollary III.5.3 ([Sei97, Theorem 1])**.** *For* $\mathbf{g} \in \widetilde{G}$ *and* $b \in \mathrm{HF}_*(M, \omega)$*,*

$$\mathrm{HF}(\mathbf{g})(b) = \mathrm{HF}(\mathbf{g})(u) *_{\mathrm{PP}} b = \Phi(q(\mathbf{g})) *_{\mathrm{PP}} b$$

Proof. This follows from proposition III.2.2 and the theorem. $\square$

In particular, $\mathrm{HF}(\mathbf{g})(u) = \Phi(q(\mathbf{g}))$.

Corollary III.5.4 ([Sei97, Corollary 3])**.** *q defines a homomorphism* $\widetilde{G} \to \mathrm{QH}^*(M, \omega)^\times$.

Proof. We can compute using proposition III.2.1 and proposition III.2.2 that

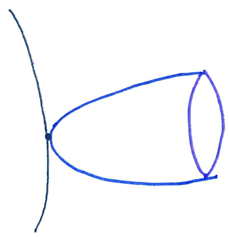
$$\begin{aligned} q(\mathbf{g}_1\mathbf{g}_2) &= \Phi^{-1}\mathrm{HF}(\mathbf{g}_1\mathbf{g}_2)(u) \\ &= \Phi^{-1}\mathrm{HF}(\mathbf{g}_1)\mathrm{HF}(\mathbf{g}_2)(u) \\ &= \Phi^{-1}\mathrm{HF}(\mathbf{g}_1)\Phi(q(\mathbf{g}_2)) \\ &= \Phi^{-1}\Phi(q(\mathbf{g}_1))\Phi(q(\mathbf{g}_2)) \\ &= q(\mathbf{g}_1) * q(\mathbf{g}_2) \end{aligned}$$

$\square$

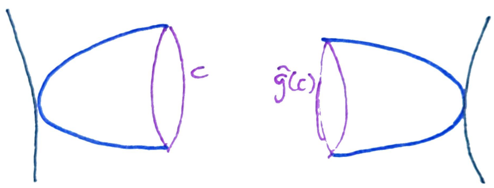
Moreover, $q(\mathbf{g})$ depends only on the class of $\mathbf{g}$ in $\pi_0(\widetilde{G})$, and so it descends to a homomorphism $\pi_0(\widetilde{G}) \to \mathrm{QH}^*(M, \omega)^\times$. Combining these, we see that $q(\mathbf{g})$ defines a left action of $\pi_0(\widetilde{G})$ on $\mathrm{QH}^*(M, \omega)$ by automorphisms of the form

$$b \mapsto q(\mathbf{g}) * b$$

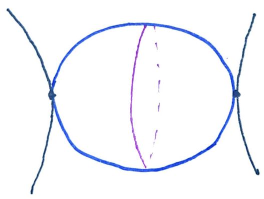
To conclude, we will sketch how the quantum cohomology and Floer homology pictures we have described are related. Recall that Φ is defined by a picture of the following form



Thus, if we take two of the above pictures, but twist the one on the right using the $\widetilde{G}$ action, we get



Finally, gluing the spheres together using the clutching construction, we obtain



which is represents the automorphism defind by the Seidel element. See [MS12, Remark 12.1.7] for more details.

III.6 Kähler manifolds

For the next chapter, we will need the fact that when we are working with Kähler manifolds, we have a more explicit description of the Seidel representation.

Proposition III.6.1 ([Sei97, Proposition 7.11]). *Let (E, J) be a compact complex manifold, $\pi : E \to \mathbb{P}^1$ a holomorphic map with no critical points, $\widetilde{\Omega} \in \Omega^2(E)$ a closed form, such that $\Omega_z = \widetilde{\Omega}|_{E_z}$ makes E_z into a Kähler manifold, $i : (M, \omega) \to (E_{z_0}, \Omega_{z_0})$ is an isomorphism for some $z_0 \in \mathbb{P}^1$. Moreover, assume that*

(i) The space $\mathcal{S}$ of holomorphic sections s of π, with $c_1(\mathrm{V}E)(s) \leq 0$ is connected. In particular, they all lie within the same Γ-equivalence class S_0,

(ii) for any $s \in \mathcal{S}$, $\mathrm{H}^{0,1}(\mathbb{P}^1, s^*\mathrm{V}E) = 0$,

(iii) let $w : \mathbb{P}^1 \to E$ be a holomorphic map, with $\mathrm{im}(w) \subseteq E_z$ for some $z \in \mathbb{P}^1$. Then

(a) $c_1(\mathrm{T}E)(w) \geq 0$.

(b) if w is non-constant, and $c_1(\mathrm{T}E)(w) + c_1(\mathrm{V}E)(S_0) \leq 0$, then $s(z) \notin \mathrm{im}(w)$ for all $s \in \mathcal{S}$.

Assuming all of this, $\mathcal{S}$ is a smooth compact manifold, and

$$Q(E, \Omega, S_0) = (\mathrm{ev}_{z_0})_*[\mathcal{S}] \cdot 1.$$

For a holomorphic vector bundle $E \to X$, $\mathrm{H}^{p,q}(X, E)$ denotes the Dolbeault cohomology. If Ω^p_X denotes the sheaf of holomorphic p-forms on X, then $\mathrm{H}^{p,q}(X, E) = \mathrm{H}^q(X, E \otimes \Omega^p_X)$. In particular, taking $p = 0$, we have $\mathrm{H}^{0,1}(X, E) = \mathrm{H}^1(X, E)$, where $\mathrm{H}^1(X, E)$ is the sheaf cohomology of the vector bundle $E \to X$. See [Huy04, Section 2.6] for more details about Dolbeault cohomology.

