Complex Analysis

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May 11, 2022

Contents

1 Differentiation

Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$.

Definition 1.1 (Differentiable)
f is holomorphic at $w \in U$ if

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

exists. We call the result the derivative of f at w.

Definition 1.2 (Holomorphic)
f is holomorphic at $a \in U$ if there exists $\varepsilon > 0$ such that f is differentiable at all $z \in D(a, \varepsilon)$.

f is holomorphic in U if f is holomorphic at every point in U. Equivalently, f is differentiable at every point in U.

Proposition 1.3. The map $f \mapsto f'$ is linear.

Proposition 1.4 (Product rule).

$$(fg)' = f'g + fg'$$

Proposition 1.5 (Chain rule).

$$(f \circ g)'(z) = f'(g(z))g'(z)$$

Let $f = u + iv$, where $u, v : U \to \mathbb{R}$, and in addition, we identify $\mathbb{C} \cong \mathbb{R}^2$, so we consider U to be an open subset of $\mathbb{R}^2$.

Theorem 1.6 (Cauchy-Riemann). f is differentiable at $w = c + id \in U$ if and only if u, v are differentiable at (c, d), and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at } (c, d)$$

Furthermore, $f'(w) = u_x + iv_x$.

Proof. f is differentiable at $w = c + id$, with derivative $p + iq$

$\iff$

$$\lim_{z \to w} \frac{f(z) - f(w)}{z - w} = p + iq$$

$\iff$

$$\lim_{z \to w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0$$

$\iff$

$$\lim_{(x,y) \to (c,d)} \frac{u(x, y) - u(c, d) - p(x - c) + q(y - d)}{\left\|(x, y) - (c, d)\right\|} = 0$$

and

$$\lim_{(x,y) \to (c,d)} \frac{v(x, y) - v(c, d) - q(x - c) - p(y - d)}{\left\|(x, y) - (c, d)\right\|} = 0$$

$\iff$

u is differentiable at (c, d) with $Du(c, d)(x, y) = px - qy$, and v is differentiable at (c, d) with $Dv(c, d)(x, y) = qx + py$.

$\iff$

u, v differentiable at (c, d) with $u_x = v_y = p$ and $u_y = -v_x = q$. □

Corollary 1.7. If $f : U \to \mathbb{C}$ has continuous partial derivatives that satisfy the Cauchy-Riemann equations, then f is differentiable ta U.

Proof. Continuous partial derivatives implies that f is differentiable. □

Definition 1.8 (Domain)

A domain U is a nonempty, open, path connected subset of $\mathbb{C}$.

Corollary 1.9. If U is a domain, $f : U \to \mathbb{C}$ holomorphic on U, and $f' = 0$ in U. Then f is constant.

Proof. By Cauchy-Riemann $Du = 0$ and $Dv = 0$, so u, v are constant. □

Definition 1.10 (Entire)
If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, then we say that f is entire.

1.1 Power series

Theorem 1.11. Suppose $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ has radius of convergence R. Then f is holomorphic in $D(a, R)$, with derivative

$$f'(z) = \sum_{n=0}^{\infty} nc_n(z-a)^{n-1}$$

which has the same radius of convergence R.

Proof. Without loss of generality, $a = 0$. The power series for f' has radius of convergence $R_1 \in [0, \infty]$.

Fix $z \in D(0, R)$, and choose ρ such that $|z| < \rho < R$. Then

$$n|c_n||z|^{n-1} = n|c_n|\left|\frac{z}{\rho}\right|^{n-1} \rho^{n-1} \leq |c_n|\rho^{n-1}$$

for n large, since $n\left|\frac{z}{\rho}\right|^{n-1} \to 0$ as $n \to \infty$. So $R \leq R_1$, as this means that $\sum nc_nz^{n-1}$ converges in $D(0, R)$.

As $|c_n||z^n| \leq n|c_n||z^n| = |z|(nc_n|z^{n-1}|)$, so if $\sum n|c_n||z|^{n-1}$ converges, so does $\sum |c_n||z^n|$, which means that $R \geq R_1$, so $R = R_1$.

To prove that f is differentiable, fix $z \in D(0, R)$, and let

$$g(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \neq z \\ \displaystyle\sum_{n=1}^{\infty} nc_nz^{n-1} & \text{if } w = z \end{cases}$$

We want to show that g is continuous as z. Define

$$h_n(w) = \begin{cases} \dfrac{c_n\left(w^n - z^n\right)}{w - z} & \text{if } w \neq z \\ nc_nz^{n-1} & \text{if } w = z \end{cases}$$

Then $g(w) = \sum_{n=1}^{\infty} h_n(w)$. h_n is continuous at z, as it is the derivative of $w \mapsto c_nw^n$. Since

$$\frac{w^n - z^n}{w - z} = z^{n-1} + wz^{n-2} + \cdots + w^{n-2}z + w^{n-1}$$

Then for any r such that $|z| < r < R$, $w \in D(0, r)$, $|h_n(w)| \leq n|c_n|r^{n-1}$. Let $M_n = n|c_n|r^{n-1}$. Then $\sum M_n$ converges, so $\sum h_n$ converges uniformly by the Weierstrass M-test. So g is the uniform limit of continuous functions, s o it is continuous. □

Corollary 1.12. Suppose $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ has radius of convergence R. If $f \equiv 0$ in $D(a, \varepsilon)$ for some $\varepsilon > 0$, then $f \equiv 0$ in $D(a, R)$.

Proof. We must have that $c_n = 0$ for all n. □

Definition 1.13 (Exponential)

$$\exp(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

Proposition 1.14. exp is entire, with derivative exp.

Proposition 1.15. $\exp(z) \neq 0$ for all z, and $\exp(z+w) = \exp(z)\exp(w)$.

Proof. Fix $w \in \mathbb{C}$, define $F(z) = \exp(z+w)\exp(-z)$. Then

$$F'(z) = -\exp(z+w)\exp(-z) + \exp(z+w)\exp(-z) = 0$$

So F is constant, and $F(z) = F(0) = \exp(w)$. □

Proposition 1.16. For $x, y \in \mathbb{R}$,

$$\exp(x+iy) = e^x(\cos(x) + i\sin(y))$$

and

$$\exp(z) = 1 \iff z \in 2\pi i\mathbb{Z}$$

Proposition 1.17. For $z \in \mathbb{C}$ nonzero, we have $w \in \mathbb{C}$ such that $\exp(w) = z$.

Definition 1.18 (Logarithm)
Given $z \in \mathbb{C}$, we say $w \in \mathbb{C}$ is a logarithm of z if $\exp(w) = z$.

Definition 1.19 (Branch of logarithm)
Let $U \subseteq \mathbb{C} \setminus 0$ be open. Then a branch of the logarithm on U is a continuous function $\lambda : U \to \mathbb{C}$ such that

$$\exp(\lambda(z)) = z$$

for all $z \in U$.

Proposition 1.20. If λ is a branch of log on U, then λ is holomorphic on U, so $\lambda'(z) = \frac{1}{z}$.

Proof. Suppose $w \in U$. Then

$$
\begin{aligned}
\lim_{z\to w}\frac{\lambda(z)-\lambda(w)}{z-w} &= \lim_{z\to w}\frac{\lambda(z)-\lambda(w)}{\exp(\lambda(z))-\exp(\lambda(w))}\\
&= \lim_{z\to w}\frac{1}{\frac{\exp(\lambda(z))-\exp(\lambda(w))}{\lambda(z)-\lambda(w)}}\\
&= \frac{1}{\exp(\lambda(w))}\lim_{z\to w}\frac{1}{\frac{\exp(\lambda(z)-\lambda(w))-1}{\lambda(z)-\lambda(w)}}\\
&= \frac{1}{\exp(\lambda(w))}\lim_{h\to 0}\frac{1}{\frac{\exp(h)-1}{h}}\\
&= \frac{1}{w}
\end{aligned}
$$

□

Definition 1.21 (Principal branch)
The principal branch of log is the function

$$\mathrm{Log} : \mathbb{C} \setminus \{x \in \mathbb{R} : x \le 0\} \to \mathbb{C}$$

by $Log(z) = \log|z| + i\arg(z)$, where we have $\arg(z) \in (-\pi, \pi)$.

Proposition 1.22. Log is a branch of log.

