Geometry

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Contents

1 Topological surfaces

Definition 1.1 (Locally Euclidean)

A topological space X is locally Euclidean if every $p \in X$ has an open neighbourhood U of p homeomorphic to an open disc in $\mathbb{R}^2$.

Remark 1.2. Every open disc in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

Definition 1.3 (Topological surface)

A topological surface Σ is a locally Euclidean, second countable, Hausdorff topological space.

Definition 1.4 (Graph)
Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ continuous. Define the graph of f to be

$$\Gamma_f = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^{n+m}$$

Proposition 1.5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ continuous. Then Γ_f is a topological surface.

Proof. Since it is a subspace of $\mathbb{R}^3$, which is Hausdorff and second countable, it must in fact be second countable. Suffices to show that every point has a neighbourhood homeomorphic to an open disc. Let the neighbourhood be all of Γ_f, and note that

$$\pi(x, y, z) = (x, y) \quad \text{and} \quad \sigma(x, y) = (x, y, f(x, y))$$

are both continuous, and inverses to eachother. So Γ_f is homeomorphic to $\mathbb{R}^2$. □

Definition 1.6 (Unit sphere)
The unit sphere in $\mathbb{R}^{n+1}$ is

$$S^n = \left\{x \in \mathbb{R}^{n+1} : ||x|| = 1\right\}$$

Proposition 1.7. S^2 is a topologial surface.

Proof. Since $S^2 \subseteq \mathbb{R}^3$, it must be Hausdorff and second countable. Consider stereographic projection from the north pole N, that is,

$$\pi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

This defines a homeomorphism $S^2 \setminus N \cong_{\text{h}} \mathbb{R}^2$. By considering the stereographic projection from a different point, we see that S^2 is a topological surface. □

Definition 1.8 (Real projective plane)
Let $a(x) = -x$, $a \in \text{Homeo}(S^2)$. Then we define the real projective plane to be

$$\mathbb{RP}^2 = S^2/\langle a\rangle$$

Lemma 1.9. As a set, we have a bijection between lines in $\mathbb{R}^3$ through the origin and $\mathbb{RP}^2$.

Proof. Consider a line in $\mathbb{R}^3$ through the origin. It must intersect S^2 at antipodal points x and $-x$, which corresponds to the equivalence class of x. Conversely, given an equivalence class, the line through x and $-x$ goes through the origin. □

Lemma 1.10. $\mathbb{RP}^2$ is a topological surface.

Definition 1.11 (Torus)

Consider $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Then define the torus to be

$$T^2 = S^1 \times S^1 \subseteq \mathbb{C} \times \mathbb{C}$$

Lemma 1.12. T^2 is a topological surface.

Definition 1.13 (Connect sum)

Suppose Σ_1 and Σ_2 are topological surfaces, $D_1 \subseteq \Sigma_1$, $D_2 \subseteq \Sigma_2$ open discs, then we define $\Sigma_1 \# \Sigma_2$ to the surface obtained by gluing $\Sigma_1 \setminus D_1$ and $\Sigma_2 \setminus D_2$ by identifying ∂D_1 and ∂D_2.

Lemma 1.14. If Σ_1 and Σ_2 are topological surfaces, so is $\Sigma_1 \# \Sigma_2$.

1.1 Subdivisions

Definition 1.15 (Subdivision)

A subdivision of a compact topological surface Σ is (V, E, F), where $V \subseteq \Sigma$ is a finite set of vertices, $E = \{e_i : [0,1] \to \Sigma\}$ a finite set of edges, and F a set of faces, satisfying the following conditions:

(i) Each e_i is an injection on $(0, 1)$ (edges do not self intersect).

(ii) $e_i^{-1} V = \{0, 1\}$ (edges only meets vertices at end points).

(iii) $e_i[0, 1] \cap e_j[0, 1] \subseteq V$ (edges only meet at vertices).

(iv) F is the set of connected components of

$$\Sigma \setminus \left(V \cup \left(\bigcup_i e_i[0, 1] \right) \right)$$

and each $f \in F$ is homeomorphic to an open disc on R^2.

Proposition 1.16. For a face $f \in F$, $\partial F \subseteq V \cup E$.

Definition 1.17 (Triangulation)

A subdivision of Σ is a triangulation if every closed face has three edges, and for any two faces, they are either disjoint ot meet at exactly one edge or vertex.

Definition 1.18 (Euler characteristic)

The Euler characteristic of a topological surface is

$$\chi(\Sigma) = \#V - \#E + \#F$$

Theorem 1.19. Every compact topological surface has a subdivision (equivalently, a triangulation).

Theorem 1.20. Euler characteristic is independent of the choice of subdivision.

Proposition 1.21. If Σ_1, Σ_2 compact topological surfaces, then

$$\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$$

Definition 1.22 (g-holed torus)
For $g \geq 1$, the g-holed torus Σ_g is the connect sum of g copies of T^2. g is called the genus of Σ_g.

Proposition 1.23. $\chi(\Sigma_g) = 2 - 2g$.

1.2 Charts

Definition 1.24 (Chart)
Suppose $U \underset{\text{open}}{\subseteq} \Sigma$, $V \underset{\text{open}}{\subseteq} \mathbb{R}^2$, $\phi : U \cong_{\text{h}} V$. Then (U, ϕ) is a chart for Σ at $p \in U$.

Definition 1.25 (Atlas)
A collection of charts $(U_i, \phi_i)_{i \in I}$ is called an atlas if

$$\bigcup_{i \in I} U_i = \Sigma$$

Definition 1.26 (Local parametrisation)
Suppose (U, ϕ) is a chart. Define $\sigma = \phi^{-1} : V \cong_{\text{h}} U$. Then σ is called a local parametrisation of Σ.

Definition 1.27 (Transition map)
For (U_1, ϕ_1), (U_2, ϕ_2) charts, the transition map from ϕ_1 to ϕ_2 is

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \cong_{\text{h}} \phi_2(U_1 \cap U_2)$$

2 Abstract smooth surfaces

Definition 2.1 (Smooth)
For $V \underset{\text{open}}{\subseteq} \mathbb{R}^m$ and $W \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $f : V \to W$ is smooth if it is infinitely differentiable.

Definition 2.2 (Smooth at)

For $Z \subseteq \mathbb{R}^m$, we say that $f : Z \to \mathbb{R}^n$ continuous is smooth at $p \in Z$ if there exists an open ball B, $p \in B$, and $F : B \to \mathbb{R}^n$ smooth such that $F|_{B \cap Z} = f|_{B \cap Z}$.

Definition 2.3 (Diffeomorphism)

For $V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ and $W \underset{\text{open}}{\subseteq} \mathbb{R}^n$, a homeomorphism $f : V \cong_{\text{h}} W$ is a diffeomorphism if f and f^{-1} are smooth.

Definition 2.4 (Abstract smooth surface)

A topological surface Σ is an abstract smooth surface if it has an atlas $(U_i, \phi_i)_{i \in I}$, such that all the transition maps

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \cong_{\text{d}} \phi_2(U_1 \cap U_2)$$

are diffeomorphisms.