Sketch proof. In proposition III.4.3, we needed a generic choice of $\widehat{J}$ for the evaluation map to define a pseudocycle and to define the element $Q(E, \widetilde{\Omega}, S)$. The $\widehat{J}$ corresponds to the almost complex structure J. However, the J which we have chosen may not be *regular*. That is, in general, we may not have transversality.

On the other hand, to define the pseudocycle, we just need to understand the possible limiting behaviour. But in this case, we have a result which is similar to Gromov's compactness theorem. Thus, for $\gamma \in \Gamma$ fixed, to compute $[\mathrm{ev}_{z_0}(S(j, \widehat{J}, \gamma + S))]$, we just need $\widehat{J}$ to be regular for the limits. In this case, we only need to consider γ for which

$$0 \leq n + c_1(\mathrm{V}E)(S) + c_1(\gamma) \leq n$$

which is implied by the assumptions in the proposition. ☐

Chapter IV

Application I - Element of infinite order

In this chapter, we will show that on a rational ruled surface, there exists an element in $\pi_1(\mathrm{Ham})$ with infinite order. To do this, we will construct a Hamiltonian fibre bundle, which in this case is naturally a complex manifold and a fibre bundle over $\mathbb{P}^1$. We will then appeal to proposition III.6.1, which gives us a way to compute the Seidel element. To complete the proof, we note that the Seidel element has infinite order, and so the element in $\pi_1(\mathrm{Ham})$ must also have infinite order.

This is the second example from [Sei97, Section 11].

IV.1 Rational ruled surface

Let $\mathcal{O}(d)$ denote the degree d line bundle over $\mathbb{P}^1$. That is, $\mathcal{O}(-1)$ is the tautological bundle, and $\mathcal{O} = \mathcal{O}(0)$ is the trivial bundle. Moreover, $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$, and any line bundle[1] $\mathcal{L} \to \mathbb{P}^1$ is isomorphic to $\mathcal{O}(d)$ for some d. In this case, we define $\deg(\mathcal{L}) = d$. More generally, we can define for any vector bundle $\mathcal{E} \to \mathbb{P}^1$, we can define[2]

$$\deg(\mathcal{E}) = \left\langle c_1(\mathcal{E}), [\mathbb{P}^1] \right\rangle \in \mathbb{Z}.$$

Let Σ_d denote the total space of the fibre bundle $p : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(d))$, which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. This is also called the *d-th Hirzebruch surface*. We have the following standard facts about Σ_d.

Lemma IV.1.1 ([Bea96, Proposition IV.1]). *(i) Let $s : \mathbb{P}^1 \to \Sigma_d$ be a section of p. Then $s^*V\Sigma_d$ is a line bundle over $\mathbb{P}^1$, and $\deg(s^*V\Sigma_d) \geq -d$. There is a unique section s_- such that equality holds, and all others have $\deg(s^*V\Sigma_d) \geq 2 - d$.*

(ii) Let $w : \mathbb{P}^1 \to \Sigma_2$ be a holomorphic map. Then $c_1([w]) \geq 0$. Moreover, all non-constant w with $c_1(w) < 2$ factors as

$$\begin{tikzcd} \mathbb{P}^1 \arrow[rr, "w"] \arrow[dr, "u"'] & & \Sigma_2 \\ & \mathbb{P}^1 \arrow[ur, "s_-"'] & \end{tikzcd}$$

with u holomorphic. Equivalently, u is a rational function on $\mathbb{P}^1$.

Remark IV.1.2. The statement in [Bea96, Proposition IV.1] is about the Picard group of Σ_d. Since Σ_d is a surface, the Weil divisors are curves. The lemma is a restatement in terms of complex geometry.

More precisely, $\mathrm{Pic}(\Sigma_d) = \mathbb{Z}f \oplus \mathbb{Z}h$, where f is the class of a fibre and h is the class of $\mathcal{O}(1)$. Then $f^2 = 0, fh = 1, h^2 = d$. s_- corresponds to the unique curve with self-intersection $-d$, in class $h - df$.

In particular, we will study the symplectic structure of $M = \Sigma_2$. First, we will construct a Kähler structure on M.

[1] Throughout, line bundle will refer to *holomorphic* line bundles.

[2] In general, we would define the degree of a vector bundle to be additive on short exact sequences. The fact that we are working over $\mathbb{P}^1$ means that the degree agrees with the first Chern class.

Lemma IV.1.3. *We have an isomorphism of complex surfaces*

$$M \cong \mathbb{V}(z_0^2 w_1 - z_1^2 w_2) \subseteq \mathbb{P}^1 \times \mathbb{P}^2 \tag{IV.1}$$

where $\mathbb{P}^1$ has coordinates $[z_0 : z_1]$ and $\mathbb{P}^2$ has coordinates $[w_0 : w_1 : w_2]$. In particular, we can consider M as a complex submanifold of $\mathbb{P}^1 \times \mathbb{P}^2$.

Proof. This follows from trivialising $\mathcal{O} \oplus \mathcal{O}(2)$ over the standard affines in $\mathbb{P}^1$, as it has transition functions of the form

$$(x, y) \mapsto (x, z^2 y)$$

and so the projectivisation has transition functions of the form

$$[x : y] \mapsto [x : z^2 y].$$

□

Thus, if we let $\mathrm{pr}_1, \mathrm{pr}_2$ be the projection maps to $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively, then for $\lambda > 1$,

$$\omega_\lambda = (\lambda - 1)\,\mathrm{pr}_1^* \,\omega_{\mathbb{P}^1} + \mathrm{pr}_2^*\, \omega_{\mathbb{P}^2}$$

is a Kähler form on M, where $\omega_{\mathbb{P}^k}$ is the Fubini-Study form on $\mathbb{P}^k$.