Proposition 1.23.

$$\mathrm{Log}(1+z) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{n} \quad \text{for} \quad |z| < 1$$

Proof. Define for $|z| < 1$,

$$F(z) = \mathrm{Log}(1+z) - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{n}$$

Then $F' = 0$, so $F = 0$. □

1.2 Conformal maps

Proposition 1.24. Let $f : U \to |C$ be holomorphic at $w \in U$, $f'(w) \neq 0$. Let $\gamma_1, \gamma_2 : [-1, 1] \to U$ be C^1 curves such that $\gamma_1(0) = \gamma_2(0) = w$, $\gamma_1'(0), \gamma_2'(0) \neq 0$. Then

$$\arg(\gamma_1'(0)) - \arg(\gamma_2'(0)) = \arg((f\circ\gamma_1)'(0)) - \arg((f\circ\gamma_2)'(0))$$

Definition 1.25 (Conformal)
$f : U \to \mathbb{C}$ is conformal at $w \in U$ if $f'(w) \neq 0$.

Definition 1.26 (Conformal equivalence)

$f : U \to \tilde{U}$ is a conformal equivalence if f is bijective and holomorphic, with $f'(z) \neq 0$ for all $z \in U$.

Proposition 1.27. Möbius maps are conformal.

2 Complex integration

Definition 2.1 (Complex (Riemann) integral)
Suppose $f : [a, b] \to \mathbb{R}$, with $\text{Re}(f), \text{Im}(f)$ integrable. Then define

$$\int_a^b f(t)\mathrm{d}t = \int_a^b \text{Re}(f(t))\mathrm{d}t + i\int_a^b \text{Im}(f(t))\mathrm{d}t$$

Proposition 2.2.

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leq \int_a^b |f(t)|\mathrm{d}t \leq (b-a)\sup_{t\in[a,b]}|f(t)|$$

Proof. If $\int_a^b f(t)\mathrm{d}t = 0$ we are done. If not, say $\int_a^b f(t)\mathrm{d}t = re^{i\theta}$. Let $M = \sup_{t\in[a,b]}|f(t)|$. Then

$$\left|\int_a^b f(t)\mathrm{d}t\right| = r = e^{-i\theta}\int_a^b f(t)\mathrm{d}t = \int_a^b \text{Re}(e^{-i\theta}f(t))\mathrm{d}t + i\int_a^b \text{Im}(e^{-i\theta}f(t))\mathrm{d}t$$

Since the left hand side of the equality is real, we must have that

$$\left|\int_a^b f(t)\mathrm{d}t\right| = \int_a^b \text{Re}(e^{-i\theta}f(t))\mathrm{d}t \leq \int_a^b \left|e^{-i\theta}f(t)\right|\mathrm{d}t = \int_a^b |f(t)|\mathrm{d}t$$

and the final inequality follows from real analysis. □

Remark 2.3. Equality holds if and only if f is constant.

Definition 2.4 (Curve integral)
Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuous, $\gamma : [a, b] \to U$ a C^1 curve. Then the integral of f along γ is

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

Proposition 2.5. Integral is independent of parametrisation.

Proof. Chain rule. □

Definition 2.6 (Length)
Define the length of a curve by

$$\text{Length}(\gamma) = \int_a^b |\gamma'(t)| dt$$

Proposition 2.7.

$$\left| \int_\gamma f(z) dz \right| \leq \text{Length}(\gamma) \sup_\gamma |f|$$

Theorem 2.8 (FTC)**.** Suppose $F : U \to \mathbb{C}$ is C^1, then

$$\int_\gamma F'(z) dz = F(\gamma(b)) - F(\gamma(a))$$

Proof. By real FTC. □

Corollary 2.9. If γ is a closed curve, then

$$\int_\gamma F'(z) dz = 0$$

Theorem 2.10 (FTC II)**.** Let $U \subseteq \mathbb{C}$ be a domain, $f : U \to \mathbb{C}$ continuous, and for every closed curve γ in U,

$$\int_\gamma f(z) dz = 0$$

Then f has an antiderivative in U.

Proof. Fix $a_0 \in U$. For $w \in U$, define a curve $\gamma_w : [0,1] \to U$ such that $\gamma_w(0) = a_0$ and $\gamma_w(1) = w$. Since U is path connected, one exists. Furthermore, we can take γ_w polygonal and piecewise C^1. Define

$$F(w) = \int_{\gamma_w} f(z) dz$$

Note that F is independent of the choice of γ, since if $\gamma_w, \tilde{\gamma}_w$ are both curves from a_0 to w, then $\gamma_w + (-\tilde{\gamma}_w)$ is a closed curve. Fix $w \in U$. Since U is open, we have $r > 0$ such that $D(w,r) \subseteq U$. For $h \in \mathbb{C}$ with $0 < |h| < r$, define $\delta_h(t) = w + th$ for $t \in [0,1]$. Now note that $\gamma = \gamma_w + \delta_h + (-\gamma_{w+h})$ is a closed curve, so $\int_\gamma f(z) dz = 0$ by assumption.

Hence we have that

$$F(w+h) = \int_{\gamma_{w+h}} f(z) dz = \int_{\gamma_w} f(z) dz + \int_{\delta_h} f(z) dz = F(w) + \int_{\delta_h} f(z) dz = F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w)) dz$$

Suffices to show the error term is $o(h)$.

$$\frac{1}{|h|} \left| \int_{\delta_h} f(z) - f(w) dz \right| \leq \frac{1}{h} \text{Length}(\delta_h) \sup_{z \in \delta_h([0,1])} |f(z) - f(w)| = \sup_{z \in \delta_h([0,1])} |f(z) - f(w)| \to 0 \quad \text{as} \quad h \to 0$$

□

2.1 Cauchy's theorem for star domains

Definition 2.11 (Star shaped domain)
A domain $U \subseteq \mathbb{C}$ is star shaped if there exists $a_0 \in U$ such that for all $z \in U$, the segment $[a_0, z]$ is contained in U.

Definition 2.12 (Triangle)
A triangle $T \subseteq \mathbb{C}$ is the closed convex hull of three points in $\mathbb{C}$.

Definition 2.13 (Boundary of the triangle)
We define the boundary of the triangle to be oriented anticlockwise.

Corollary 2.14. If U is star shaped, $f : U \to \mathbb{C}$ is continuous, and

$$\int_{\partial T} f(z) \mathrm{d}z = 0$$

for all triangles $T \subseteq U$, then f has an antiderivative in U.

Proof. Modify proof of FTC II. □

Theorem 2.15 (Cauchy's theorem for triangles). Suppose $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$ holomorphic. Suppose $T \subseteq U$ is a triangle. Then

$$\int_{\partial T} f(z) \mathrm{d}z = 0$$

Proof. Let $\eta(T) = \int_{\partial T} f(z)\mathrm{d}z$. Subdivide T into 4 smaller triangles $T^{(i)}$ by connecting the midpoints of each edge. Then as the inner edges cancel, $\eta(T) = \eta(T^{(1)}) + \cdots + \eta(T^{(4)})$. By triangle inequality, we have i such that

$$\left|\eta(T^{(i)})\right| \geq \frac{|\eta(T)|}{4}$$

Define $T_0 = T$, $T_1 = T^{(i)}$. Then

$$|\eta(T_1)| \geq \frac{1}{4}|\eta(T_0)| \quad \text{and} \quad \text{Length}(\partial T_1) = \frac{1}{2}\text{Length}(\partial T_0)$$

Repeat the above process to get $T_0, T_1, T_2, \ldots$ such that

$$|\eta(T_n)| \geq \frac{1}{4^n}|\eta(T_0)| \quad \text{and} \quad \text{Length}(\partial T_n) = \frac{1}{2^n}\text{Length}(\partial T_0)$$

Since $\text{diam}(T_n) \to 0$, by compactness we have that $\bigcap_n T_n = \{z_0\}$. Let $\varepsilon > 0$, since f is differentiable at z_0, we have $\delta > 0$ such that

$$\forall z \in U, |z - z_0| < \delta \implies \left|f(z) - f(z_0) - f'(z_0)(z - z_0)\right| \leq \varepsilon|z - z_0|$$

Now, by FTC we have that

$$\eta(T_n) = \int_{\partial T_n} f(z)\mathrm{d}z = \int_{\partial T_n} f(z) - f(z_0) - f'(z_0)(z - z_0)\mathrm{d}z$$

Choose n such that $T_n \subseteq D(z_0, \delta)$. Then

$$
\begin{aligned}
|\eta(T_0)| &= 4^n |\eta(T_n)| \\
&\le 4^n \left| \int_{\partial T_n} f(z) \mathrm{d}z \right| \\
&= 4^n \left| \int_{\partial T_n} f(z) - f(z_0) - f'(z_0)(z - z_0) \mathrm{d}z \right| \\
&\le 4^n \operatorname{Length}(\partial T_n) \sup_{z \in \partial T_n} \left| f(z) - f(z_0) - f'(z_0)(z - z_0) \right| \\
&\le 4^n \varepsilon \operatorname{Length}(\partial T_n) \sup_{z \in \partial T_n} |z - z_0| \\
&\le 4^n \varepsilon \operatorname{Length}(\partial T_n)^2 \\
&= \varepsilon \operatorname{Length}(\partial T_0)^2
\end{aligned}
$$