Proposition 2.5. S^2 with charts given by stereographic projection is an abstract smooth surface.

Proposition 2.6. T^2 is an abstract smooth surface with parametrisation given by $(s, t) \mapsto (e^{2\pi i s}, e^{2\pi i t})$.

Definition 2.7 (Smooth map to Euclidean space)

Let Σ be an abstrac smooth surface, $f : \Sigma \to \mathbb{R}^n$ continuous, then we say that f is smooth at $p \in \Sigma$ if for all charts (U, ϕ) such that $p \in U$, we have that

$$f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^n$$

is smooth at $\phi(p) \in \phi(U) \underset{\text{open}}{\subseteq} \mathbb{R}^2$.

Definition 2.8 (Smooth map between abstract smooth surfaces)

Let Σ_1, Σ_2 be abstract smooth surfaces, $f : \Sigma_1 \to \Sigma_2$ is smooth at $p \in \Sigma_1$ if for all charts (U, ϕ) of p and (V, ψ) of $f(p)$, we have that

$$\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth at $\phi(p) \in \phi(V) \underset{\text{open}}{\subseteq} \mathbb{R}^2$.

Definition 2.9 (Diffeomorphism)

A diffeomorphism $f : \Sigma_1 \cong_{\text{d}} \Sigma_2$ between abstract smooth surfaces is a smooth homeomorphism with smooth inverse.

Definition 2.10 (Pullback atlas)
Suppose Σ_1, Σ_2 topological surfaces, Σ_2 is an abstract smooth surface with atlas $(U_i, \phi_i)_{i \in I}$, $f : \Sigma_1 \cong_h \Sigma_2$. Then

$$(f^{-1}U_i, \phi_i \circ f|_{f^{-1}U_i})_{i \in I}$$

defines an atlas on Σ_1 which makes it an abstract smooth surface, with the same transition maps.

3 Embedded surfaces

Definition 3.1 (Smooth surface in $\mathbb{R}^3$)
A smooth surface in $\mathbb{R}^3$ is a subspace $\Sigma \subseteq \mathbb{R}^3$ such that for all $p \in \Sigma$, thete exists $U \underset{\text{open}}{\subseteq} \Sigma$, $p \in U$ such that U is diffeomorphic to an open subset of R^2.

Theorem 3.2 (Implicit function theorem). Suppose $p = (x_0, y_0) \in U \underset{\text{open}}{\subseteq} \mathbb{R}^k \times \mathbb{R}^l$, $f : U \to \mathbb{R}^l$ has $f(p) = 0$, and $(\partial f / \partial y_j)_{j=1}^l$ an isomorphism at p. Then we have $x_0 \in V \underset{\text{open}}{\subseteq} \mathbb{R}^k$, $g : V \to \mathbb{R}^l$ C^1 with $g(x_0) = y_0$, such that for any $(x, y) \in U \cap (V \times \mathbb{R}^l)$, $f(x, y) = 0$ if and only if $g(x) = y$.

Proof. Define $F : U \to \mathbb{R}^k \times \mathbb{R}^l$ by $F(x, y) = (x, f(x, y))$. Then

$$DF = \begin{pmatrix} I & 0 \\ 0 & \left(\frac{\partial f}{\partial y_j}\right)_j \end{pmatrix}$$

which means that it is an isomorphism. The inverse function theorem they says that F is locally invertible, near $F(x_0, y_0) = (x_0, 0)$. Say we have $V \underset{\text{open}}{\subseteq} \mathbb{R}^k$ and $W \underset{\text{open}}{\subseteq} \mathbb{R}^l$ such that $(x_0, 0) \in V \times W$, and $G : V \times W \to U \subseteq \mathbb{R}^k \times \mathbb{R}^l$ be the C^1 inverse, with $F \circ G = \mathrm{id}_{V \times W}$. Suppose $G(x, y) = (\phi(x, y), \psi(x, y))$. Then

$$F(G(x, y)) = F(\phi(x, y), \psi(x, y)) = (\phi(x, y), f(\phi(x, y), \psi(x, y))) = (x, y)$$

Which means that $\phi(x, y) = x$, and $G(x, y) = (x, \psi(x, y))$, with $f(x, \psi(x, y)) = y$ when $(x, y) \in V \times W$. Thus $f(x, y) = 0$ if and only if $y = \psi(x, 0)$. Define $g : V \to \mathbb{R}^l$ by $x \mapsto \psi(x, 0)$. □

Definition 3.3 (Allowable parametrisation)
Let $\Sigma \subseteq \mathbb{R}^3$, $V \underset{\text{open}}{\subseteq} \mathbb{R}^2$, $U \underset{\text{open}}{\subseteq} \Sigma$, $\sigma : V \to U$ is an allowable parametrisation of Σ at $p \in U$ if σ is a homeomorphism, and $\mathrm{rank}(D\sigma(x)) = 2$ for all $x \in V$.

Theorem 3.4. For $\Sigma \in \mathbb{R}^3$, the following are equivalent.

(i) Σ is an smooth surface.

(ii) Σ is locally the graph of a smooth function over one of the x-y, x-z, y-z planes.

(iii) Σ is locally the zero set of a smooth function with nonzero derivative.

(iv) Σ is locally the image of an allowable parametrisation.

Proof. We will first show that (ii) implies all of the others. Suppose (ii) holds, and locally, $\Sigma = \{(x, y, g(x, y)) : (x, y) \in V\}$, then π_{XY} gives a chart. So (i) holds. Similarly, Σ is the zero locus of $z - g(x, y) = 0$, which has nonzero derivative, so (iii) holds.

$$\sigma(x, y) = (x, y, g(x, y))$$

defines an allowable parametrisation, and (iv) holds.

Now suppose (i) holds. By the definition, each chart defines a diffeomorphism to an open subset of $\mathbb{R}^2$, so the inverse defines an allowable parametrisation, and (iv) holds.

Suppose (iii) holds. Without loss of generality, suppose $D_z f(x_0, y_0, z_0) \neq 0$. The implicit function theorem gives us a neighbourhood $V \underset{\text{open}}{\subseteq} \mathbb{R}^2$, with $(x_0, y_0) \in V$, $g : V \to \mathbb{R}$ smooth, $g(x_0, y_0) = z_0$ and locally, Σ is the graph of g. So (ii) holds.

Suppose (iv) holds, and we have an allowable parametrisation. Say

$$D\sigma = \begin{pmatrix} \frac{\partial \sigma_1}{\partial x} & \frac{\partial \sigma_2}{\partial x} \\ \frac{\partial \sigma_1}{\partial y} & \frac{\partial \sigma_2}{\partial y} \\ \frac{\partial \sigma_1}{\partial z} & \frac{\partial \sigma_3}{\partial z} \end{pmatrix}$$

As this has rank 2, without loss of generality the first two rows form an invertible matrix. Consider $\pi_{XY} \circ \sigma : V \to \mathbb{R}^2$. The inverse function theorem says this is locally invertible, so Σ is locally given by a graph. So (ii) holds. □

Definition 3.5 (Surface of revolution)

Suppose $\gamma = (f, 0, g) : [a, b] \to \mathbb{R}^3$, with γ injective, $\gamma' \neq 0$ and $f > 0$. The surface of revolution has (local) parametrisation

$$\sigma(u, v) = \begin{pmatrix} f(u)\cos(v) \\ f(u)\sin(v) \\ g(u) \end{pmatrix}$$

where $\sigma : (a, b) \times (\theta, \theta + 2\pi) \to \mathbb{R}^3$.