IV.2 Hamiltonian fibre bundle

Next, we will construct a Hamiltonian fibre bundle. We have an action of $S^1 \subseteq \mathbb{C}$ on $\mathcal{O}(2)$ by scalar multiplication, which induces a Hamiltonian circle action g on (M, ω_λ). Consider the following vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$

$$V = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathrm{pr}_1^* \,\mathcal{O}(2) \otimes \mathrm{pr}_2^* \,\mathcal{O}(-1)$$

where pr_i are the projection maps. In this case, we have a bundle $\pi : E \to \mathbb{P}^1$ given by the composition

$$E = \mathbb{P}(V) \xrightarrow{\pi_V} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1$$

The fibres $E_z = \pi^{-1}(z)$ are given by

$$E_z = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2) \otimes \mathcal{O}(-1)_z).$$

We have a standard Hermitian metric on the tautological bundle $\mathcal{O}(-1)$, and so a unit-length element $\xi \in \mathcal{O}(-1)$ determines a biholomorphism $E_z \to M$. Any two such maps differ by a complex isometry, and so the symplectic form Ω_z on E_z is independent of the choice of ξ. With this, we obtain a Hamiltonian fibre bundle (E, Ω) over $\mathbb{P}^1$.

IV.3 Holomorphic sections

Let s be a holomorphic section of π. Then we can decompose s as:

- $s_1 = \pi_V \circ s : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is a section of pr_2,
- s_2 is a section of $F = \mathbb{P}(s_1^* V) \to \mathbb{P}^1$,

Here, since $s(z) \in \mathbb{P}(V_{s_1(z)})$ for all $z \in \mathbb{P}^1$, s_2 is just given by s. This gives us an exact sequence

$$0 \longrightarrow s_2^* \mathrm{V}F \longrightarrow s^* \mathrm{V}E \xrightarrow{\mathrm{d}\pi_V} s_1^*(\ker(\mathrm{d}\,\mathrm{pr}_2)) \longrightarrow 0 \tag{IV.2}$$

of vector bundles over $\mathbb{P}^1$. Since s_1 is a section of pr_2, we have that $s_1(z) = (u(z), z)$ for a map $u : \mathbb{P}^1 \to \mathbb{P}^1$. With this,

$$s_1^*(\ker(\mathrm{d}\,\mathrm{pr}_2)) = u^* \mathrm{T}\mathbb{P}^1 = u^* \mathcal{O}(2) = \mathcal{O}(2d)$$

and

$$s_1^* V = \mathcal{O} \oplus u^* \mathcal{O}(2) \otimes \mathcal{O}(-1) = \mathcal{O} \oplus \mathcal{O}(2d - 1).$$

For $d > 0$, F is isomorphic to Σ_{2d-1}, and when $d = 0$, F is isomorphic to Σ_1.

First suppose $d > 0$. By lemma IV.1.1(i), $\deg(s_2^* \mathrm{V}F) \geq 1 - 2d$ and $\deg(u^* \mathrm{T}\mathbb{P}^2) = 2d$. But degree is additive on short exact sequences, and so we must have that

$$\deg(s^* \mathrm{V}E) = \deg(s_2^* \mathrm{V}F) + \deg(s_1^* V) \geq 1.$$

Now consider the case $d = 0$. In this case, by lemma IV.1.1 again, we see that $\deg(s^* \mathrm{V}E) > 0$ unless s_2 is the unique section with $\deg(s_2^* \mathrm{V}F) = -1$. Any such section must be of the form

$$s_c(z) = [1 : 0] \in \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2)_c \otimes \mathcal{O}(-1)_z) \subseteq E_z$$

for some $c \in \mathbb{P}^1$.

Thus, we have that $\mathcal{S} = \{s_c\}_{c \in \mathbb{P}^1}$. For z fixed, the evaluation map $\mathcal{S} \to E_z \cong M$ has image

$$C^+ = \{([z_0 : z_1], [1 : 0 : 0])\} \subseteq M$$

where we use the embedding into $\mathbb{P}^1 \times \mathbb{P}^2$ from lemma IV.1.3.

Proposition IV.3.1. *The conditions of proposition III.6.1 are satisfied.*

Proof. (i) Since we have a natural identification of $\mathcal{S}$ with $\mathbb{P}^1$, it is connected.

(ii) For $s \in \mathcal{S}$, eq. (IV.2) becomes

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow s^* \mathrm{V}E \longrightarrow \mathcal{O} \longrightarrow 0$$

Using the long exact sequence for sheaf cohomology, we get that $\mathrm{H}^{0,1}(\mathbb{P}^1, s^* \mathrm{V}E) = 0$.

(iii) (a) This is lemma IV.1.1(ii)

(b) Now consider the curve

$$C^- = \{([z_0 : z_1], [0 : z_1^2 : z_0^2])\} \subseteq M.$$

This is the unique curve on M with self-intersection -2. Let s_- be the corresponding section of p. Then

$$\deg(s_-^* \mathrm{V}\Sigma_2) = -2.$$

Now let $w : \mathbb{P}^1 \to E_z$ be a non-constant holomorphic map with $c_1(\mathrm{T}E)(w) < 2$. By lemma IV.1.1(ii), $\mathrm{im}(w) \subseteq C^-$. On the other hand, for all $s \in \mathcal{S}$, $s(z) \in C^+$. Noting that $C^- \cap C^+ = \varnothing$, we see that $s(z) \notin \mathrm{im}(w)$.