But $\varepsilon > 0$ is arbitrary. So $\eta(T_0) = 0$. □

Theorem 2.16. Let $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$ continuous, $S \subseteq U$ finite, f holomorphic on $U \setminus S$. Then for evert triangle $T \subseteq U$, we have that

$$\int_{\partial T} f(z) \mathrm{d}z = 0$$

Proof. By the above process, subdivide T into $N = 4^n$ triangles, say $T_1, \dots, T_N$. Then the interiors cancel, so

$$\int_{\partial T} f(z) \mathrm{d}z = \sum_{j=1}^{N} \int_{\partial T_j} f(z) \mathrm{d}z$$

Let $J = \{j : T_j \cap S = \varnothing\}$. By Cauchy theorem for triangles, for all $j \in J$, $\int_{\partial T_j} f(z) \mathrm{d}z = 0$. So we have that

$$\int_{\partial T} f(z) \mathrm{d}z = \sum_{j \notin J} \int_{\partial T_j} f(z) \mathrm{d}z$$

Note that each point in S is in at most 6 triangles, so

$$\left| \int_{\partial T} f(z) \mathrm{d}z \right| \le 6|S| \left| \sup_{z \in T} f(z) \mathrm{d}z \right| \frac{\operatorname{Length}(\partial T)}{2^n} \to 0 \quad \text{as} \quad n \to \infty$$

□

Corollary 2.17. Let $U \subseteq \mathbb{C}$ be a star domain, $f : U \to \mathbb{C}$ continuous, holomorphic in $U \setminus S$, where $S \subseteq U$ finite. Then for any closed curve γ in U,

$$\int_{\gamma} f(z) \mathrm{d}z = 0$$

Proof. f has an antiderivative, so result follows by FTC for star domains. □

2.2 Cauchy integral formula for a disc

Definition 2.18 (Boundary of a disc)
For $D(a, r)$, we define the boundary $\partial D(a, r)$ to be the path

$$t \mapsto a + re^{2\pi i t}$$

Lemma 2.19 (Fundamental integral)**.** Let $r > 0$, $w \in D(a, r)$. Then

$$\int_{\partial D(a,r)} \frac{1}{z-w} \mathrm{d}z = 2\pi i$$

Proof.

$$\frac{1}{z-w} = \frac{1}{z-a+a-w} = \frac{1}{z-a}\frac{1}{1-\frac{w-a}{z-a}} = \sum_{j=0}^{\infty} \frac{(w-a)^j}{(z-a)^{j+1}}$$

Since $\left|\frac{(w-a)}{(z-a)}\right| = |w-a|r < 1$. Furthermore, by the Weierstrass M-test, the series converges uniformly. So we have that

$$\int_{\partial D(a,r)} \frac{1}{z-w} \mathrm{d}z = \sum_{j=0}^{\infty} (w-a)^j \int_{\partial D(a,r)} \frac{1}{(z-a)^{j+1}} \mathrm{d}z$$

By computing the integral explicitly for $j = 0$, and using FTC for $j \geq 1$ we find the required result. □

Theorem 2.20 (Cauchy integral formula for a disc)**.** Let $D = D(a, r)$, $f : D \to \mathbb{C}$ holomorphic. Then for nay $0 < \rho < r$, $w \in D(a, \rho)$, we have that

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z-w} \mathrm{d}z$$

Proof. Fix w, define $h : D \to \mathbb{C}$ by

$$h(z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

Then h is continuous on D and holomorphic in $D \setminus \{w\}$. By Cauchy's theorem for star domains, we have that

$$\int_{\partial D(a,\rho)} h(z) \mathrm{d}z = 0$$

Substituting the definition of h, we get that

$$f(w) \int_{\partial D(a,\rho)} \frac{1}{z-w} \mathrm{d}z = \int_{\partial D(a,\rho)} \frac{f(z)}{z-w} \mathrm{d}z$$

Result then follows by the fundamental integral. □

Corollary 2.21 (Mean value property)**.** Suppose $f : D(a, R) \to \mathbb{C}$ holomorphic, $0 < \rho < R$. Then

$$f(a) = \int_0^1 f(a + \rho e^{2\pi i t}) \mathrm{d}t$$

Proof. By Cauchy integral formula for a disc. □

Proposition 2.22. If $f : \mathbb{C} \to \mathbb{C}$ is entire, for some $K \geq 0$, $\alpha < 1$, we have that

$$|f(z)| \leq K(1 + |z|^{\alpha})$$

for all $z \in \mathbb{C}$, then f is constant.

Proof. Given $w \in \mathbb{C}$, $\rho > |w|$, by the Cauchy integral formula, we have that

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z-w} \mathrm{d}z$$

Then

$$\begin{aligned}
|f(w) - f(0)| &= \frac{1}{2\pi} \left| \int_{\partial D(a,\rho)} \frac{f(z)}{z-w} - \frac{f(z)}{z} \mathrm{d}z \right| \\
&\leq \frac{|w|}{2\pi} \operatorname{Length}(\partial D(a,\rho)) \sup_{z \in \partial D(a,\rho)} \left| \frac{f(z)}{z(z-w)} \right| \\
&\leq \frac{|w| K (1+\rho)^\alpha}{2\pi\rho(\rho - |w|)} = \frac{|w| K (1+\rho^\alpha)}{\rho - |w|}
\end{aligned}$$

Letting $\rho \to \infty$, we get $f(w) = f(0)$. □

Theorem 2.23 (Liouville)**.** If $f : \mathbb{C} \to \mathbb{C}$ is entire, $|f(z)| \leq K$ for all $z \in \mathbb{C}$, then f is constant.

Proof. Immediate by above proposition. □

Theorem 2.24 (Fundamental theorem of algebra)**.** Every non constant polynomial with complex coefficients has a root over $\mathbb{C}$.

Proof. Let $n = \deg(p) \geq 1$, and without loss of generality, p monic, so $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Then for $z \neq 0$, we have that

$$p(z) = z^n \left(1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right)$$

which means that

$$|p(z)| \geq |z^n| \left(1 - \left(\frac{|a_{n-1}|}{|z|} + \cdots + \frac{|a_0|}{|z^n|} \right) \right)$$

So $|p(z)| \to \infty$ as $|z| \to \infty$. So we have $R > 0$ such that if $|z| > R$, $|p(z)| > 1$. Furthermore, suppose for contradiction p has no root over $\mathbb{C}$. Define $g(z) = \frac{1}{p(z)}$. Then g is entire. For $|z| > R$, $|g| < 1$, and by compactness and continuity of g, g is also bounded on $\overline{D}(0, R)$. But this means that g is constant, so p is constant. Contradiction. □

Theorem 2.25 (Local maximum modulus)**.** Suppose $f : D(a, R) \to \mathbb{C}$ is holomorphic, $|f(z)| \leq |f(a)|$ for all $z \in D(a, R)$. Then f is constant.

Proof. By the mean value property, we have that for any $0 < \rho < R$,

$$f(a) = \int_0^1 f(a + \rho e^{2\pi i t}) \mathrm{d}t$$

So

$$|f(a)| = \left| \int_0^1 f(a + \rho e^{2\pi i t}) \mathrm{d}t \right| \leq \sup_{t \in [0,1]} \left| f(a + \rho e^{2\pi i t}) \right| \leq |f(a)|$$

So equality holds. The first inequality gives us that $f(a + \rho e^{2\pi i t}) = c_\rho$ constant. The second one gives that $|c_\rho| = |f(a)|$, so $|f(z)|$ is constant, and by Cauchy-Riemann, f is constant. □

2.3 Power series

Theorem 2.26. Let $f : D(a, R) \to \mathbb{C}$ be holomorphic. Then

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z$$

Proof. Fix $0 < \rho < R$. Then for $w \in D(a, \rho)$, we have that

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z-w} \mathrm{d}z = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} f(z) \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} \mathrm{d}z = \sum_{n=0}^{\infty} c_n(\rho)(w-a)^n$$

where

$$c_n(\rho) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z$$

This gives us a power series representation of f, which means that f is infinitely differentiable, with

$$c_n(\rho) = \frac{f^{(n)}(a)}{n!}$$

So $c_n(\rho)$ is independent of ρ. □

Corollary 2.27. Let $f : U \to \mathbb{C}$ be holomorphic. Then f is analytic.