4 Orientability

Definition 4.1 (Orientation preserving linear map)

A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is orientation preserving if $\det(T) > 0$. The subgroup of orientation preserving linear maps is denote by $\mathrm{GL}_n^+(\mathbb{R})$.

Definition 4.2 (Orientation preserving)

Let $V, W \underset{\text{open}}{\subseteq} \mathbb{R}^2$, then $f : V \to W$ is orientation preserving if $Df(x) \in \mathrm{GL}_2^+(\mathbb{R})$ for all $x \in V$.

Definition 4.3 (Orientable surface)

An abstract smooth surface Σ is orientable if there exists an atlas $(U_i, \phi_i)_{i \in I}$ such that the transition maps are orientation preserving diffeomorphisms. A choice of such an atlas is called an orientation.

Proposition 4.4. Orientability is a homeomorphism (and thus diffeomorphism) invariant.

Proof. Suppose Σ_2 orientable, $f : \Sigma_1 \cong_h \Sigma_2$. Then by passing to the pullback atlas, we have an orientable atlas for Σ_1, as the transition maps are the same. □

Proposition 4.5. An abstract smooth surface is orientable if and only if it does not contain a subspace homeomorphic to the Möbius band.

4.1 Embedded surfaces

Definition 4.6 (Tangent plane)
Let Σ be a smooth surface in $\mathbb{R}^3$, and $p \in \Sigma$. Fix an allowable parametrisation σ near p with $\sigma(0) = 0$. Define the tangent plane

$$T_p\Sigma = \operatorname{Im}(D\sigma(0))$$

and the affine tangent plane $T_p\Sigma + p$.

Lemma 4.7. $T_p\Sigma$ is well defined. That is, it is independent of the choice of Σ.

Proof. Suppose σ and $\tilde{\sigma}$ are choices of parametrisation at p. Then $\sigma^{-1} \circ \tilde{\sigma}$ is a diffeomorphism, so $D(\sigma^{-1} \circ \tilde{\sigma})(0)$ is an isomorphism $\operatorname{Im}(D\tilde{\sigma}(0)) \to \operatorname{Im}(D\sigma(0))$. □

Proposition 4.8.
$$T_p\Sigma = \operatorname{span}\{\gamma'(0) : \gamma \text{ smooth paths in } \Sigma \text{ with } \gamma(0) = p\}$$

Definition 4.9 (Normal)
Suppose Σ is a smooth surface in $\mathbb{R}^3$, $p \in \Sigma$. Then the normal direction at p is $(T_p\Sigma)^\perp$.

Definition 4.10 (Two sided)
A smooth surface in $\mathbb{R}^3$ is two sided if it admits a continuous choice of unit normal vector.

Proposition 4.11. A smooth surface in $\mathbb{R}^3$ is orientable if and only if it is two sided.

Proof. First suppose Σ is orientable. We will show that we can define a continuous positive normal in one e, which agrees on the intersections.

Suppose $\sigma : V \to U$ is an allowable parametrisation with $\sigma(0) = p$. Define the positive normal at p to be

$$n_\sigma(p) = \frac{\sigma_u \times \sigma_v}{\left\|\sigma_u \times \sigma_v\right\|}$$

Then the bases $\sigma_u, \sigma_v, n_\sigma(p)$ are related to the standard basis by a change of basis matrix with positive determinant. Suppose $\tilde{\sigma}$ is another parametrisation, with $\tilde{\sigma}(0) = p$. Since Σ is orientable, we have an orientation preserving transition map ϕ such that $\sigma = \tilde{\sigma} \circ \phi$. This means that

$$\sigma_u \times \sigma_v = \det(D\phi(0))\tilde{\sigma}_u \times \tilde{\sigma}_v \tag{*}$$

and as $\det(D\phi(0)) > 0$, $n_\sigma(p) = n_{\tilde{\sigma}}(p)$. So this defines a continuous choice of unit normal vector for Σ.

Conversely, suppose Σ is two sided, and we have a global continuous choice of normal. Consider the subatlas where at each point p, we only have parametrisations σ where $\sigma_u, \sigma_v, n(p)$ is a positively oriented basis. Then this defines an orientation, by (*). □

Lemma 4.12. Suppose Σ is a smooth surface in $\mathbb{R}^3$, $f : \mathbb{R}^3 \to \mathbb{R}^3$ smooth such that $f(\Sigma) = \Sigma$. Then for $p \in \Sigma$, we have that

$$DA(p) : \mathbb{R}^3 \to \mathbb{R}^3$$

sends $T_p(\Sigma)$ to $T_{A(p)}(\Sigma)$.

Proof. By chain rule. □

5 Geometry

Definition 5.1 (Length)
Suppose $\gamma : (a, b) \to \mathbb{R}^3$ smooth, then we define the length of γ to be

$$\text{Length}(\gamma) = \int_a^b \left\| \gamma'(t) \right\| \mathrm{d}t$$

Proposition 5.2. Length is independent of e.

Proposition 5.3. Suppose γ is a C^1 curve with $\gamma'(t) \neq 0$ for all $t \in (a, b)$, then γ can be parametrised by arc length s, so $\left\| \gamma'(s) \right\| = 1$.

5.1 First fundamental form

Definition 5.4 (First fundamental form)
The first fundamental form of Σ in paramtrisation σ is the quadratic form

$$\begin{pmatrix} \mathrm{d}u & \mathrm{d}v \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}$$

where $E = \langle \sigma_u, \sigma_u \rangle$, $F = \langle \sigma_u, \sigma_v \rangle$, $G = \langle \sigma_v, \sigma_v \rangle$.

Proposition 5.5. Suppose $\gamma(t) = \sigma(u(t), v(t))$. Then

$$\text{Length}(\gamma) = \int_a^b \left(E(u')^2 + 2Fu'v' + G(v')^2 \right)^{1/2} \mathrm{d}t$$

Proof. Chain rule. □

Proposition 5.6. The first fundamental form is the Euclidean inner product on $T_p\Sigma$ given in terms of the basis σ_u, σ_v.

Definition 5.7 (Isometric)
Suppose Σ_1, Σ_2 are smooth surfaces in $\mathbb{R}^3$, we say that $f : \Sigma_1 \to \Sigma_2$ is an isometry if for all curves γ,

$$\text{Length}(\gamma) = \text{Length}(f \circ \gamma)$$

Lemma 5.8. Smooth surfaces Σ_1, Σ_2 in $\mathbb{R}^3$ are locally isometric near $p \in \Sigma_1$, $q \in \Sigma_2$ if and only if there exists allowable parametrisations

$$\sigma : V \to U \underset{\text{open}}{\subseteq} \Sigma_1 \quad \text{and} \quad \tilde{\sigma} : V \to \tilde{U} \underset{\text{open}}{\subseteq} \Sigma_2$$

for which the first fundamental forms are equivalent as functions on V.

