Algebraic topology

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May 30, 2023

Contents

1 Fundamental group

1.1 Homotopy

Definition 1.1 (homotopy)
Suppose $f_0, f_1 : X \to Y$ are maps, we say that f_0 and f_1 are homotopic if there exists $H : X \times I \to Y$ such that

$$H(\cdot, 0) = f_0 \quad \text{and} \quad H(\cdot, 1) = f_1$$

We write $f_0 \sim f_1$, or $f_0 \overset{H}{\sim} f_1$

Lemma 1.2. Homotopy is an equivalence relation.

Proof. Reflexivity and symmetry are clear, for transitivity use the gluing lemma. □

Lemma 1.3. If $f_0 \sim f_1$ and $g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$.

Notation 1.4. We write $c_{X,p} : X \to Y$ for the constant map $c(x) = p$.

Definition 1.5 (contractible)
A topological space X is contractible if $\mathrm{id}_X \sim c_{X,p}$ for some $p \in X$.

Proposition 1.6. If Y is contractible, then any $f_0, f_1 : X \to Y$ are homotopic.

Proposition 1.7. Any contractible space is path connected.

Definition 1.8 (homotopy equivalent)
Two topological spaces X, Y are homotopy equivalent if there exists maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. We write $X \sim Y$.

Proposition 1.9. Homotopy equivalence is an equivalence relation.

Proposition 1.10. A topological space X is contractible if and only if $X \sim *$.

Definition 1.11 (homotopy rel)
Suppose $f_0, f_1 : X \to Y$ are maps, $A \subseteq X$. Then we say that f_0 and f_1 are homotopy equivalent rel A if $f_0 \overset{H}{\sim} f_1$ and $H(x, t) = f_0 = f_1$ for all $t \in I, x \in A$.

Lemma 1.12. Homotopy rel is an equivalence relation.

Lemma 1.13. Suppose $f_0 \sim f_1$ rel A, $g_0 \sim g_1$ rel $f(A)$, then $g_0 \circ f_0 \sim g_1 \circ f_1$ rel A.

1.2 Paths and the fundamental group

Notation 1.14 (homotopy rel end points). If $\gamma_0, \gamma_1 : I \to X$ are paths, we write $\gamma_0 \sim_e \gamma_1$ for $\gamma_0 \sim \gamma_1$ rel $\{0, 1\}$.

Lemma 1.15. If $f_0, f_1 : I \to I$, $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$, then $f_0 \sim_e f_1$.

Proof. I is convex. □

Proposition 1.16. Suppose $f : I \to I$, $\gamma : I \to X$ path, then

(i) If $f(0) = 0$ and $f(1) = 1$, then $f \circ \gamma \sim_e \gamma$.

(ii) If $f(0) = f(1) = 0$, then $\gamma \circ f \sim c_{I,\gamma(0)}$.

Proof. Immediate from the lemma. □

Definition 1.17

Let X be a space, $p, q \in X$, then write $\Omega(X, p, q)$ be the set of paths from p to q. Write $\Omega(X, p) := \Omega(X, p, p)$ for the set of loops based at p.

Notation 1.18. Suppose $\gamma \in \Omega(X, p, q)$ and $\eta \in \Omega(X, q, r)$, we can compose them to get $\gamma\eta \in \Omega(X, p, r)$. Furthermore, let $\gamma^{-1} \in \Omega(X, q, p)$ be the reverse of γ.

Lemma 1.19. Suppose $\gamma_0, \gamma_1 \in \Omega(X, p, q)$ with $\gamma_0 \sim_e \gamma_1$, and $\eta_0, \eta_1 \in \Omega(X, q, r)$ with $\eta_0 \sim_e \eta_1$, then $\gamma_0\eta_0 \sim_e \gamma_1\eta_1$.

Proposition 1.20. Let $\gamma \in \Omega(X, p, q)$, $\gamma' \in \Omega(X, q, r)$ and $\gamma'' \in \Omega(X, r, s)$, then

(i) $c_{I,p}\gamma \sim_e \gamma \sim_e \gamma c_{I,q}$,

(ii) $\gamma^{-1}\gamma \sim_e c_{I,q}$ and $\gamma\gamma^{-1} \sim_e c_{I,p}$,

(iii) $(\gamma\gamma')\gamma'' \sim_e \gamma(\gamma'\gamma'')$

Definition 1.21 (fundamental group)

For a topological space X, $x_0 \in X$, define

$$\pi_1(X, x_0) = \frac{\Omega(X, x_0)}{\sim_e}$$

This is a group with

(i) $[\gamma_0][\gamma_1] = [\gamma_0\gamma_1]$

(ii) $\mathrm{id} = 1 = [c_{I,x_0}]$

(iii) $[\gamma]^{-1} = [\gamma^{-1}]$

Proposition 1.22 (functoriality). π_1 defines a functor from pointed spaces to groups.