Corollary 2.28.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z$$

Theorem 2.29 (Morera)**.** Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ is continuous, and for every closed curve γ in U, $\int_\gamma f(z) \mathrm{d}z = 0$. Then f is holomorphic in U.

Proof. f has an antiderivative F. Then $f = F'$ is holomorphic. □

2.4 Zeroes of a holomorphic function

Theorem 2.30 (Principle of isolated zeroes)**.** Suppose $f : D(a, R) \to \mathbb{C}$ is holomorphic, $f \not\equiv 0$. Then there exists $r > 0$ such that $f(z) \neq 0$ whenever $0 < |z - a| < r$.

Proof. If $f(a) \neq 0$ we are done by continuity. If $f(a) = 0$, then we have $m \geq 1$ such that $f(z) = z^m g(z)$, where $g : D(a, R) \to \mathbb{C}$ holomorphic, $g(a) \neq 0$. Then we are done by continuity of g. □

Theorem 2.31 (Unique analytic continuation)**.** Suppose U, V domains, $U \subseteq V$, $g_1, g_2 : V \to \mathbb{C}$ analytic, $g_1 = g_2$ on U. Then $g_1 = g_2$.

Proof. Let $h = g_1 - g_2$. Then $h = 0$ on U. Define

$$V_0 = \{z \in V : h \equiv 0 \text{ in some } D(z,r)\} \quad \text{and} \quad V_1 = \left\{z \in V : h^{(n)}(z) \neq 0 \text{ for some } n \geq 0\right\}$$

By the principle of isolated zeroes, V_0 and V_1 partition V. By construction, V_0 open, and by continuity of the derivatives, V_1 is open. Since V is connected and V_0 nonempty, we must in fact have $V = V_0$. ☐

Proposition 2.32 (Identity principle)**.** Suppose $f, g : U \to \mathbb{C}$ holomorphic, and suppose

$$S = \{z \in U : f(z) = f(z)\}$$

has a limit point. Then $f = g$.

Proof. Let $h(z) = f(z) - g(z)$. Then by the principle of isolated zeroes, h must be identically zero. ☐

Corollary 2.33 (Global maximum modulus principle)**.** Suppose U is bounded, $f : \overline{U} \to \mathbb{C}$ continuous, f holomorphic on U. Then $|f|$ attains its maximum value on $\partial U = \overline{U} \setminus U$.

Proof. $\overline{U}$ is compact, so $|f|$ is bounded and attains its maxima. Say for all $z \in \overline{U}$, $|f(z)| \leq |f(w)|$. If $w \notin U$, then $w \in \partial U$ and we are done.

On the other hand, if $w \in U$, choose $D = D(w,r) \subseteq U$. Then by local maximum modulus principle, f is constant on D, so by identity principle (or unique analytic continuation), f is constant on U. By continuity, f is constant on $\overline{U}$. ☐

Theorem 2.34 (Cauchy integral formula for derivatives)**.** Suppose $f : D(a,R) \to \mathbb{C}$ holomorphic, then for any $0 < \rho < R$, $w \in D(a,\rho)$, we have that

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^{n+1}} \mathrm{d}z$$

Proof. By induction on n. $n = 0$ is the Cauchy integral formula.

For $n = 1$, let $g(z) = \frac{f(z)}{z-w}$. This is holomorphic on $D(a,R) \setminus \{w\}$, with $g'(z) = \frac{f'(z)}{z-w} - \frac{f(z)}{(z-w)^2}$. Since $\partial D(a,\rho) \subseteq D(a,R) \setminus \{w\}$, we have that

$$\int_{\partial D(a,\rho)} g'(z) \mathrm{d}z = 0$$

which means that

$$\int_{\partial D(a,\rho)} \frac{f'(z)}{z-w} = \int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^2} \mathrm{d}z$$

Using the Cauchy integral formula for f', we have that

$$f'(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f'(z)}{z-w} \mathrm{d}z = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^2}$$

For $n \geq 2$, let $n = k+1$ and $g(z) = \frac{f(z)}{(z+w)^{k+1}}$. Then $g'(z) = \frac{f'(z)}{(z+w)^{k+1}} - \frac{(k+1)f(z)}{(z+w)^{k+2}}$. Similarly, we have that

$$\int_{\partial D(a,\rho)} g'(z) \mathrm{d}z = 0$$

which means that

$$\int_{\partial D(a,\rho)} \frac{f'(z)}{(z-w)^{k+1}} \mathrm{d}z = (k+1) \int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^{k+2}} \mathrm{d}z$$

which by the induction hypothesis gives the required result. □

3 Uniform limits

Proposition 3.1. (f_n) converges locally uniformly on U if and only if (f_n) converges on every compact subset $K \subseteq U$.

Theorem 3.2. Let $U \subseteq \mathbb{C}$ be open, $f_n : U \to \mathbb{C}$ holomorphic, $f_n \to f$ locally uniformly on U. Then f is holomorphic, and $f_n^{(k)} \to f^{(k)}$ locally uniformly.

Proof. For $a \in U$, let $r > 0$ be such that $\overline{D}(a,r) \subseteq U$. Then $f_n \to f$ uniformly on $D(a,r$, which means that f is continuous on $\overline{D}(a,r)$ as the uniform limit of continuous functions. Let γ be a closed curve in $D(a,r)$, then by Cauchy for star domains, we have that

$$\int_\gamma f_n(z)\mathrm{d}z = 0$$

for all n. As $\int_\gamma f_n(z)\mathrm{d}z \to \int_\gamma f(z)\mathrm{d}z$ by uniform convergence, we have that $\int_\gamma f(z)\mathrm{d}z = 0$. So by Morera, f is holomorphic. By the Cauchy integral formula for derivatives, we have that

$$f^{(k)}(w) - f_n^{(k)}(w) = \frac{k!}{2\pi i} \int_{\partial D(a,r/2)} \frac{f(z)-f_n(z)}{(z-w)^{k+1}} \mathrm{d}z$$

which means that

$$\begin{aligned} \left|f^{(k)}(w) - f_n^{(k)}(w)\right| &= \frac{1}{2\pi} \left| \int_{\partial D(a,r/2)} \frac{f(z)-f_n(z)}{(z-w)^{k+1}} \mathrm{d}z \right| \\ &\leq \frac{1}{2\pi} 2\pi \left(\frac{r}{2}\right)^2 \sup_{z\in\partial D(a,r/2)} \left| \frac{f(z)-f_n(z)}{(z-w)^{k+1}} \right| \\ &\leq C_k \sup_{z\in\partial D(a,r/2)} |f(z)-f_n(z)| \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

for some constant C_k. □

4 Winding numbers and topology

4.1 Winding numbers

Definition 4.1 (Continuous choice of argument)

For a curve $\gamma : [a,b] \to \mathbb{C}$, $w \in \mathbb{C}$, we can write $\gamma(t) = w + r(t)e^{i\theta(t)}$ as long as $w \notin \mathrm{Image}(\gamma)$. If γ continuous, then we can choose θ continuous, and we call θ a continuous choice of argument.

Definition 4.2 (Winding number)

Define the winding number, or index of γ about w to be

$$I(\gamma; w) = \frac{\theta(b)-\theta(a)}{2\pi}$$

Proposition 4.3. For a closed curve, $I(\gamma; w)$ is an integer.

Proof.

$$e^{i\theta(b)-i\theta(a)} = 1 \iff \theta(b) - \theta(a) \in 2\pi\mathbb{Z}$$

□

Proposition 4.4. A continuous choice of θ exists, and for different choices, we get the same value of $I(\gamma; w)$.