Proof. By definition, the first fundamental form determines lengths. So suffices to show that the lengths determine the FFF of a parametrisation.

Given $\sigma : V \to U \subseteq \Sigma$, without loss of generality, suppose $V = D(0, \delta)$ for some $\delta > 0$ and $\sigma_0 = p$. Consider the curve $\gamma_\varepsilon : [0, \varepsilon] \to U$, given by $\gamma(t) = \sigma(t, 0)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\text{Length}(\gamma_\varepsilon))\bigg|_{\varepsilon=0} = \sqrt{E(0,0)}$$

So lengths determine E at p. Similarly F and G are determined by curves of the form $\sigma(t, t)$ and $\sigma(0, t)$. □

Lemma 5.9. Suppose $\sigma, \tilde{\sigma}$ are parametrisations, $f = \tilde{\sigma}^{-1} \circ \sigma$ the transition map, then

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (Df)^{\mathsf{T}} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} (Df)$$

Proof.

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (D\sigma)^{\mathsf{T}}(D\sigma)$$

and $\sigma = \tilde{\sigma} \circ f$, so the result follows by chain rule. □

Definition 5.10 (Angle)
Suppose $v, w \in T_p\Sigma$, with $v = D\sigma(p)(v_0)$ and $w = D\sigma(p)(w_0)$. Let θ is the angle between v and w, σ is an allowable parametrisation, with first fundamental form (as a bilinear form) $I(x, y)$. Then

$$\cos(\theta) = \frac{I(v_0, w_0)}{\sqrt{I(v_0, v_0)}\sqrt{I(w_0, w_0)}}$$

Lemma 5.11. σ is conformal, i.e. angle preserving if and only if $E = G$, $F = 0$.

Proof. Suppose $\gamma, \tilde{\gamma}$ curves in V, $\sigma : V \to U$ an allowable parametrisation, $\gamma(0) = \tilde{\gamma}(0) = 0$, $\sigma(0) = p$. Then the curves $\sigma \circ \gamma$ and $\sigma \circ \tilde{\gamma}$ meet at angle θ on $T_p\Sigma$, where

$$\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F\left(\dot{u}\dot{\tilde{v}} + \dot{\tilde{u}}\dot{v}\right) + G\dot{v}\dot{\tilde{v}}}{\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right)^{1/2}\left(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2\right)^{1/2}}$$

Choosing curves $\gamma(t) = (t, 0)$ and $\tilde{\gamma}(t) = (0, t)$ forces $F = 0$, and choosing $\gamma(t) = (t, t)$ and $\tilde{\gamma}(t) = (t, -t)$ force $E = G$.

Converely, the first fundamental form is just a pointwise rescaling og the Euclidean FFF, which preserves angles. □

Definition 5.12 (Area)
Let Σ be a smooth surface in $\mathbb{R}^3$, $\sigma : V \to U$ an allowable parametrisation. Then define

$$\text{Area}(U) = \int_V \left(EG - F^2\right)^{1/2} \mathrm{d}A$$

Proposition 5.13. Suppose $\sigma, \tilde{\sigma}$ are allowable parametrisations, with transition map $\phi = \sigma^{-1} \circ \tilde{\sigma}$. Then

$$\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = |\det(D\phi)|\sqrt{EG - F^2}$$

Corollary 5.14. Area is independent of parametrisation.

Proof. Above formula and change of variables for area integrals from vector calculus. □

5.2 Second fundamental form

Definition 5.15 (Second fundamental form)
The second fundamental form of the smooth surface $\Sigma \subseteq \mathbb{R}^3$, at the parametrisation σ, is the quadratic form

$$\begin{pmatrix} \mathrm{d}u & \mathrm{d}v \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \end{pmatrix}$$

where $L = \langle n, \sigma_{uu} \rangle$, $M = \langle n, \sigma_{uv} \rangle$, $N = \langle n, \sigma_{vv} \rangle$, where n is the positive unit normal with direction $\sigma_u \times \sigma_v$.

Lemma 5.16. The second fundamental form is also given by

$$-(Dn)^{\mathsf{T}}(D\sigma) = \begin{pmatrix} -\langle n_u, \sigma_u \rangle & -\langle n_u, \sigma_v \rangle \\ -\langle n_v, \sigma_u \rangle & -\langle n_v, \sigma_v \rangle \end{pmatrix}$$

Proof. By definition of n, $\langle n, \sigma_u \rangle = \langle n, \sigma_v \rangle = 0$. Taking derivatives with respect to u and v gives the required result. □

Lemma 5.17. Suppose $\sigma, \tilde{\sigma}$ allowable parametrisations, $\phi = \sigma^{-1} \circ \tilde{\sigma}$ the transition map. Then

$$\begin{pmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{pmatrix} = \pm (D\phi)^{\mathsf{T}} \begin{pmatrix} L & M \\ M & N \end{pmatrix} (D\phi)$$

where the sign is given by $\operatorname{sign}(\det(D\phi))$.

5.3 Gaussian curvature

Definition 5.18 (Gauss map)
Let $\Sigma \subseteq \mathbb{R}^3$ be a smooth oriented surface, the Gauss map $n : \Sigma \to S^2$ sends a point p to the positive unit normal $n(p)$ at p.

Lemma 5.19. n is smooth.

Proposition 5.20. Suppose I_p and II_p are the bilinear forms representing the first and second fundamental forms respectively. Then

$$II_p(v, w) = I_p(-Dn(p)(v), w)$$

Proof. Fix $v \in T_p\Sigma$. Suppose we have a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$, with $\gamma(0) = p$, $\gamma'(0) = v$. Then $n \circ \gamma : (-\varepsilon, \varepsilon) \to S^2$ has $(n \circ \gamma)(0) = n(p)$. Furthermore, by the chain rule,

$$Dn(p)(v) = Dn(\gamma(0))(D\gamma(0)) = D(n \circ \gamma)(0) \in T_{n(p)}S^2 = T_p\Sigma$$

Which means that $Dn(p) \in \mathrm{End}(T_p\Sigma)$. Regarding Dn as a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with respect to the basis σ_u, σ_v (i.e. $a = \langle Dn_u, \sigma_u \rangle$ etc.), we have that

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = -\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Finally, by noting that $I_p(-Dn(p)(\sigma_u), \sigma_u) = -(aE + bF) = -L$ etc., we get the required result. □

Lemma 5.21. $Dn(p) \in T_p\Sigma$ is a self adjoint linear map with respect to the inner product I_p.

Proof. Both the first and second fundamental forms are symmetric. □

Definition 5.22 (Gaussian curvature)
Suppose Σ is a smooth surface in $\mathbb{R}^3$, then the Gaussian curvature is the function $\kappa : \Sigma \to \mathbb{R}$, given by

$$\kappa : p \mapsto \det(Dn(p))$$

Proposition 5.23. Suppose we have parametrisation σ, with first fundamental form $I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and second fundamental form $II_p = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$, then

$$\kappa = \frac{\det(II_p)}{\det(I_p)} = \frac{LN - M^2}{EG - F^2}$$

Proof. Write $S = Dn(p)$ in the basis σ_u, σ_v, then we have that $B = -S^{\mathsf{T}}A$, so $S = (-BA^{-1})^{\mathsf{T}}$. Taking determinants gives the required result. □

Definition 5.24 (Flat)
A surface $\Sigma \subseteq \mathbb{R}^3$ is flat if $\kappa \equiv 0$.