Proof. We only need to check the map on morphisms. Suppose $f : (X, x_0) \to (Y, y_0)$ is a map, then we have an induced map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, defined by $f_*([\gamma]) = [f \circ \gamma]$. It is easy to check that $(g \circ f)_* = g_* \circ f_*$ and $\mathrm{id}_* = \mathrm{id}$. □

Proposition 1.23 (homotopy invariance). If $f_0 \sim f_1$ rel $\{x_0\}$, then $f_{0*} = f_{1*}$.

Definition 1.24 (retraction, strong deformation retraction)
Suppose $A \subseteq X$, $\iota : A \to X$ is the inclusion map. Then

(i) $r : X \to A$ is a retraction if $r \circ \iota = \mathrm{id}_A$.

(ii) $r : X \to A$ is a strong deformation retraction if r is a retraction, and $\iota \circ r \sim \mathrm{id}_X$ rel A.

Proposition 1.25. If r is a retraction, then ι_* is injective and r_* is surjective.

Proposition 1.26. A strong deformation retraction defines a homotopy equivalence.

1.3 Nullhomotopy and extensions

Definition 1.27 (nullhomotopic)
A map $f : X \to Y$ is nullhomotopic if $f \sim c_{X,p}$ for some $p \in Y$.

Proposition 1.28. Let $f : S^1 \to Y$, then f extends to all of D^2 if and only if f is nullhomotopic.

Proof. One direction is clear. For the other use the homotopy to define an extension radially. □

Definition 1.29 (closed loop)
Let $\gamma \in \Omega(X, x_0)$. Define $\overline{\gamma} : S^1 \to X$ by

$$\overline{\gamma}(e^{2\pi i t}) = \gamma(t)$$

Lemma 1.30.

1. If $\gamma_0 \sim_e \gamma_1$, then $\overline{\gamma}_0 \sim \overline{\gamma}_1$.

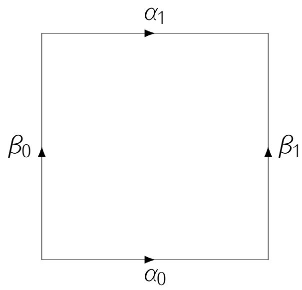
2. $\overline{\gamma\gamma'} \sim \overline{\gamma'\gamma}$.

Proof. (i) is clear. For (ii), notice that the antipodal map $S^1 \to S^1$ is homotopic to the identity. ☐

Let $\Phi : D^2 \to I^2$ be a homeomorphism, then $h : \partial(I^2) \to X$ extends to I^2 if and only if $h \circ \Phi$ extends to D^2. But we have seen this holds if and only if $h \circ \Phi$ is nullhomotopic.

For $i \in \{0, 1\}$, define

$$\alpha_i(t) = h(t, i) \quad \text{and} \quad \beta_i(t) = h(i, t)$$



Then $h \circ \Phi = \overline{\alpha_0 \beta_1 \alpha_1^{-1} \beta_1^{-1}}$.

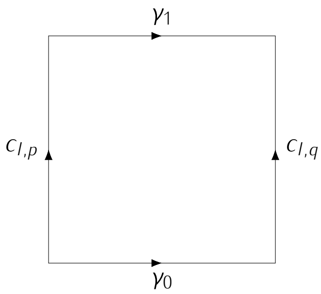
Proposition 1.31. Suppose $\gamma_0, \gamma_1 \in \Omega(X, p, q)$ are paths, then the following are equivalent.

(i) $\gamma_0 \sim_e \gamma_1$,

(ii) $\overline{\gamma_0 \gamma_1^{-1}}$ is nullhomotopic,

(iii) $[\gamma_0 \gamma_1^{-1}] = 1$ in $\pi_1(X, p)$.