Proof. Existence follows from taking local choices and using compactness. For uniqueness, note that

$$\frac{\theta(t) - \tilde{\theta}(t)}{2\pi} \in \mathbb{Z}$$

is a continuous integer valued function from a connected set, so must be constant. □

Lemma 4.5. If $w \in \mathbb{C}$, $\gamma : [a, b] \to \mathbb{C} \setminus \{w\}$ piecewise C^1, then we have θ piecewise C^1, and if γ is closed, then

$$I(\gamma; w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w} \mathrm{d}z$$

Proof. Let

$$h(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - w} \mathrm{d}s$$

The integrand is bounded, and continuous at all but finitely many points, so h is continuous. Furthermore, by FTC, h is piecewise C^1, with $h'(t) = \dfrac{\gamma'(t)}{\gamma(t) - w}$ when γ' is continuous. This gives us an ODE for $\gamma - w$,

$$(\gamma(t) - w)' - (\gamma(t) - w)h(t) = 0$$

Using the integrating factor $e^{-h(t)}$, we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\gamma(t) - w)e^{-h(t)}\right) = 0$$

for all but finitely many t. Since $(\gamma(t) - w)e^{-h(t)}$ is continuous, it must in fact be constant. So

$$(\gamma(t) - w) = (\gamma(a) - w)e^{h(t)} = |\gamma(a) - w|e^{\mathrm{Re}(h(t))}e^{\mathrm{Im}(h(t))+\alpha}$$

for some α. Then set $\theta(t) = \alpha + \mathrm{Im}(h(t))$. We have that

$$I(\gamma; w) = \frac{\theta(b) - \theta(a)}{2\pi} = \frac{\mathrm{Im}(h(b))}{2\pi}$$

For a closed curve γ, $e^{h(b)} = 1$, so $\mathrm{Re}(h(b)) = 0$ and $\mathrm{Im}(h(b)) = \frac{h(b)}{i}$. Hence we have that

$$I(\gamma; w) = \frac{h(b)}{2\pi i} = \frac{1}{2\pi i}\int_a^b h'(s)\mathrm{d}s = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(s)}{\gamma(s) - w}\mathrm{d}s = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - w}\mathrm{d}z$$

□

Proposition 4.6. For a closed curve γ, $w \mapsto I(\gamma; w)$ is constant on each connected component of $\mathbb{C} \setminus \mathrm{Image}(\gamma)$.

Proposition 4.7. If $\gamma : [a, b] \to D(z_0, r)$ is a closed curve, then for all $w \notin D(z_0, r)$, we have that $I(\gamma; w) = 0$

Proof. Apply convex Cauchy, as $\frac{1}{z-w}$ is holomorphic in $D(z_0, r)$. □

Proposition 4.8. If $\gamma : [a, b] \to \mathbb{C}$ closed, then there exists a unique unbounded connected component Ω, and for $w \in \Omega$, $I(\gamma; w) = 0$.

Proof. By compactness of Image(γ), Image(γ) is bounded, so there can only be one unbounded connected component. Furthermore, as Image(γ) is contained in a disc, apply previous proposition to a point in Ω not in the disc. □

4.2 Homology

Lemma 4.9. Suppose $\phi : [a, b] \times [c, d] \to \mathbb{R}$ is continuous. Then

$$s \mapsto \int_c^d \phi(s, t)\mathrm{d}t \quad \text{and} \quad t \mapsto \int_a^b \phi(s, t)\mathrm{d}s$$

are continuous.

Proof. Follows from ϕ being uniformly continuous as it is continuous on a compact set. □

Lemma 4.10 (Fubini)**.** Suppose $\phi : [a, b] \times [c, d] \to \mathbb{R}$ is continuous. Then

$$\int_a^b \int_c^d \phi(s, t)\mathrm{d}t\mathrm{d}s = \int_c^d \int_a^b \phi(s, t)\mathrm{d}s\mathrm{d}t$$

Proof. Since ϕ is uniformly continuous, we have that ϕ is the uniform limit of step functions. That is, a partition of $R = [a, b) \times [c, d)$ by sets of the form

$$R_j = [a_j, b_j) \times [c_j, d_j)$$

and

$$g(x, y) = \sum_{j=1}^{n} \alpha_j \mathbb{1}_{R_j}(x, y)$$

where α_j constants. By cumputing the iterated integrals for the step functions, we get the required result. □

Lemma 4.11. Let $f : U \to \mathbb{C}$ be holomorphic, define $g : U \times U \to \mathbb{C}$ by

$$g(z, w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(z) & \text{if } z = w \end{cases}$$

Then g is continuous. Furthermore, if γ is a closed curve in U, then

$$h(w) = \int_\gamma g(z, w)\mathrm{d}z$$

is holomorphic in U.

Proof. For continuity, away from $z = w$ we can take an open ball where g is continuous. Now suppose we have $(a, a) \in U \times U$. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$D(a, \delta) \subseteq U \quad \text{and} \quad \left|f'(z) - f'(a)\right| < \varepsilon \text{ for all } z \in D(a, \delta)$$

which exist by U being open and f' being continuous respectively. Choose $(z, w) \in D(a, \delta) \times D(a, \delta)$. If $z = w$, then

$$|g(z, w) - g(a, a)| = \left|f'(z) - f'(a)\right| < \varepsilon$$

If $z \neq w$, then the path $\gamma(t) = tz + (1 - t)w$ is contained in $D(a, \delta)$ for $t \in [0, 1]$ by convexity. So

$$f(z) - f(w) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(tz - (1 - t)w)\mathrm{d}t = (z - w) \int_0^1 f'(tz + (1 - t)w)\mathrm{d}t$$

This means that

$$\begin{aligned} |g(z, w) - g(a, a)| &= \left|\frac{f(z) - f(w)}{z - w} - f'(a)\right| \\ &= \left|\int_0^1 f'(tz + (1 - t)w) - f'(a)\mathrm{d}t\right| \\ &\leq \sup_{t \in [0,1]} \left|f'(tz + (1 - t)w) - f'(a)\right| \\ &< \varepsilon \end{aligned}$$

So g is continuous at (a, a). To show that h is holomorphic, we will apply Morera. First, we must show that h is continuous. Fix $w_0 \in U$, and a sequence $w_n \to w_0$. Choose $\delta > 0$ such that $\overline{D}(w_0, \delta) \subseteq U$. g is continuous on $U \times U$, so it is uniformly continuous on $\mathrm{Image}(\gamma) \times \overline{D}(w_0, \delta)$ compact.

If $g_n(z) = g(z, w_n)$ for $z \in \mathrm{Image}(\gamma)$, then $g_n \to g_0$ uniformly on $\mathrm{Image}(\gamma)$. So

$$h(w_n) = \int_\gamma g_n(z)\mathrm{d}z \to \int_\gamma g_0(z)\mathrm{d}z = h(w_0)$$

So h is continuous. Now say $\gamma : [a, b] \to D(w_0, \delta)$ is any closed curve, and $\beta : [c, d] \to D(w_0, \delta)$ is any closed curve. Then

$$\begin{aligned} \int_\beta h(w)\mathrm{d}w &= \int_\beta \int_\gamma g(z, w)\mathrm{d}z\mathrm{d}w \\ &= \int_c^d \int_a^b g(\gamma(t), \beta(s))\gamma'(t)\beta'(s)\mathrm{d}t\mathrm{d}s \\ &= \int_a^b \int_c^d g(\gamma(t), \beta(s))\gamma'(t)\beta'(s)\mathrm{d}s\mathrm{d}t \\ &= \int_\gamma \int_\beta g(z, w)\mathrm{d}w\mathrm{d}z \\ &= \int_\gamma 0\mathrm{d}z \\ &= 0 \end{aligned}$$

where since $g(z, w)$ is continuous and holomorphic everywhere except z, by convex Cauchy we get that $\int_\beta g(z, w)\mathrm{d}w = 0$. By Morera, this then means that h is holomorphic. □

Definition 4.12 (Homologous to zero)

Let $U \subseteq \mathbb{C}$ be open, $\gamma : [a, b] \to U$ be a closed curve. Then γ is homologous to zero in U if $I(\gamma; w) = 0$ for all $w \notin U$.

Theorem 4.13 (Cauchy integral formula). Let $U \subseteq \mathbb{C}$ be a nonempty open set, γ closed curve in U homologous to zero in U. Suppose $f : U \to \mathbb{C}$ holomorphic, and $w \in U \setminus \text{Image}(\gamma)$, then

$$I(\gamma; w)f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\mathrm{d}z$$

Proof. Note that the statement is equivalent to showing that

$$\int_\gamma g(z,w)\mathrm{d}z = 0$$

where $g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(z) & \text{if } z = w \end{cases}$. Define $h(w) = \int_\gamma g(z,w)\mathrm{d}z$. Then h is holomorphic in U, and we wish to show that $h = 0$ by first extending it to an entire function $H : \mathbb{C} \to \mathbb{C}$ which has $|H| \to 0$ as $|z| \to \infty$.