Definition 5.25 (Elliptic, parabolic, hyperbolic points)
Let $\Sigma \subseteq \mathbb{R}^3$ be smooth, $p \in \Sigma$ is

- Elliptic if $\kappa(p) > 0$,

- Parabolic if $\kappa(p) = 0$,
- Hyperbolic if $\kappa(p) < 0$.

Lemma 5.26. Suppose $p \in \Sigma \subseteq \mathbb{R}^3$ elliptic. Then for a sufficiently smapp neighbourhood of p, Σ lies on one side of $T_p\Sigma + p$.

Proof. Fix a paramatrisation σ near p. Then $\kappa = \dfrac{LN - M^2}{EG - F^2}$, and as I_p is positive definite, $EG - F^2 > 0$, so $\operatorname{sign}(\kappa) = \operatorname{sign}(LN - M^2)$.

Now note that

$$\sigma(u+h, v+l) = \sigma(u,v) + (h\sigma_u(u,v) + l\sigma_v(u,v)) + \frac{1}{2}\left(h^2\sigma_{uu}(u,v) + 2hl\sigma_{uv}(u,v) + l^2\sigma_{vv}(u,v)\right) + \mathcal{O}\left(h^3, l^3\right)$$

and as the linear term $h\sigma_u(u,v) + l\sigma_v(u,v)$ lies within $T_p\Sigma$, the signed perpendicular distance from Σ to the tangent plane $T_p\Sigma$ is given by $\left\langle n, \frac{1}{2}\left(h^2\sigma_{uu}(u,v) + 2hl\sigma_{uv}(u,v) + l^2\sigma_{vv}(u,v)\right)\right\rangle = \frac{1}{2}II_p(h,l)$.

If p is elliptic, then within a sufficiently small neighbourhood, by continuity $\kappa > 0$. Then the quadratic form II_p must be positive or negative definite, as both the eigenvalues have the same sign. This means that the signed distance always has the same sign. □

Lemma 5.27. Suppose $p \in \Sigma \subseteq \mathbb{R}^3$ hyperbolic. Then for a sufficiently smapp neighbourhood of p, Σ lies on both sides of $T_p\Sigma + p$.

Proof. From the above lemma, the eigenvalues must have different signs, so the signed distance takes both signs, so Σ lies on both sides of the tangent plane. □

Proposition 5.28. Let $\Sigma \subseteq \mathbb{R}^3$ be a compact smooth surface in $\mathbb{R}^3$. Then Σ has an elliptic point.

Proof. Σ is closed and bounded, so for R sufficiently large, $\Sigma \subseteq \overline{D}(0,R)$. Suppose R minimal. Up to rotation/translation, without loss of generality suppose the intersection occurs at $p = (0,0,z)$. Here, we have that

$$T_p\partial D(0,R) = \mathrm{T}_p\Sigma$$

and that locally, Σ is the graph of a function $f(u,v) : V \to \mathbb{R}^3$, such that $f - \sqrt{R^2 - u^2 - v^2} \leq 0$. Since $f_u = f_v = 0$ as $f(0,0) = p$ is a local maximum, we have from the Taylor series that

$$\frac{1}{2}(f_u uu^2 + 2f_{uv}uv + f_{vv}v^2) + \frac{1}{2R}(u^2 + v^2) \leq 0$$

for sufficiently small u, v. So locally, $\begin{pmatrix} L & M \\ M & N \end{pmatrix}$ is negative definite and $\kappa(p) > 0$. □

Theorem 5.29. Let Σ be a smooth surface in $\mathbb{R}^3$, $p \in \Sigma$ has $\kappa(p) \neq 0$. Let U be a small neighbourhood of p, and we have a decreasing sequence (A_i) of open neighbourhoods of p, such that for all $\varepsilon > 0$, there exists I such that for all $i \geq I$, $A_i \subseteq D(p,\varepsilon)$. Then

$$|\kappa(p)| = \lim_{\varepsilon \to 0} \frac{\operatorname{Area}(n(A_i))}{\operatorname{Area}(A_i)}$$

Proof. Fix a parametrisation $\sigma : V \to U$ near p, with $\sigma(0) = p$. Define $V_i = \sigma^{-1}A_i$. Since A_i shrinks to p, $\bigcap V_i = \{0\}$. By definition, we have that

$$\text{Area}(A_i) = \int_{V_i} \sqrt{EG - F^2}\text{d}u\text{d}v = \int_{V_i} \left\|\sigma_u \times \sigma_v\right\|\text{d}u\text{d}v$$

By chain rule, we have that $Dn(p) \in \text{End}(T_p\Sigma)$, with $\sigma_u \mapsto n_u$ and $\sigma_v \mapsto n_v$. Since $\kappa(p) = \kappa(\sigma(0)) \neq 0$, $n \circ \sigma : V \to S^2$ defines an allowable parametrisation of an open neighbourhood of $n(p)$ by the inverse function theorem. Therefore, we have that

$$\text{Area}(n(A_i)) = \int_{V_i} \left\|n_u \times n_v\right\|\text{d}u\text{d}v$$

provided that V_i lies in the neighbourhood of 0 such that $n \circ \sigma$ is a diffeomorphism. But we have that

$$\begin{aligned}\int_{V_i} \left\|n_u \times n_v\right\|\text{d}u\text{d}v &= \int_{V_i} \left\|Dn(\sigma_u) \times Dn(\sigma_v)\right\|\text{d}u\text{d}v \\ &= \int_{V_i} |\det(Dn)|\left\|\sigma_u \times \sigma_v\right\|\text{d}u\text{d}v \\ &= \int_{V_i} |\kappa|\left\|\sigma_u \times \sigma_v\right\|\text{d}u\text{d}v\end{aligned}$$

Since κ is continuous, given $\varepsilon > 0$, we have $\delta > 0$ such that if $x \in D(0, \delta) \subseteq V$, then $|\kappa(\sigma(x)) - \kappa(p)| < \varepsilon$. Taking i large enough, this gives us the required result. □

Theorem 5.30 (Theorema egregium)**.** Suppose $f : \Sigma_1 \cong_{\text{d}} \Sigma_2$ is an isometry. Then

$$\kappa(p) = \kappa(f(p))$$

for all $p \in \Sigma_1$.