□

Corollary IV.3.2.

$$Q(E, \Omega, S_0) = [C^+] \cdot 1$$

IV.4 Element with infinite order

Let $x^\pm \in \mathrm{H}_2(M; \mathbb{Z})$ denote the classes of $C^\pm$, and $y^\pm \in \mathrm{H}_2(M; \mathbb{Z}/2)$ the reduction modulo 2.

Lemma IV.4.1. $(y^+)^2 = [M](t^{\frac{1}{2}(x^+ - x^-)} - t^{\frac{1}{2}(x^+ + x^-)})$.

Proof. In [McD87], McDuff shows that (M, ω_λ) is symplectomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, with symplectic form $\omega_{\mathbb{P}^1} \oplus \lambda \omega_{\mathbb{P}^1}$. The symplectomorphism sends $x^\pm$ to $a \pm b$, where $a = [\mathbb{P}^1 \times \mathrm{pt}]$ and $b = [\mathrm{pt} \times \mathbb{P}^1]$. The result then follows by computing the quantum cohomology ring of $\mathbb{P}^1 \times \mathbb{P}^1$, for example, as done in section V.6. □

Proposition IV.4.2. *For all $\lambda > 1$, $[g] \in \pi_1(\mathrm{Ham}(M, \omega))$ has infinite order.*

Proof. Let $\widetilde{g} : \widetilde{\mathcal{L}M} \to \widetilde{\mathcal{L}M}$ be the lift of g corresponding to the Γ-equivalence class S_0. Then

$$q(\mathbf{g}^2) = Q(E, \Omega, S_0)^2 = [M] t^{\frac{1}{2}(x^+ - x^-)}(1 - t^{x^-})$$

and

$$q(\mathbf{g}^{2m}) = [M] t^{\frac{m}{2}(x^+ - x^-)}(1 - t^{x^-})^m$$
$$q(\mathbf{g}^{2m+1}) = y^+ t^{\frac{m}{2}(x^+ - x^-)}(1 - t^{x^-})^m$$

Recall from eq. (III.1) that we have an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

Applying π_0, we get

$$\Gamma \longrightarrow \pi_0(\widetilde{G}) \longrightarrow \pi_0(G) = \pi_1(\mathrm{Ham}(M, \omega))$$

Moreover, consider the map

$$\begin{aligned} \tau : \Gamma &\to \mathrm{QH}^*(M, \omega)^\times \\ \gamma &\mapsto q(1, \gamma) = [M]t^\gamma \end{aligned}$$

We have a commuting diagram [Sei97, Equation 1.4]

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d, "="'] & \pi_0(\widetilde{G}) \arrow[r] \arrow[d, dashed, "q"] & \pi_1(\mathrm{Ham}(M, \omega)) \arrow[d, dashed, "\overline{q}"] \\ \Gamma \arrow[r, "\tau"'] & \mathrm{QH}^*(M, \omega)^\times \arrow[r] & \mathrm{QH}^*(M, \omega)^\times / \tau(\Gamma) \end{tikzcd}$$

Next, note that $\omega_\lambda(x^-) = \lambda - 1$, and so the class of x^- in Γ has infinite order. Thus, $q(\mathbf{g}^k) \notin \tau(\Gamma)$ for all k, and so $\overline{q}([g]^k) \neq 1$ for all k. Hence $[g]$ has infinite order. $\square$

Chapter V

Application II - Quantum cohomology of symplectic toric manifolds

In the previous chapter, we studied the element we obtain from applying the Seidel representation to a Hamiltonian circle element. In this chapter, we will study the Seidel element for a Hamiltonian circle action on a symplectic toric manifold. As an application, we will apply this theory to compute the quantum cohomology of toric complex surfaces. In particular, we will show that in nice cases, the quantum cohomology can be computed directly from the combinatorial data of the toric variety.

The main reference for this section is [MT06]. However, the examples in section V.5 and section V.6 have been written by the author, as a direct application of the techniques in [MT06].

V.1 Symplectic toric manifolds

In this section, we will first briefly give a sketch of the theory of symplectic toric manifolds. For a more detailed reference, see [Sil08; Aud04].

Definition V.1.1 ([Sil08, Definition 27.4]). A symplectic manifold (M^{2n}, ω) is *toric* if there exists an effective action of $T = T^n$ on M by symplectomorphisms, with moment map $\mu : M \to \mathfrak{t}^* \cong \mathbb{R}^n$.

We will denote this data by (M, ω, T, μ).

Here, the action is *effective* if

$$\bigcap_{p \in M} \mathrm{Stab}_T(p) = 1$$

Theorem V.1.2 (Atiyah-Guillemin-Sternberg, [Sil08, Theorem 27.1]). *Suppose (M, ω, T, μ) is a compact toric symplectic manifold. Then $\Delta = \mu(M)$ is a convex polytope. In particular, Δ is the convex hull of the images of the fixed points of the T-action.*

Δ is called the *moment polytope* of M. Akin to the case of normal toric varieties, we have a correspondence between toric symplectic manifolds and a certain class of convex polytopes.

Definition V.1.3 ([Sil08, Definition 28.1]). A convex polytope $\Delta \subseteq \mathbb{R}^n$ is *Delzant* if it is

(i) *simple*: there are n edges meeting at each vertex, say the edges are given by $p + tu_i$, $0 \leq t \leq T_i$.

(ii) *rational*: the u_i can be chosen to be in $\mathbb{Q}^n$ or equivalently, $\mathbb{Z}^n$,

(iii) *smooth*: the u_i can be chosen to be a basis of $\mathbb{Z}^n$.