Let $V = \{w \in \mathbb{C} \setminus \text{Image}(\gamma) : I(\gamma; w) = 0\}$. Since γ is homologous to zero in U, we have that $\mathbb{C} = U \cup V$. Since $I(\gamma; w)$ is locally constant, V is open. For $w \in U \cap V$,

$$h(w) = \int_\gamma \frac{f(z)-f(w)}{z-w}\mathrm{d}z = \int_\gamma \frac{f(z)}{z-w}\mathrm{d}z = h_1(w)$$

where $h_1 : V \to \mathbb{C}$ holomorphic. Hence the function $H : \mathbb{C} \to \mathbb{C}$.

$$H(z) = \begin{cases} h(w) & \text{if } w \in U \\ h_1(w) & \text{if } w \in V \end{cases}$$

is well defined and holomorphic. Since $\text{Image}(\gamma)$ is compact, we have $R > 0$ such that $\text{Image}(\gamma) \subseteq D(0,R)$. Since the winding number is locally constant, $\mathbb{C} \setminus D(0,R) \subseteq V$. So for $|w| > R$, we have that

$$|H(w)| = |h_1(w)| = \left|\int_\gamma \frac{f(z)}{z-w}\mathrm{d}z\right| \leq \frac{\text{Length}(\gamma)}{|w|-R}\sup_{z \in \text{Image}(\gamma)}|f(z)|$$

which shows that $|H(w)| \to 0$ as $|w| \to \infty$. This means that H is bounded, so constant by Liouville, and must be identically zero. □

Theorem 4.14 (Cauchy's theorem). Suppose U is a nonempty open set, γ closed curve in U homologous to zero in U, and $f : U \to \mathbb{C}$ holomorphic. Then

$$\int_\gamma f(z)\mathrm{d}z = 0$$

Proof. Equivalent to Cauchy integral formula. □

4.3 Homotopy

Definition 4.15 (Null homotopic)

$\gamma : [a,b] \to U$ is null homotopic in U if it is homotopic ot a constant curve in U.

Lemma 4.16. If γ, δ closed piecewise C^1 curves, $|\gamma(t) - \delta(t)| < |w - \gamma(t)|$ for all t, then $I(\gamma; w) = I(\delta; w)$.

Theorem 4.17. If γ_0, γ_1 are homotopic closed curves, and $w \in \mathbb{C} \setminus U$. Then $I(\gamma_0; w) = I(\gamma_1; w)$.

Proof. Let $H : [0,1] \times [a,b] \to U$ be a homotopy from γ_0 to γ_1. Since $K = H([0,1] \times [a,b])$ is compact, we have $\varepsilon > 0$ such that for all $z \in K$, $w \notin D(z, 3\varepsilon)$. Furthermore, H is uniformly continuous, so choose $n \in \mathbb{N}$ such that

$$\left|s - s'\right| + \left|t - t'\right| < \frac{1}{n} \implies \left|H(s,t) - H(s',t')\right| < \varepsilon$$

For $k = 0, \dots, n$, define $\Gamma_k(t) = H(\frac{k}{n}, t)$. In particular, $\Gamma_0 = \gamma_0$ and $\Gamma_n = \gamma_1$. Then by construction, for all $t \in [a,b]$, $k \geq 1$, we have that

$$|\Gamma_{k-1}(t) - \Gamma_k(t)| < \varepsilon < 3\varepsilon < |w - \Gamma_{k-1}(t)|$$

Let $\tilde{\Gamma}_k(t)$ be the polygonal approximation with nodes at $\Gamma_k(t)$ at $0, (b-a)/n, \dots, 1$. Suppose we chose n such that

$$\left|s - s'\right| + \left|t - t'\right| < \frac{\max(1, b-a)}{n} \implies \left|H(s,t) - H(s',t')\right| < \varepsilon$$

Then we have that for $t \in [a,b]$,

$$\left|\tilde{\Gamma}_{k-1}(t) - \tilde{\Gamma}_k(t)\right| \leq \left|\tilde{\Gamma}_{k-1}(t) - \Gamma_k(t)\right| + \left|\tilde{\Gamma}_k(t) - \Gamma_k(t)\right| < 2\varepsilon$$

and

$$|w - \Gamma_{k-1}(t)| \leq \left|w - \tilde{\Gamma}_{k-1}(t)\right| + \left|\Gamma_{k-1}(t) - \tilde{\Gamma}_{k-1}(t)\right|$$

which means that

$$\left|w - \tilde{\Gamma}_{k-1}(t)\right| \geq |w - \Gamma_{k-1}(t)| - \left|\Gamma_{k-1}(t) - \tilde{\Gamma}_{k-1}(t)\right| > 2\varepsilon$$

Which gives us that $I(\tilde{\Gamma}_{k-1}; w) = I(\tilde{\Gamma}_k; w)$ by the lemma. Finally, checking that $I(\tilde{\Gamma}_0; w) = I(\gamma_0; w)$ and $I(\tilde{\Gamma}_n; w) = I(\gamma_1; w)$ gives the required result. □

Corollary 4.18. If γ is null homotopic then it is homologous to zero.

Corollary 4.19. If γ_1, γ_2 homotopic curves, $f : U \to \mathbb{C}$ holomorphic, then

$$\int_{\gamma_1} f(z) \mathrm{d}z = \int_{\gamma_2} f(z) \mathrm{d}z$$

Proof. By theorem and Cauchy's integral formula. □

Definition 4.20 (Simply connected)
A domain U is simply connected if every closed curve in U is null homotopic.

Theorem 4.21 (Cauchy's theorem for simply connected domains)**.** If U is simply connected, γ closed curve in U and $f : U \to \mathbb{C}$ holomorphic, then

$$\int_\gamma f(z) \mathrm{d}z = 0$$

5 Singularities

Definition 5.1 (Isolated singularity)
Let $U \subseteq \mathbb{C}$ be open, $f : U \setminus \{a\} \to \mathbb{C}$ be holomorphic. Then f has an isolated singularity at a.

Definition 5.2 (Removable singularity)
An isolated singularity a is removable if f can be extended to a holomorphic function $U \to \mathbb{C}$.

Proposition 5.3. Let $U \subseteq \mathbb{C}$ be open, $a \in U$, $f : U \setminus \{a\} \to \mathbb{C}$ holomorphic. Then the following are equivalent.

(i) a is a removable singularity.

(ii) $\lim_{z \to a} f(z)$ exists in $\mathbb{C}$.

(iii) There exists $D(a, \varepsilon) \subseteq \mathbb{C}$ such that $|f(z)|$ is bounded on $D'(a, \varepsilon)$.

(iv) $\lim_{z \to a} (z - a) f(z) = 0$

Proof. Suppose a is removable. Then we have $g : U \to \mathbb{C}$ extending f. Then

$$\lim_{z \to a} f(z) = \lim_{z \to a} g(z) = g(a)$$

So (i) implies (ii). By definitions, (ii) implies (iii), and (iii) implies (iv). Suppose (iv) holds. Consider

$$h(z) = \begin{cases} (z-a)^2 f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a \end{cases}$$

Then

$$\lim_{z \to a} \frac{h(z) - h(a)}{z - a} = \lim_{z \to a} (z - a) f(z) = 0$$

So h is differentiable at a, with $h'(a) = 0$. Thus h is holomorphic on U. Considering the Taylor series of h, we have that $h(z) = (z-a)^2 g(z)$ where $g : U \to \mathbb{C}$ holomorphic. So g extends f, and a si removable. □

Definition 5.4 (Pole)
Suppose $a \in U$ is an isolated singularity of f, a is a pole of f if

$$\lim_{z \to a} |f(z)| = \infty$$

Proposition 5.5. Let $f : U \setminus \{a\} \to \mathbb{C}$ holomorphic. Then the following are equivalent.

(i) a is a pole.

(ii) There exists $\varepsilon > 0$, $h : D(a, \varepsilon) \to \mathbb{C}$ with $h(a) = 0$, $h'(a) \neq 0$ for $z \in D'(a, \varepsilon)$ and $f(z) = \frac{1}{h(z)}$ for $z \in D'(a, \varepsilon)$.

(iii) There exists $k \geq 1$ such that $g : U \to \mathbb{C}$ holomorphic, $g(a) \neq 0$, and $f(z) = (z-a)^{-k} g(z)$ for $z \in U \setminus \{a\}$.

Proof. Suppose (i) holds. Then we have $\varepsilon > 0$ such that for $z \in D'(a, \varepsilon)$, $|f(z)| \neq 1$. So $h(z) = \frac{1}{h(z)}$ is holomorphic and bounded in $D'(a, \varepsilon)$. This means that h has a removable singularity at a.

Now suppose (ii) holds. By the Taylor series, we have $k \geq 1$ and $h_1 : U \to D(a, \varepsilon)$ holomorphic, $h_1(z) \neq 0$ for all $z \in D(a, \varepsilon)$. Let $g_1(z) = \frac{1}{h_1(z)}$. Then $f(z) = (z - a)^{-k} g_1(z)$ in $D'(a, \varepsilon)$. Now define

$$g(z) = \begin{cases} g_1(z) & \text{if } z \in D(a, \varepsilon) \\ (z - a)^k f(z) & \text{if } z \in U \setminus \{a\} \end{cases}$$

□

Definition 5.6 (Order)

k above is unique, and called the order of the pole.