Theorem 5.31 (Gauss-Bonnet)**.** Suppose Σ is a compact surface in $\mathbb{R}^3$, then

$$\int_{\Sigma} \kappa(p)\text{d}A = 2\pi\chi(\Sigma)$$

6 Geodesics

Definition 6.1 (Energy)

For a smooth curve $\gamma : [a, b] \to \mathbb{R}^3$, the energy of γ is

$$E(\gamma) = \int_a^b \left\|\gamma'(t)\right\|^2\text{d}t$$

Definition 6.2 (One parameter variation)

Suppose $\gamma : [a, b] \to \mathbb{R}^3$ is a smooth curve, a one parameter variation $\Gamma : (-\varepsilon, \varepsilon) \times [a, b] \to \mathbb{R}^3$ of γ is a smooth map such that $\Gamma(0, \cdot) = \gamma$, $\Gamma(\cdot, a) = \gamma(a)$ and $\Gamma(\cdot, b) = \gamma(b)$.

We write $\gamma_s(t) = \Gamma(s, t)$.

Definition 6.3 (Geodesic)
A smooth curve $\gamma : [a, b] \to \mathbb{R}^3$ is a geodesic if for every variation γ_s, we have that

$$\frac{\mathrm{d}}{\mathrm{d}s}(E(\gamma_s))\Big|_{s=0} = 0$$

That is, γ is a critical point of the energy functional.

Theorem 6.4 (Geodesic equations). A smooth $\gamma : [a, b] \to U \underset{\text{open}}{\subseteq} \Sigma$, where we have a parametrisation $\sigma : V \to U$, is a geodesic if and only if it satisfies the geodesic equations.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) &= \frac{1}{2}E_u\dot{u}^2 + F_u\dot{u}\dot{v} + \frac{1}{2}G_u\dot{v}^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) &= \frac{1}{2}E_v\dot{u}^2 + F_v\dot{u}\dot{v} + \frac{1}{2}G_v\dot{v}^2 \end{aligned}$$

Proof. Suppose γ_s is a one parameter variation. For s small, we can write

$$\gamma_s(t) = \sigma(u(s, t), v(s, t))$$

Let $R = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$, where dot denotes derivative with respect to t. Then

$$E(\gamma_s) = \int_a^b R(s, t)\mathrm{d}t$$

By result from Analysis and Topology, we have that

$$\frac{\mathrm{d}E(\gamma_s)}{\mathrm{d}s} = \int_a^b \frac{\partial R}{\partial s}\mathrm{d}t$$

Computing the derivative,

$$\frac{\partial R}{\partial s} = \left(E_u\dot{u}^2 + F_u\dot{u}\dot{v} + G_u\dot{v}^2\right)\frac{\partial u}{\partial s} + \left(E_v\dot{u}^2 + F_v\dot{u}\dot{v} + G_v\dot{v}^2\right)\frac{\partial v}{\partial s} + 2(E\dot{u} + F\dot{v})\frac{\partial \dot{u}}{\partial s} + 2(F\dot{u} + G\dot{v})\frac{\partial \dot{v}}{\partial s}$$

Noting that $\frac{\partial u}{\partial s} = \frac{\partial v}{\partial s} = 0$ for $t = a, b$, since we have fixed end points and integrating by parts, we get that

$$\frac{\mathrm{d}}{\mathrm{d}s}(E(\gamma_s))\Big|_{s=0} = \int_a^b A\frac{\partial u}{\partial s} + B\frac{\partial v}{\partial s}\mathrm{d}t$$

where

$$\begin{aligned} A &= (E_u\dot{u}^2 + F_u\dot{u}\dot{v} + G_u\dot{v}^2) - 2\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) \\ B &= (E_v\dot{u}^2 + F_v\dot{u}\dot{v} + G_v\dot{v}^2) - 2\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) \end{aligned}$$

□

Remark 6.5. Note the above is just a special case of the Euler-Lagrange equations.

Proposition 6.6. If γ has constant speed, and locally minimises length, then γ is a geodesic.

Proof. The Cauchy-Schwarz inequality gives us that

$$\text{Length}(\gamma)^2 \leq E(\gamma)(b-a)$$

where equality holds if and only if $\sqrt{R} = \|\gamma'\|$ is constant. □

Corollary 6.7. If γ globally minimises energy, then it minimises length, and is parametrised at unit speed.

Proposition 6.8. Suppose $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface, $\gamma : (a, b) \to \Sigma$ is a geodesic if and only if $\ddot{\gamma}$ is everywhere normal to the surface Σ.

Proof. Since orthogonality and the geodesic equations are both local properties, we can with with a parametrisation $\sigma : V \to U$, with $\gamma(t) = \sigma(u(t), v(t))$. Then $\dot{\gamma} = \sigma_u \dot{u} + \sigma_v \dot{v}$, so $\ddot{\gamma}$ is normal to Σ if it is normal to $T_p\Sigma = \text{span}\{\sigma_u, \sigma_v\}$. This is true if and only if

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u \dot{u} + \sigma_v \dot{v}), \sigma_u \right\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u \dot{u} + \sigma_v \dot{v}), \sigma_v \right\rangle = 0$$

We only consider the first one, since the second one will be similar. Rearranging, it is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_u \rangle - \left\langle \sigma_u \dot{u} + \sigma_v \dot{v}, \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u) \right\rangle = 0$$

Using $E = \langle \sigma_u, \sigma_u \rangle$ etc, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) - \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_{uu}\dot{u} + \sigma_{uv}\dot{v} \rangle = 0$$

which means that

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) - \left(\dot{u}^2 \langle \sigma_u, \sigma_{uu} \rangle + \dot{u}\dot{v}\left(\langle \sigma_u, \sigma_{uv} \rangle + \langle \sigma_v, \sigma_{vu} \rangle\right) + \dot{v}^2 \langle \sigma_v, \sigma_{uv} \rangle\right) = 0$$

Computing E_u, F_u and G_u, we see that this is precisely the first geodesic equation. □

Lemma 6.9. Suppose Σ is a smooth surface in $\mathbb{R}^3$, $\Pi \subseteq \mathbb{R}^3$ plane, where $C = \Pi \cap \Sigma$ is a smooth embedded curve, and Σ is preserved under reflection across Π. Then when parametrised at constant speed, C is a geodesic.

Definition 6.10 (Meridian)

Suppose Σ is a surface of revolution, then a curve formed by the intersection of Σ and a plane through the z-axis is known as a meridian.

Definition 6.11 (Parallel)

Suppose Σ is a surface of revolution, then a curve formed by the intersection of Σ and a plane perpendicular to the the z-axis is known as a parallel.

Proposition 6.12. All meridians are geodesics.

Lemma 6.13. A parallel is a geodesic if and only if it is at a critical point of $f = r = \sqrt{x^2 + y^2}$.

Proposition 6.14 (Clairaut's relation)**.** Suppose γ is a geodesic, $\rho(t) = \sqrt{x^2 + y^2}$ and $\theta(t)$ is the angle between γ and the parallel passing through $\gamma(t)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho \cos(\theta)) = 0$$

Proof.

$$\cos(\theta) = \frac{\langle \sigma_v, \sigma_u \dot{u} + \sigma_v \dot{v} \rangle}{\|\sigma_v\| \|\sigma_u \dot{u} + \sigma_v \dot{v}\|}$$

Without loss of generality, γ is parametrised by arc length, so $\|\sigma_u \dot{u} + \sigma_v \dot{v}\| = 1$. Furthermore, $\|\sigma_v\| = \rho$. Then the second geodesic equation gives us the required result. ☐

6.1 Geodesic normal coordinates

Proposition 6.15. Suppose $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface, for $p \in \Sigma$, $v \in T_p\Sigma$, $v \neq 0$, we have $\varepsilon > 0$ and a geodesic $\gamma : [0, \varepsilon) \to \Sigma$ such that $\gamma(0) = 0$ and $\gamma'(0) = v$. Moreover, γ depends smoothly on (ρ, v).