Theorem V.1.4 (Delzant, [Sil08, Theorem 28.2]). *There is a one-to-one correspondence between Delzant polytopes in $\mathbb{R}^n$ and compact symplectic toric manifolds of dimension $2n$.*

The final thing which we will mention is how to compute the cohomology ring of a toric variety from its moment polytope. Let (M, ω, T, μ) be a compact toric symplectic manifold, with moment polytope Δ. Since the T is abelian, we can assume without loss of generality that $\mathfrak{t} = \mathbb{R}^n$, with lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$.

Let $D_1, \dots, D_N$ be the codimension 1 faces of Δ, with outward pointing normals $\eta_1, \dots, \eta_N \in \mathbb{Z}^n$. Let Σ be the set of subsets $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, N\}$ such that $D_{i_1} \cap \dots \cap D_{i_k} \neq \varnothing$.

A subset $I \subseteq \{1, \ldots, N\}$ is *primitive* if $I \notin \Sigma$ but every proper subset of I is in Σ. Define two ideals in $\mathbb{Q}[x_1, \ldots, x_N]$:

$$\mathrm{P}_\Delta = \left\langle \sum_i \langle \xi, \eta_i \rangle x_i \;\middle|\; \xi \in \mathbb{Z}^n \right\rangle$$
$$\mathrm{SR}_\Delta = \langle x_{i_1} \cdots x_{i_k} \mid I = \{i_1, \ldots, i_k\} \text{ is primitive} \rangle .$$

Note that for each i, $V_i = \Phi^{-1}(D_i)$ is a (real) codimension 2 submanifold of M. In fact, in the toric *varieties* case, $\Phi^{-1}(D_i)$ corresponds to a complex codimension 1 subvariety, giving us a divisor. In either case, let $u_i = \mathrm{PD}[V_i] \in \mathrm{H}^2(M; \mathbb{Q})$. Then the map

$$\frac{\mathbb{Q}[x_1, \ldots, x_N]}{\mathrm{P}_\Delta + \mathrm{SR}_\Delta} \to \mathrm{H}^*(M; \mathbb{Q})$$
$$x_i \mapsto u_i$$

is an isomorphism of rings. The proof of this is by Morse theory. For a generic choice of ξ, the map $\langle \mu, \xi \rangle$ defines a Morse function on M. The critical points correspond to the vertices of Δ, and the indices correspond to the dimension of the faces.

In particular, this means that $\mathrm{H}^*(M; \mathbb{Q})$ is determined by the intersection of toric divisors, which we can compute using combinatorial data. See [Aud04, Section VII.3.b] for more details.

Finally, we can compute the symplectic form and the first Chern class of a toric manifold.

Proposition V.1.5.

$$[\omega] = \sum_i d_i u_i \quad \textit{and} \quad c_1(M) = \sum_i u_i$$

where $d_i = \langle \eta_i, D_i \rangle$. This is well-defined since η is the normal to D_i.

V.2 Novikov rings

For this chapter, we will need a different definition of the Novikov ring. Let Λ' be the ring of elements of the form

$$\sum_{k \in \mathbb{R}} r_k t^k$$

with $r_k \in \mathbb{Q}$, subject to the finiteness condition that for all $C > 0$, there is finitely many $k < C$ such that $r_k \neq 0$. That is, we replace t^A with $t^{\omega(A)}$. Now define $\Lambda = \Lambda'[q, q^{-1}]$, and $\mathrm{QH}^*(M, \omega) = \mathrm{H}^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda$.

For $a, b \in \mathrm{H}^*(M; \mathbb{Q})$, we define

$$a * b = \sum_{A \in \pi_2(M)} (a * b)_A \otimes q^{c_1(A)} t^{\omega(A)}$$

where as before,

$$\int_\gamma (a * b)_A = \Psi_A(\mathrm{PD}(a), \mathrm{PD}(b), \gamma).$$

Note that when (M, ω) is monotone, i.e. $\omega(A) = \lambda c_1(A)$ for all $A \in \pi_2(M)$, then $q^{c_1(A)} t^{\omega(A)} = (q t^\lambda)^{c_1(A)}$, and so we actually have *less* information in this case.

V.3 Hamiltonian circle actions

As q is now used as a variable in the Novikov ring, we will denote the Seidel map by $S : \pi_1(\mathrm{Ham}(M, \omega)) \to \mathrm{QH}^*(M, \omega)^\times$.

Let (M, ω, T, μ) be a compact toric symplectic manifold. Let Δ, D_i and η_i be as in section V.1. Let $\mu_i : M \to \mathbb{R}$ be given by

$$\mu_i(p) = \langle \mu(p), \eta_i \rangle .$$

μ_i defines a moment map for a circle action $\mathbf{g}_i$, given by the tangent vector $\eta_i \in \mathfrak{t}$.

We say that (M, ω) is *Fano* if there are no non-constant J-holomorphic spheres u with $c_1([u]) \leq 0$. In particular, (M, ω) being monotone is a sufficient condition, since in this case, $c_1([u])$ is a positive multiple of $\omega([u]) = \mathrm{E}(u) > 0$.

Theorem V.3.1 ([MT06, Theorem 1.10(ii)]). *If M is Fano, then*

$$S(\mathbf{g}_i) = u_i \otimes q^{-1} t^{-d_i}$$

We will use this without proof, and instead focus on applying this result to compute the quantum cohomology of symplectic toric manifolds.