Definition 5.7 (Meromorphic function)

If U open, $S \subseteq U$ discrete, $f : U \setminus S \to \mathbb{C}$ holomorphic, and each $a \in S$ is a removable singularity or a pole, then f is meromorphic.

Definition 5.8 (Essential singularity)

An isolated singularity a is essential if it is not removable and not a pole.

5.1 Laurent expansions

Theorem 5.9. Let $A = \{z \in \mathbb{C} : r < |z - a| < R\}$, $0 \leq r < R \leq \infty$, $f : A \to \mathbb{C}$ holomorphic. Then f has a unique convergent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n = \sum_{n=1}^{-\infty} c_{-n} (z - a)^{-n} + \sum_{n=0}^{\infty} c_n (z - a)^n$$

where the coefficients are given by for $r < \rho < R$,

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z - a)^{n+1}} \mathrm{d}z$$

and if $r < \rho \leq \rho' < R$, the series for f converges uniformly on $\{z : \rho \leq |z - a| \leq \rho'\}$

Proof. Fix $w \in A$, let $g(z) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$. Then g is continuous on A, and is holomorphic on $A \setminus \{w\}$, so holomorphic in A. Choose ρ_1, ρ_2 such that $r < \rho_1 < |w - a| < \rho_2 < R$. Within A, $\partial D(a, \rho_1)$ and $\partial D(a, \rho_2)$ are homotopic, so

$$\int_{\partial D(a,\rho_1)} g(z) \mathrm{d}z = \int_{\partial D(a,\rho_2)} g(z) \mathrm{d}z$$

Substituting the definition of g, we get that

$$\int_{\partial D(a,\rho_1)} \frac{f(z)}{z - w} \mathrm{d}z - 2\pi i I(\partial D(a, \rho_1); w) f(w) = \int_{\partial D(a,\rho_2)} \frac{f(z)}{z - w} \mathrm{d}z - 2\pi I I(\partial D(a, \rho_2); w) f(w)$$

Since $I(\partial D(a, \rho_1); w) = 0$ and $I(\partial D(a, \rho_2); w) = 1$, we get that

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho_2)} \frac{f(z)}{z - w} \mathrm{d}z - \frac{1}{2\pi i} \int_{\partial D(a,\rho_1)} \frac{f(z)}{z - w} \mathrm{d}z$$

For the first one, note that

$$\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$$

and for the second,

$$\frac{1}{z-w} = -\sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

Hence $f(w) = \sum_{n=0}^{\infty} c_n(w-a)^n + \sum_{n=1}^{\infty} c_{-n}(w-a)^{-n}$, where

$$c_n = \begin{cases} \frac{1}{2\pi i}\int_{\partial D(a,\rho_2)} \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z & \text{if } n \geq 0 \\ \frac{1}{2\pi i}\int_{\partial D(a,\rho_1)} \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z & \text{if } n < 0 \end{cases}$$

Since $\partial D(a,\rho_1)$ and $\partial D(a,\rho_2)$ are homotopic to $\partial D(a,\rho)$, we have that

$$c_n = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z$$

and this gives us uniqueness of the expansion. Now suppose we have (any) c_n such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

Then choose $r < \rho \leq \rho' < R$. Then we have that

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

converges for all $z \in A$, so it has radius of convergence $\geq R$, which means that it converges uniformly on $\overline{D}(a,\rho')$. Letting $\zeta = (z-a)^{-1}$,

$$\sum_{n=1}^{\infty} c_{-n}\zeta^n$$

converges foe all $z \in A$, so it has radius of convergence $> \frac{1}{r}$, and converges uniformly for $|\zeta| \leq \frac{1}{\rho}$. This means that

$$\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$$

converges uniformly for $|z-a| \geq \rho$. This means that

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

converges uniformly for $\rho \leq |z-a| \leq \rho'$. Thus for any $m \in \mathbb{Z}$, we have that

$$\int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{m+1}}\mathrm{d}z = \sum_{n=-\infty}^{\infty} c_n \int_{\partial D(a,\rho)} (z-a)^{n-m-1}\mathrm{d}z = 2\pi i c_n$$

Which gives us uniqueness of the expansion. □

Corollary 5.10. We have $f_1 : D(a,R) \to \mathbb{C}$ and $f_2 : \mathbb{C} \setminus \overline{D}(a,r) \to \mathbb{C}$ holomorphic such that $f = f_1 + f_2$ in A.

Proposition 5.11. Suppose we have $f : D'(a, R) \to \mathbb{C}$ holomorphic, with series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

If $c_n = 0$ for all $n < 0$, then a is a removable singularity.

Proposition 5.12. Suppose we have $f : D'(a, R) \to \mathbb{C}$ holomorphic, with series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

If $c_n \neq 0$ for finiely many $n < 0$, then a is a pole.

Proposition 5.13. Suppose we have $f : D'(a, R) \to \mathbb{C}$ holomorphic, with series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

If $c_n \neq 0$ for infinitely many $n < 0$, then a is an essential singularity.

5.2 Residue

Definition 5.14 (Residue)
Suppose $f : D'(a, R) \to \mathbb{C}$ holomorphic, with series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

Then the residue of f at a is $\mathrm{Res}_f(a) = c_{-1}$.

Definition 5.15 (Principal part)
Suppose $f : D'(a, R) \to \mathbb{C}$ holomorphic, with series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

Then the principal part of f at a is

$$f_p(z) = \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n}$$

Theorem 5.16 (Residue theorem)**.** Let U be open, $a_1, \dots, a_k \in U$, $f : U \setminus \{a_1, \dots, a_n\} \to \mathbb{C}$ holomorphic. Suppose γ is a closed curve homologous to zero in U, $a_j \notin \mathrm{Image}(\gamma)$, then

$$\frac{1}{2\pi i} \int_\gamma f(z) \mathrm{d}z = \sum_{j=1}^{k} I(\gamma; a_j) \, \mathrm{Res}_f(a_j)$$

Proof. Let

$$f_{p,j} = \sum_{n=1}^{\infty} c_{-n,j}(z - a_j)^{-n}$$

be the principal part of f at a_j. Then $f_{p,j}$ is holomorphic on $\mathbb{C} \setminus \{a_j\}$, so it is holomorphic on $\mathbb{C} \setminus \{a_1, \ldots, a_k\}$. Let $h = f - \sum_{j=1}^{k} f_{p,j}$. Then h is holomorphic on $U \setminus \{a_1, \ldots, a_k\}$.

Fix j, then $f - f_{p,j}$ has a removable singularity at a_j, and for $l \neq j$, $f_{p,j}$ is holomorphic at a_l, so h has a removable singularity at a_j. Which means that h can be extended to a holomorphic function $h : U \to \mathbb{C}$. By Cauchy's theorem,

$$\int_\gamma f(z)\mathrm{d}z = 0$$

Which means that

$$\frac{1}{2\pi i} \int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{k} \frac{1}{2\pi i} \int_\gamma f_{p,j}(z)\mathrm{d}z = \sum_{j=1}^{k} I(\gamma; a) \operatorname{Res}_f(a_j)$$

□

Proposition 5.17. If $f = \frac{g}{h}$, g, h holomorphic at a, $g(a) \neq 0$ and $h(a) = 0$, $h'(a) \neq 0$, then

$$\operatorname{Res}_f(a) = \frac{g(a)}{h'(a)}$$

Lemma 5.18 (Jordan's lemma)**.** Let f be a continuous complex valued function on the semicircle $\gamma_R = \partial D(0, R) \cap \{z : \operatorname{Re}(z) \geq 0$, $\gamma_R(t) = Re^{it}$ for $t \in [0, \pi]$. Then for $\alpha > 0$,

$$\left| \int_{\gamma_R} f(z)e^{i\alpha z}\mathrm{d}z \right| \leq \frac{\pi}{\alpha} \sup_{z \in \gamma_R} |f(z)|$$

Proof. Let $M_R = \sup_{z \in \gamma_R} |f(z)|$. Then we have that

$$\begin{aligned}
\left| \int_{\gamma_R} f(z)e^{i\alpha z}\mathrm{d}z \right| &= \left| \int_0^\pi f(Re^{it})e^{-\alpha R \sin(t) + \alpha R i \cos(t)} iRe^{it}\mathrm{d}t \right| \\
&\leq RM_R \int_0^\pi e^{-\alpha R \sin(t)}\mathrm{d}t \\
&= 2RM_R \int_0^{\pi/2} e^{-\alpha R \sin(t)}\mathrm{d}t \\
&\leq 2RM_R \int_0^{\pi/2} e^{-\frac{2\alpha R t}{\pi}}\mathrm{d}t \\
&= \frac{\pi M_R}{\alpha}(1 - e^{-2\alpha R}) \\
&< \frac{\pi}{\alpha} M_R
\end{aligned}$$