Proof. The geodesic equations can be written as

$$f[u, v] = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix}$$

which is a smooth function in u, v, and $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is invertible, so we can write the geodesic equations as a system

$$\begin{aligned} \dot{u} &= p \\ \dot{v} &= q \\ \dot{p} &= A(u, v, p, q) \\ \dot{q} &= B(u, v, p, q) \end{aligned}$$

where A, B smooth. By the mean value inequality, a bound on $\|DA\|$ and $\|DB\|$ gives us the Lipschitz condition we require for Picard-Lindelöf. ☐

Definition 6.16 (Geodesic normal coordinates)

Fix $p > 0$, let γ be a geodesic starting at p parametrised by arc length. γ_t be the geodesic such that $\gamma_t(0) = \gamma(t)$, $\gamma_t'(0) \in T_p\Sigma$ orthogonal to $\gamma'(t)$ and γ_t parametrised by arc length.

Define $\sigma : [0, \varepsilon) \times [0, \delta)$ by $\sigma(u, v) = \gamma_v(u)$.

Proposition 6.17. For ε, δ sufficiently small, σ defines an allowable parametrisation of an open set of Σ when restricted to $\mathrm{Int}([0, \varepsilon) \times [0, \delta))$.

Proof. Smoothness follows by Picard-Lindelöf. At $(0, 0)$, by construction, σ_u, σ_v orthogonal so linearly independent. So by continuity on a small open set it defines a local diffeomorphism. ☐

Proposition 6.18.

$$G(0, v) = 1 \quad \text{and} \quad G_u(0, v) = 0$$

Corollary 6.19. Any smooth surface $\Sigma \subseteq \mathbb{R}^3$ has local parametrisations for which the first fundamental form has the form

$$\mathrm{d}u^2 + G(u, v)\mathrm{d}v^2$$

where $E = 1$, $F = 0$.

Proof. Compute the first fundamental form for the geodesic normal coordinates. □

Proposition 6.20. If the first fundamental form has the form

$$\mathrm{d}u^2 + G\mathrm{d}v^2$$

then the curvature is given by

$$\kappa = \frac{-(\sqrt{G})_{uu}}{\sqrt{G}}$$

Proposition 6.21. Suppose $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface, if $\kappa \equiv 0$ then Σ is locally isometric to the Euclidean plane.

Proof. Passing to geodesic normal coordinates, we have that

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left(\sqrt{G}\right) = 0$$

Solving the differential equations and using the boundary conditions for G we get that $G \equiv 1$. □

Proposition 6.22. If $\Sigma \subseteq \mathbb{R}^3$ is a smooth surface, and $\kappa \equiv 1$, then Σ is locally isometric to the sphere $(S^2, \mathrm{d}u^2 + \cos(u)\mathrm{d}v^2)$.

Proof. Passing to geodesic normal coordinates and solving the differential equation we have that $G \equiv \cos^2(u)$. □

7 Abstract smooth surface

Definition 7.1 ((Abstract) Riemannian metric)
Let $V \underset{\text{open}}{\subseteq} \mathbb{R}^2$, an (abstract) Riemannian metric on V is a smooth map

$$V \mapsto \{\text{positive definite symmetric bilinear forms}\} \subseteq \mathrm{Mat}_2(\mathbb{R})$$

In terms of matrices, we have that

$$v \mapsto \begin{pmatrix} E(v) & F(v) \\ F(v) & G(v) \end{pmatrix}$$

where $E(v), G(v) > 0$, $E(v)G(v) - F(v)^2 > 0$.

Definition 7.2 (Length)
If $\gamma = (u, v) : [a, b] \to V$ smooth, define

$$\text{Length}(\gamma) = \int_a^b \left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right)^{1/2} \mathrm{d}t$$

Definition 7.3 (Riemannian metric)
Suppose Σ is an abstract smooth surface, $(U_i, \phi_i)_{i\in I}$ smooth atlas for Σ. A Riemannian metric g or $\mathrm{d}s^2$ is a Riemannian metric on each V_i such that if $\sigma, \tilde{\sigma}$ parametrisations, $f = \tilde{\sigma}^{-1} \circ \sigma$ transition map, then we require

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (Df)^{\mathsf{T}} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} (Df)$$

That is Df defines an isometry of the open sets V and $\tilde{V}$.

Definition 7.4 (Isometry)
Suppose (Σ_1, g_1) and (Σ_2, g_2) are abstract smooth surfaces with abstract Riemannian metrics, then a diffeomorphism

$$f : \Sigma_1 \cong_{\mathrm{d}} \Sigma_2$$

is an isometry if it preserves the lengths of all curves.

Definition 7.5 (Length metric)
Given a Riemannian metric g on a connected abstract smooth surface Σ, define the length metric

$$d(p, q) = \inf_{\gamma:p\to q \text{ piecewise smooth}} \text{Length}(\gamma)$$

Proposition 7.6. The length metric defines a metric.

Proof. Σ is path connected, and piecewise smoothness follows by compactness. □

Proposition 7.7. The length metric defines the same topology as the one from the topological surface structure.

8 Hyperbolic geometry

8.1 Models of hyperbolic geometry

Definition 8.1 (Disc model)
Let $D = D(0, 1) \subseteq \mathbb{C}$, define the hyperbolic metric by

$$g = \frac{4(\mathrm{d}u^2 + \mathrm{d}v^2)}{(1 - u^2 - v^2)^2} = \frac{4|\mathrm{d}z|^2}{(1 - |z|^2)^2}$$

Proposition 8.2. The subgroup of the Möbius group that preserves the unit disc D is

$$\mathcal{M}(D) = \{T \in \mathcal{M} : T(D) = D\} = \left\{z \mapsto e^{i\theta} \frac{z - a}{1 - \overline{a}z} : |a| < 1\right\}$$

Lemma 8.3. The hyperbolic metric g is invariant under $\mathcal{M}(D)$. That is, $\mathcal{M}(D)$ acts by isometries.

Proof. Suffices to consider the generators. $z \mapsto e^{i\theta} z$ is a rotation and preserves the metric. For the second type, let $w = \dfrac{z - a}{1 - \overline{a}z}$, and by computing we find that

$$\frac{|\mathrm{d}w|^2}{(1 - |w|^2)^2} = \frac{|\mathrm{d}z|^2}{(1 - |z|^2)^2}$$

□

Lemma 8.4. Every pair of points in (D, g) is joined by a unique geodesic.

Proof. By a Möbius map, we can consider one point being the origin. Then by computing, we find that the diameter of the circle is the unique geodesic. □

Lemma 8.5. Geodesics in (D, g) are diameters of the disc, and circular arcs perpendicular to ∂D.