V.4 Quantum cohomology

In this section, we will show that there exists an ideal $\widetilde{\mathrm{SR}}_\Delta \subseteq \mathbb{Q}[x_1, \ldots, x_N] \otimes \Lambda$, such that the map $x_i \mapsto u_i = \mathrm{PD}(V_i)$ induces an isomorphism

$$\frac{\mathbb{Q}[x_1, \ldots, x_N] \otimes \Lambda}{\mathrm{P}_\Delta + \widetilde{\mathrm{SR}}_\Delta} \cong \mathrm{QH}^*(M, \omega) \tag{V.1}$$

Let σ be a face of Δ. Then

$$\sigma = D_{j_1} \cap \cdots \cap D_{j_\ell}$$

for some $j_1, \ldots, j_\ell$. The dual cone is given by

$$\sigma^\vee = \mathrm{Cone}(\eta_{j_1}, \ldots, \eta_{j_\ell}) = \left\{ \sum_{p=1}^{\ell} a_p \eta_{j_p} \;\middle|\; a_p \in \mathbb{R}_{\geq 0} \right\}.$$

The collection $\Sigma = \{\sigma^\vee\}_{\sigma \preccurlyeq \Delta}$ forms a complete fan in $\mathbb{R}^n$, and so any $z \in \mathbb{R}^n$ lies within the interior of some (necessarily unique) $\sigma^\vee$.

Let $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$. Then $\eta_{i_1} + \cdots + \eta_{i_k} \in \sigma^\vee$ for some $\sigma \preccurlyeq \Delta$. Say $\sigma = D_{j_1} \cap \cdots \cap D_{j_\ell}$. Then there exists unique positive integers $m_1, \ldots, m_\ell$ such that

$$\eta_{i_1} + \cdots + \eta_{i_k} = m_1 \eta_{j_1} + \cdots + m_\ell \eta_{j_\ell}. \tag{V.2}$$

When I is primitive, $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_\ell\}$ are disjoint. Let $v_i \in \mathrm{H}_2(M; \mathbb{Q})$ be such that $u_i(v_j) = \delta_{ij}$, and let

$$\beta_I = v_{i_1} + \cdots + v_{i_k} - m_1 v_{j_1} - \cdots - m_\ell v_{j_\ell}.$$

Then

$$\begin{aligned} c_1(\beta_I) &= k - m_1 - \cdots - m_\ell \\ \omega(\beta_I) &= d_{i_1} + \cdots + d_{i_k} - m_1 d_{j_1} - \cdots - m_\ell d_{j_\ell}. \end{aligned}$$

On the other hand, eq. (V.2) shows that we have two circle actions which are the same. Since the Seidel representation is a homomorphism, we must have that

$$u_{i_1} \cdots u_{i_k} \otimes q^{-k} t^{-d_{i_1} - \cdots - d_{i_k}} = u_{j_1}^{m_1} \cdots u_{j_\ell}^{m_\ell} \otimes q^{-m_1 - \cdots - m_\ell} t^{-m_1 d_{j_1} - \cdots - m_\ell d_{j_\ell}}.$$

Rearranging, we obtain that

$$u_{i_1} \cdots u_{i_k} \otimes 1 - u_{j_1}^{m_1} \cdots u_{j_\ell}^{m_\ell} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)} = 0.$$

Define the ideal $\widetilde{\mathrm{SR}}_\Delta \subseteq \mathbb{Q}[x_1, \ldots, x_N] \otimes \Lambda$ by

$$\widetilde{\mathrm{SR}}_\Delta = \left\langle x_{i_1} \cdots x_{i_k} \otimes 1 - x_{j_1}^{m_1} \cdots x_{j_\ell}^{m_\ell} \otimes q^{c_1(\beta_I)} t^{\omega(\beta_I)} \mid I \text{ primitive} \right\rangle.$$

We will show that this satisfies eq. (V.1). Note if $\ell = 0$, we set $x_{j_1}^{m_1} \cdots x_{j_\ell}^{m_\ell} = 1$. This is the case if $\eta_{i_1} + \cdots + \eta_{i_k} = 0$.

Define a valuation v on $\mathbb{Q}[x_1, \ldots, x_n] \otimes \Lambda$ by

$$v\left(\sum_{d,k} a_{d,k} \otimes q^d t^k \right) = \min\{k \mid \text{exists } d \text{ such that } a_{d,k} \neq 0\}$$

Lemma V.4.1 ([MT06, Lemma 5.1]). *The natural map $\Theta : \mathbb{Q}[x_1, \dots, x_N] \otimes \Lambda \to \mathrm{QH}^*(M, \omega)$ is surjective.*

Moreover, if $p_1, \dots, p_m \in \mathbb{Q}[x_1, \dots, x_N]$ generate $\ker(\mathbb{Q}[x_1, \dots, x_N] \to \mathrm{H}^*(M, \omega))$, and if $q_1, \dots, q_m \in \mathbb{Q}[x_1, \dots, x_N] \otimes \Lambda$ are such that $\Theta(q_i) = 0$ and $\nu(p_i - q_i) > 0$ for all i, then $q_1, \dots, q_m$ generate $\ker(\Theta)$.

Assuming this, eq. (V.1) is immediate.

Sketch proof. The main idea in the proof is that we have a "division algorithm". Let $z \in \mathrm{QH}^*(M)$. We would like to find $\tilde{z} \in \mathbb{Q}[x_1, \dots, x_N] \otimes \Lambda$ such that

$$\nu(z - \Theta(\tilde{z})) \geq \nu(z) + \hbar$$
$$\nu(\tilde{z}) \geq \nu(z)$$

Let

$$z = \sum_{i=1}^{s} z_i \otimes q^{d_i} t^{k_i} + r$$

where $\nu(r) \geq \nu(z) + \hbar$, $z_i \in \mathrm{H}^*(M)$, $d_i \in \mathbb{Z}$ and $k_i \geq \nu(z)$. Here, $\hbar$ is the constant from corollary I.4.13, where we can assume without loss of generality that $\nu(p_i - q_i) > \hbar$.

