Inverse function theorem

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In this document, we go through the proof of the inverse function theorem as an example application of the contraction mapping theorem.

Theorem (Inverse function theorem)**.** Let $f : U \to \mathbb{R}^n$ be C^1, where $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. Suppose we have that $a \in U$ such that $f'(a)$ is invertible.

Then we have neighbourhoods V, W of $a, f(a)$ respectively such that $f|_V : V \to W$ is a bijection with C^1 inverse.

Step 1: Simplification

First of all, we will show that without loss of generality, we may assume that $a = 0, f(a) = 0$ and $f'(a) = \mathrm{id}$. To do this, we want to define a function h such that for a small change $a + x$, $h(x)$ is the local linear approximation to x, as this would be zero for $x = 0$, and would have derivative id. Formally, let $T = Df(a)$, and $h(x) = T^{-1}(f(a + x) - f(a))$. By the chain rule, we have that

$$h'(x) = T^{-1} \circ Df(a + x)$$

which is the composition of continuous functions, so continuous. In addition, we have that

$$h(0) = 0 \quad \text{and} \quad h'(0) = \mathrm{id}$$

Since $f(a + x) = T(h(x)) + f(a)$, suffices to prove the result for h.

Step 2: Continuity

By the continuity of Df, and U being open, we have $r > 0$ such that

$$\overline{D}(0, r) \subseteq U \quad \text{and} \quad \left\|Df(x) - \mathrm{id}\right\| \leq \frac{1}{2} \text{ for all } x \in \overline{D}(0, r)$$

Note that here we take the closed ball, as we will be applying the contraction mapping theorem later, which requires a *complete* metric space.

Step 3: Contraction mapping theorem

Let $W = D(0, r/2)$. We wish to show that for all $w \in W$, the equation $f(x) = w$ has a solution, with $\|x\| < r$. To do this, we will use the contraction mapping theorem. Fix $w \in W$, and let $q : \overline{D}(0, r) \to \overline{D}(0, r)$ be defined by

$$q(x) = (w - f(x)) + x = w - (f(x) - x)$$

Then x is a fixed point of q if and only if $f(x) = w$. First, we must show that $q(x) \in \overline{D}(0, r)$. To do this, note that

$$\left\|q(x)\right\| \leq \|w\| + \left\|p(x) - p(0)\right\|$$

where $p(x) = f(x) - x$. Here, we have that $Dp(x) = Df(x) - \mathrm{id}$, so $\left\|Dp(x)\right\| \leq 1/2$ for all $x \in \overline{D}(0, r)$. The mean value inequality then gives us that

$$\|p(x) - p(0)\| \leq \frac{1}{2}\|x - 0\| \leq \frac{r}{2}$$

which gives us that $\|q(x)\| < r$ as required. Also by the mean value inequality, we have that

$$\|q(x) - q(y)\| = \|p(x) - p(y)\| \leq \frac{1}{2}\|x - y\|$$

So q is a contraction map, which has a unique fixed point z. Finally, we note that $\|z\| = \|q(z)\| < r$, so in fact $z \in D(0, r)$. Since w was arbitrary, we must have that

$$D(0, r/2) \subseteq f(D(0, r))$$

Step 4: Constructing the inverse

Let $V = D(0, r) \cap f^{-1}(W)$. By the previous step, we have that $f|_V : V \to W$ is surjective as a fixed point exists, and injective as the fixed point is unique. So $f|_V$ is a bijection. Let $g : W \to V$ be the inverse. Showing g is C^1 is beyond the scope of the course, however we will show that g is continuous. In fact, we will show that g is Lipschitz.

From the mean value inequality, we have that

$$\|f(x) - f(y)\| = \|(p(x) + x) - (p(y) + y)\| \geq \|x - y\| + \|p(x) = p(y)\| \geq \frac{1}{2}\|x - y\|$$

Setting $x = g(w)$ and $y = g(z)$, we get that g is 2-Lipschitz.