Proof. By above, diameters are geodesics, and as Möbius maps are conformal, and sending circles/lines to circles/lines, we get the required result. □

Corollary 8.6. If $p, q \in D$, then

$$d(p, q) = 2\,\mathrm{artanh}\left(\left|\frac{p - q}{1 - \overline{p}q}\right|\right)$$

Definition 8.7 (Half plane model)
Let $\mathcal{H}$ be the (open) upper half plane, define the metric

$$g = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} = \frac{|\mathrm{d}z|^2}{\mathrm{Im}(z)^2}$$

Lemma 8.8. (D, g) and $(\mathcal{H}, g)$ are isometric.

Proof. The Möbius map

$$T(w) = \frac{w - i}{w + i}$$

defines an isometry $D \to \mathcal{H}$. □

Corollary 8.9. In $(\mathcal{H}, g)$ every pair of points are joined by a unique geodesic, where the geodesics are vertical lines or arcs of circles with centres on the real axis.

8.2 Inversion

Definition 8.10 (Inverse points)
Let $\Gamma \subseteq \hat{\mathbb{C}}$ be a line or circle. $z, z' \in \hat{\mathbb{C}}$ are inverse points if every line/circle orthogonal to Γ, passing through z also passes through z'.

Lemma 8.11. For every circle $\Gamma \subseteq \hat{C}$, $z \in \mathbb{C}$, there is a unique inverse point z' for z.

Proof. Since Möbius maps are conformal and preserves circles, without loss of generality we may assume that $\Gamma = \mathbb{R} \cup \{\infty\}$. Then $J(z) = \overline{z}$ works. □

Proposition 8.12. If $\Gamma = S^1 = \{z : |z| = 1\}$, then inversion is given by $z \mapsto 1/z$.

Proposition 8.13. A composition of two inversions is a Möbius map.

Lemma 8.14. An orientation preserving element of $(\mathbb{H}^2, g)$ is an element of $\mathcal{M}(\mathbb{H}^2)$.

Definition 8.15 (Elliptic, parabolic, hyperbolic)
Suppose $\alpha \in \mathcal{M}(\mathbb{H}^2)$ is a non identity element, then

- α is elliptic if α fixes $p \in \mathbb{H}$.
- α is parabolic if α fixes a unique $p \in \partial\mathbb{H}$.
- α is hyperbolic if α fixes two points on $\partial\mathbb{H}$.

8.3 Geometry

Definition 8.16 (Hyperbolic line)
Geodesics in the hyperbolic plane are called hyperbolic lines.

Definition 8.17 (Parallel, ultraparallel, intersecting)
Suppose l_1, l_2 are lines in $\mathbb{H}$, then

- l_1 and l_2 are parallel if they meet at $\partial\mathbb{H}$ but not in $\mathbb{H}$.
- l_1 and l_2 are ultraparallel if they do not meet in $\mathbb{H} \cup \partial\mathbb{H}$.
- l_1 and l_2 are intersecting if they meet in $\mathbb{H}$.

Definition 8.18 (Hyperbolic triangle)
A hyperbolic triangle is the region bound by three hyperbolic lines, no two of which are ultraparallel.

Definition 8.19 (Ideal vertices)
For a hyperbolic polygon, vertices on ∂H are ideal vertices.

Proposition 8.20 (Hyperbolic cosh formula). Suppose we have a hyperbolic triangle with side lengths A, B, C, and opposite angles α, β, γ,

$$\cosh(C) = \cosh(A)\cosh(B) - \sinh(A)\sinh(B)\sin(\gamma)$$

Proposition 8.21 (Area of triangle). The area of a hyperbolic triangle with angles α, β, γ is $\pi - \alpha - \beta - \gamma$.

Lemma 8.22. Suppose $n \geq 3$, then we have a regular n-gon with interior angle $\frac{2\pi}{n}$.

Proof. Consider an ideal n-gon with has interior angle 0, and by intermediate value theorem we have such an n-gon. □

Theorem 8.23. For $g \geq 2$, there is an abstract Riemannian metric on the compac surface of genus g with curvature -1.

Proof. Consider a $4g$-gon with gluing pattern $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$. □

Lemma 8.24. For $l_\alpha, l_\beta, l_\gamma > 0$, there exists a right angle hyperbolic hexagon with side lengths (in order) $l_\alpha, \bullet, l_\beta, \bullet, l_\gamma, \bullet$.

Proof. For all $t > 0$, there exists (up to isometry) a unique configuration of ultraparallel lines, with a unique perpendicular and length $t > 0$. Taking $t \to \infty$ we can get the required result. □

8.4 Decompositions

Definition 8.25 (Pair of pants)
A pair of pants is a topological space homeomorphic to S^2 with three open discs removed.

Proposition 8.26. For $l_\alpha, l_\beta, l_\gamma > 0$, there exists a hyperbolic pair of pants with boundary lengths $l_\alpha, l_\beta, l_\gamma$.

Proof. Glue two copies of the same hyperbolic right angle hexagon together. □

Proposition 8.27. Any compact surface of genus $g \geq 2$ can be made by gluing pairs of pants.

Theorem 8.28 (Local Gauss-Bonnet). Let Σ be an abstract smooth surface with Riemannian metric g. Let R be a geodesic polygon in Σ. Then

$$\int_R \kappa \mathrm{d}A = \sum \alpha_i - (n-2)\pi$$

where the α_i are the interior angles, and n is the number of sides.

Theorem 8.29 (Global Gauss-Bonnet). Let Σ be a compact smooth surface with Riemannian metric g. Then

$$\int_\Sigma \kappa \mathrm{d}A = 2\pi\chi(\Sigma)$$

Lemma 8.30. A compact smooth surface has a subdivision into geodesic polygons.

Corollary 8.31. Local Gauss-Bonnet implies global Gauss-Bonnet.

9 Moduli

Recall we have a flat metric on T^2 by quotienting $[0,1]^2$. But this choice is not unique. For any parallelogram, we can glue it into a torus. Furthermore, in general these are not isometric.

Considering the set of all parallelograms, quotienting out by dilations and isometries of $\mathbb{R}^2$, we have a parallelogram with vertices $(0,0), (1,0), (x,y), (1+x,y)$, where $y > 0$. This then gives us a map

$$\mathcal{H} \to \frac{\text{Flat metrics on } T^2}{\text{Dilations}}$$

Lemma 9.1. $\mathrm{SL}_2(\mathbb{Z})$ acts on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by isometries.

Theorem 9.2. The map

$$\mathcal{H} \to \frac{\text{Flat metrics on } T^2}{\text{Dilations}}$$

defines a bijection map

$$\frac{\mathcal{H}}{\mathrm{SL}_2(\mathbb{Z})} \equiv \frac{\text{Flat metrics on } T^2}{\text{Dilations, Diffeomorphisms}^+}$$

where Diffeomorphisms$^+$ is the group of orientation preserving diffeomorphisms. $\dfrac{\mathcal{H}}{\mathrm{SL}_2(\mathbb{Z})}$ is called the moduli space of flat metrics on T^2.