□

Corollary 5.19. If f is continuous on $\{z : \operatorname{Re}(z) > 0, |z| > r\}$, and $\sup_{z \in \gamma_R} |f(z)| \to 0$ as $R \to \infty$, then

$$\left| \int_{\gamma_R} f(z)e^{i\alpha z}\mathrm{d}z \right| \to 0 \quad \text{as} \quad R \to \infty$$

Lemma 5.20. Let $f : D'(a, R) \to \mathbb{C}$ be holomorphic, $z = a$ be a simple pole, $\gamma_\varepsilon(t) = a + \varepsilon e^{-it} : [\alpha, \beta] \to \mathbb{C}$. Then

$$\lim_{\varepsilon \downarrow 0} \int_{\gamma_\varepsilon} f(z) \mathrm{d}z = (\beta - \alpha) i \operatorname{Res}_f(a)$$

Proof.

$$f(z) = \frac{c}{z - a} + g(z)$$

and by computing the separate integrals, we get the required result, since

$$\int_{\gamma_\varepsilon} g(z) \mathrm{d}z \to 0 \quad \text{and} \quad \int_{\gamma_\varepsilon} \frac{c}{z - a} \mathrm{d}z = (\beta - \alpha) ic$$

□

5.3 Argument principle

Proposition 5.21. Suppose f has a zero (pole) of order $k \geq 1$ at $z = a$. Then f'/f has a simple pole at $z = a$, with

$$\operatorname{Res}_f(a) = \begin{cases} k & \text{if } a \text{ is a zero} \\ -k & \text{if } a \text{ is a pole} \end{cases}$$

Proof. We only prove the case for a zero. Then we have $D(a, r)$ such that

$$f(z) = (z - a)^k g(z)$$

where $g : D(a, r) \to \mathbb{C}$ is holomorphic and $g(a) \neq 0$. Then we have that

$$f'(z) = k(z - a)^{k-1} + (z - a)^k g'(z)$$

So

$$\frac{f'}{f} = \frac{k}{z - a} + \frac{g'}{g}$$

and g'/g is holomorphic at a, which gives the required result.
For a pole, use $f(z) = (z - a)^{-k} g(z)$ instead. □

Definition 5.22 (Order)
For a zero/pole a of f, write $\operatorname{ord}_f(a)$ for the order.

Theorem 5.23 (Argument principle)**.** Let f be meromorphic on U with finitely many zeroes $a_1, \ldots, a_k$, finitely many poles $b_1, \ldots, b_l$. Let γ be a closed curve homologous to zero in U, $a_i, b_i \notin \operatorname{Image}(\gamma)$. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \sum_{j=1}^{k} I(\gamma; a_j) \operatorname{ord}_f(a_j) - \sum_{j=1}^{l} I(\gamma; b_j) \operatorname{ord}_f(b_j)$$

Proof. Residue theorem with $g(z) = f'(z)/f(z)$ and previous proposition. □

Definition 5.24 (Bound)

Let Ω be a domain, γ a closed curve in $\mathbb{C}$. Then γ bounds Ω if for all $w \in \Omega$, $I(\gamma; w) = 1$ and for all $w \in \mathbb{C} \setminus (\Omega \cup \text{Image}(\gamma))$, $I(\gamma; w) = 0$.

Corollary 5.25. If γ bounds a domain Ω, f meromorphic in $U \supseteq \Omega \cup \text{Image}(\gamma)$, with no zeroes/poles on $\text{Image}(\gamma)$, N zeros and P poles in Ω with multiplicity, then N, P are finite, and

$$N - P = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = I(\Gamma; 0) = I(f \circ \gamma; 0)$$

Proof. Since Ω is bounded, $\overline{\Omega}$ is compact, and $\overline{\Omega} \subseteq U$. Let S be the set of singularities of f. If $\overline{\Omega} \cap S$ is infinite, by compactness $\overline{\Omega} \cap S$ has a limit point. Contradiction as the singularities are isolated. So P is finite. Similarly by compactness and the principle of isolated zeroes, N is finite. The integral follows by the argument principle. □

Definition 5.26 (Local degree)

Let $f : D(a, R) \to \mathbb{C}$ be holomorphic, f non constant. Then the local degree of f at a is $\deg_f(a)$, which is the order of the zero of $f(z) - f(a)$ at $z = a$.

Theorem 5.27 (Local degree)**.** Let $f : D(a, R) \to \mathbb{C}$ be holomorphic non constant, $\deg_f(a) = d$. Then there exists $r_0 > 0$ such that

$$\forall r \in (0, r_0], \exists \varepsilon > 0, \forall w, 0 < |f(a) - w| < \varepsilon \implies f(a) = w \text{ has } d \text{ roots in } D'(a, r)$$

Proof. Let $g(z) = f(z) - f(a)$. Then g is non constant, so $g' \not\equiv 0$ in $D(a, R)$. Applying the principle of isolated zeroes to g and g', we have $r_0 \in (0, R)$ such that $g(z), g'(z) \neq 0$ for all $z \in D'(a, r_0)$.

Fix $r \in (0, r_0]$ and for $t \in [0, 1]$ define $\gamma(t) = a + re^{2\pi i t}$, $\Gamma(t) = g(\gamma(t))$. Since $\text{Image}(\gamma)$ is compact so closed, and $0 \notin \text{Image}(\gamma)$ since g is nonzero. So we have $\varepsilon > 0$ such that $D(0, \varepsilon) \subseteq \mathbb{C} \setminus \text{Image}(\gamma)$.

Fix w with $0 < |f(a) - w| < \varepsilon$. Then $w - f(a) \in D(0, \varepsilon) \subseteq \mathbb{C} \setminus \text{Image}(\gamma)$. As $z \mapsto I(\Gamma; z)$ is locally constant, it is constant on $D(0; \varepsilon)$, so $I(\Gamma; w - f(a)) = I(\Gamma; 0)$. Then we have that

$$I(\Gamma; w - f(a)) = \frac{1}{2\pi i} \int_0^1 \frac{g'(\gamma(t))\gamma'(t)}{g(\gamma(t)) - (w - f(a))} \mathrm{d}t = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f'(z)}{f(z) - w} \mathrm{d}z = d$$

by the argument principle, since g has one zero, with multiplicity d. Thus by the argument principle, $f(z) - w$ has d roots in $D(a, r)$ as well. Since $w \neq f(a)$, none of the zeros are at a. Since $f' \neq 0$ in $D(a, r)$, the zeroes are simple, so distinct. □

Corollary 5.28 (Open mapping theorem)**.** A non constant holomorphic function is an open map.

Proof. Suppose $f : U \to \mathbb{C}$ holomorphic, $V \subseteq U$ open, $b = f(a) \in f(V)$. Then we have $r > 0$ such that $D(a, r) \subseteq V$. Applying the local degree theorem, there exists $\varepsilon > 0$ such that

$$w \in D'(f(a), \varepsilon) \implies w \in f(D'(a, r))$$

So $D(f(a), \varepsilon) \subseteq f(V)$, so $f(V)$ is open. □

Theorem 5.29 (Rouché)**.** Let γ be a closed curve bounding a region Ω, f, g holomorphic on U open, with

$U \supseteq \Omega \cup \text{Image}(\gamma)$. If $|f(z) - g(z)| < |g(z)|$ for $z \in \text{Image}(\gamma)$, then f, g have the same number of zeroes in Ω (with multiplicity).

Proof. Note that the inequalities imply that f, g nonzero on $\text{Image}(\gamma)$. So we have V open, $V \supseteq \text{Image}(\gamma)$ such that f, g nonzero on V. Let $h : f/g$. Then h is holomorphic and never zero. Since $g \not\equiv 0$ in Ω, we have that the zeroes of g in $\Omega \cup V$ are isolated, so h is meromorphic on $\Omega \cup V$, with no zeroes or poles on $\text{Image}(\gamma)$. Furthermore, f, g have finitely many zeroes on Ω.

In addition, for $z \in \text{Image}(\gamma)$, $|h(z) - 1| < 1$, so letting $\Gamma = h \circ \gamma$, we have that $\text{Image}(\Gamma) \subseteq D(1, 1)$, so $I(\Gamma; 0) = 0$.

By counting the zeroes and poles of h we get the required result. □