Since θ is surjective, let $\tilde{z}_i$ be such that $\theta(\tilde{z}_i) = z_i$. Then $\nu(\Theta(\tilde{z}_i) - \theta(\tilde{z}_i)) \geq \hbar$, and so we can set

$$\tilde{z} = \sum_{i=1}^{s} \tilde{z}_i \otimes q^{d_i} t^{k_i}$$

□

V.5 Projective space

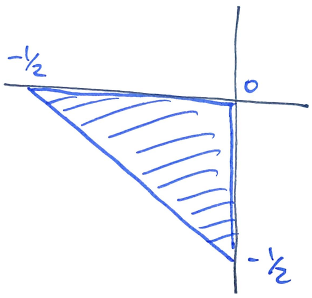
For simplicity, we will restrict ourselves to $\mathbb{P}^2$. The same ideas will work for $\mathbb{P}^n$, just with more things to keep track of. $\mathbb{P}^2$ is a toric manifold, via

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2]$$

and we have an associated moment map $\mu : \mathbb{P}^2 \to \mathbb{R}^2$, defined by

$$\mu[z_0 : z_1 : z_2] = -\frac{1}{2} \left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right).$$

The moment polytope is given by



In this case, we can take

$$\eta_1 = (1, 0) \qquad \eta_2 = (0, 1) \quad \text{and} \quad \eta_3 = (-1, -1)$$

and we have that

$$d_1 = 0 \qquad d_2 = 0 \quad \text{and} \quad d_3 = \frac{1}{2}.$$

The only primitive subset of $\{1, 2, 3\}$ is $\{1, 2, 3\}$ itself, and as $\eta_1 + \eta_2 + \eta_3 = 0$, we have that

$$\widetilde{\mathrm{SR}}_\Delta = \left\langle x_1 x_2 x_3 \otimes 1 - 1 \otimes q^3 t^{-1/2} \right\rangle.$$

For $\xi = (\xi_1, \xi_2) \in \mathbb{Z}^2$,

$$\sum_i \langle \xi, \eta_i \rangle x_i = \xi_1 x_1 + \xi_2 x_2 - (\xi_1 + \xi_2) x_3 = \xi_1(x_1 - x_3) + \xi_2(x_2 - x_3)$$

and so $\mathrm{P}_\Delta = \langle x_1 - x_3, x_2 - x_3 \rangle$. With this, we obtain that

$$\mathrm{QH}^*(\mathbb{P}^2, \omega) = \frac{\mathbb{Q}[x] \otimes \Lambda}{\langle x^3 \otimes 1 - 1 \otimes q^3 t^{1/2} \rangle}.$$

More generally,

$$\mathrm{QH}^*(\mathbb{P}^n, \omega) = \frac{\mathbb{Q}[x] \otimes \Lambda}{\langle x^{n+1} \otimes 1 - 1 \otimes q^{n+1} t^{1/2} \rangle}.$$

Comparing this to section I.9, where we computed $\mathrm{QH}^*(\mathbb{P}^n)$ directly, we can see that we get same result, just with different coefficient rings.

V.6 Product of lines

Consider $M = \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathrm{pr}_1, \mathrm{pr}_2 : M \to \mathbb{P}^1$ denote the projection maps. On M, we have the symplectic form $\mathrm{pr}_1^* \omega + \mathrm{pr}_2^* \omega$.

Then

$$TM = \mathrm{pr}_1^* T\mathbb{P}^1 \oplus \mathrm{pr}_2^* T\mathbb{P}^1$$

and so

$$c_1(M) = c_1(\mathrm{pr}_1^* T\mathbb{P}^1) + c_1(\mathrm{pr}_2^* T\mathbb{P}^1) = \mathrm{pr}_1^* c_1(\mathbb{P}^1) + \mathrm{pr}_2^* c_1(\mathbb{P}^1).$$

Since $\mathbb{P}^1$ is monotone, so is $\mathbb{P}^1 \times \mathbb{P}^1$.

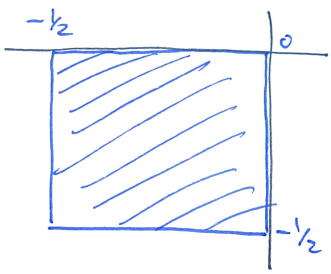
M is a toric manifold, via

$$(t_1, t_2) \cdot ([x_0 : x_1], [y_0 : y_1]) = ([x_0 : t_1 x_1], [y_0 : t_2 y_1])$$

and we have moment map

$$\mu([x_0 : x_1], [y_0 : y_1]) = -\frac{1}{2} \left(\frac{|x_1|^2}{|x_0|^2 + |x_1|^2}, \frac{|y_1|^2}{|y_0|^2 + |y_1|^2} \right).$$

The moment polytope is then given by



We can take as normals

$$\eta_1 = (1, 0) \qquad \eta_2 = (0, 1) \qquad \eta_3 = (-1, 0) \quad \text{and} \quad \eta_4 = (0, -1).$$

Then we have that

$$d_1 = 0 \qquad d_2 = 0 \qquad d_3 = \frac{1}{2} \quad \text{and} \quad d_4 = \frac{1}{2}.$$

For $\xi = (\xi_1, \xi_2) \in \mathbb{Z}^2$,

$$\sum_i \langle \xi, \eta_i \rangle x_i = \xi_1 x_1 + \xi_2 x_2 - \xi_1 x_3 - \xi_2 x_4 = \xi_1(x_1 - x_3) + \xi_2(x_2 - x_4)$$

and so $\mathrm{P}_\Delta = \langle x_1 - x_3, x_2 - x_4 \rangle$.

The primitive subsets are $\{1,3\}$, $\{2,4\}$, and $\eta_1+\eta_3=\eta_2+\eta_4=0$, and we obtain that

$$\widetilde{\mathrm{SR}}_\Delta = \left\langle x_1x_3\otimes 1 - 1\otimes q^2t^{1/2}, x_2x_4\otimes 1 - 1\otimes q^2t^{1/2}\right\rangle.$$

Hence

$$\mathrm{QH}^*(M) = \frac{\mathbb{Q}[x_1,x_2]}{\left\langle x_1^2\otimes 1 - 1\otimes q^2t^{1/2}, x_2^2\otimes 1 - 1\otimes q^2t^{1/2}\right\rangle}.$$

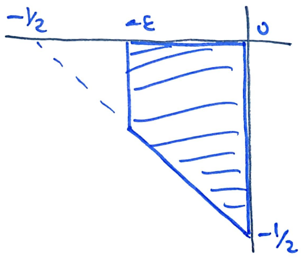
Comparing this to the result in the previous section, we see that this can also be seen as the tensor product $\mathrm{QH}^*(\mathbb{P}^1)\otimes_{\mathbb{Q}}\mathrm{QH}^*(\mathbb{P}^1)$.

Remark V.6.1. In [MT06, Example 5.7], they do the same thing but with $(\mathbb{P}^1\times\mathbb{P}^1, \omega\oplus\lambda\omega)$. That is the same example as in chapter IV, but in this case the different symplectic form changes the quantum cohomology ring. Further discussion on the differences in the symplectic topology of these two cases can be found in [MS17, Example 13.4.2].

V.7 One-point blowup of projective plane

Next, let M be the size $\frac{1}{2}-\varepsilon$ blowup of $\mathbb{P}^2$ at a point. This is [MT06, Example 5.6], although there are some differences between the method here and in [MT06].

This has polytope



and we have

$$\eta_1=(1,0)\qquad \eta_2=(0,1)\qquad \eta_3=(-1,0)\quad\text{and}\quad \eta_4=(-1,-1)$$

and

$$d_1=0\qquad d_2=0\qquad d_3=\varepsilon\quad\text{and}\quad d_4=\frac{1}{2}.$$

Computing, we find that

$$\mathrm{P}_\Delta = \langle x_1-x_3-x_4, x_2-x_4\rangle$$

and

$$\omega = \varepsilon x_3 + \frac{1}{2}x_4$$
$$c_1 = 2x_3+3x_4.$$

For M to be monotone, we need to take $\varepsilon = 1/3$.

As for $\mathbb{P}^1\times\mathbb{P}^1$, the primitive subsets are $\{1,3\}$ and $\{2,4\}$. We have that $\eta_1+\eta_3=0$, and that $\eta_2+\eta_4=(-1,0)=\eta_3$. Thus,

$$\widetilde{\mathrm{SR}}_\Delta = \left\langle x_1x_3\otimes 1 - 1\otimes q^2t^{-1/3}, x_2x_4\otimes 1 - x_3\otimes qt^{-1/6}\right\rangle$$

and this computes the quantum cohomology of a one-point blowup of $\mathbb{P}^2$.

Remark V.7.1. Different choices of μ will give the same complex manifold, since the complex structure (i.e. the toric variety) only depends on the fan, and not on the polytope. However, the symplectic form depends on ε, and we need a specific choice of ε for monotonicity to hold.

Remark V.7.2. This is also a Hirzebruch surface, namely $\Sigma_1 = \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))$.

Chapter VI

Conclusion and further directions

In this essay, we first constructed the quantum cohomology ring $\mathrm{QH}^*(M)$. To define $\mathrm{QH}^*(M)$, we defined a family of invariants $\Psi_{A,p}$. In the literature, these are called the *Gromov-Witten invariants*. We have defined them in the very special case of the domain being $\mathbb{P}^1$, and with M being monotone. Defining them in general is a much more difficult problem, and is the subject of active research [AMS21; HS22]. Even in the contexts where the Gromov-Witten invariants have been defined, computing them is a difficult problem.

We then defined the Hamiltonian Floer homology, which gave us a different perspective on $\mathrm{QH}^*(M)$, through the isomorphism defined in section II.5. There is another version of Floer homology, called *Lagrangian Floer homology* [Aur14; Smi15]. In Lagrangian Floer homology, we define $\mathrm{HF}^*(L_1, L_2)$ where L_1, L_2 are Lagrangian submanifolds. The generators of the chain complex are intersection points of L_1 and L_2, and the boundary maps are given by counting J-holomorphic strips.

If we define $\mathrm{Ham}_L(M, \omega)$ to be the set of all Hamiltonian diffeomorphisms ϕ such that $\phi(L) = L$, then we can define a *relative Seidel representation* [HL10]

$$\pi_1(\mathrm{Ham}(M, \omega), \mathrm{Ham}_L(M, \omega)) \to \mathrm{HF}^*(L, L)$$

In the last chapter, we computed the quantum cohomology ring of toric varieties using the Seidel representation. There are also other methods, such as [Giv98; FOOO16].

In a different direction, the analytic details have mostly been omitted from this essay. The proofs can be found in [MS94; MS12]. For a background on the analytic aspects of finite-dimensional Morse theory, see [AD14; Jos08; Nic11], and [AD14] for Floer homology.

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