Proof. Consider $h : \partial(I^2) \to X$ given by

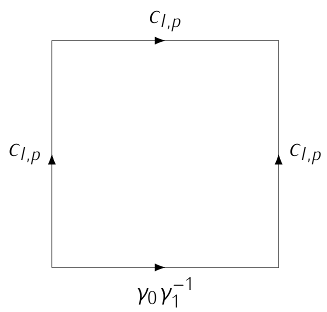


Then we have that

$$\begin{aligned} \gamma_0 \sim_e \gamma_1 &\iff h \text{ extends to } I^2 \\ &\iff h \circ \Phi \text{ extends to } D^2 \\ &\iff \overline{\gamma_0 c_{I,q} \gamma_1^{-1} c_{I,p}^{-1}} \text{ is nullhomotopic} \\ &\iff \overline{\gamma_0 \gamma_1^{-1}} \text{ is nullhomotopic} \end{aligned}$$

So (i) $\iff$ (ii).

Now consider $h' : \partial(I^2) \to X$ given by



Then we have that

$$\begin{aligned}
[\gamma_0\gamma_1^{-1}] = 1 &\iff \gamma_0\gamma_1^{-1} \sim_e c_{I,p} \\
&\iff h' \text{ extends to } I^2 \\
&\iff h' \circ \Phi \text{ extends to } D^2 \\
&\iff \overline{\gamma_0\gamma_1^{-1} c_{I,p} - 1 c_{I,p}^{-1} c_{I,p}^{-1}} \text{ is nullhomotopic} \\
&\iff \overline{\gamma_0\gamma_1^{-1}} \text{ is nullhomotopic}
\end{aligned}$$

So (ii) $\iff$ (iii). ☐

Corollary 1.32. The following are equivalent.

(i) $\gamma_0 \sim_e \gamma_1$, for all $\gamma_0, \gamma_1 \in \Omega(X, p, q)$,

(ii) any $f : S^1 \to X$ is nullhomotopic,

(iii) $\pi_1(X, x_0) = 1$

Definition 1.33 (simply connected)
We say that X is simply connected if X is path connected and any of the above conditions hold.

1.4 Change of base point

Definition 1.34 (change of base point map)
Given $u \in \Omega(X, x_0, x)$, we can define a map $u_\# : \Omega(X, x_0) \to \Omega(X, x)$ by

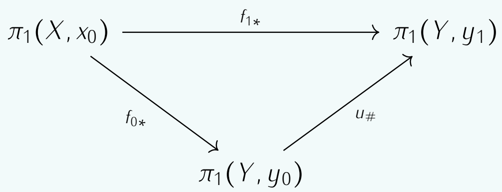
$$u_\#(\gamma) = u^{-1}\gamma u$$

Proposition 1.35. If $\gamma \sim_e \gamma'$, then $u_\#(\gamma) \sim u_\#(\gamma')$, so $u_\#$ gives a map $u_\# : \pi_1(X, x_0) \to \pi_1(X, x)$.

Proposition 1.36. $u_\#$ is an isomorphism of groups.

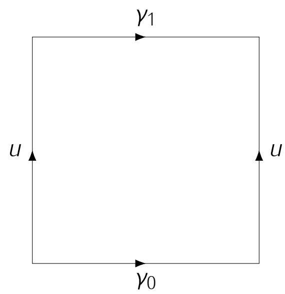
Proof. $u_\#$ is a homomorphism is as for conjugation for groups, and note that $u_\#$ is the inverse of $(u^{-1})_\#$. ☐

Theorem 1.37. Suppose $f_0, f_1 : X \to Y$, $f_0 \sim f_1$ via $H : X \times I \to Y$. Then let $u(t) = H(x_0, t)$, $y_0 = f_0(x_0)$, $y_1 = f_1(x_1)$, then



commutes.

Proof. We need to show that for all $\gamma \in \Omega(X, x_0)$, $f_{1*}[\gamma] = u_\# f_{0*}[\gamma] = u_\#[f_0 \circ \gamma]$. Let $\gamma_i = f_i \circ \gamma$, then we want to show $\gamma_1 \sim_e u^{-1}\gamma_0 u$. Consider $h : \partial(I^2) \to Y$ given by



Then

$$\begin{aligned} h \text{ extends to } I^2 &\iff \overline{\gamma_0 u \gamma_1^{-1} u^{-1}} \text{ is nullhomotopic} \\ &\iff \overline{u^{-1}\gamma_0 u \gamma_1^{-1}} \text{ is nullhomotopic} \\ &\iff u^{-1}\gamma_0 u \sim_e \gamma_1. \end{aligned}$$

But h does extend to all of I^2, given by $\tilde{H}(x, t) = H(\gamma(x), t)$. ☐

Proposition 1.38. Suppose X and Y are homotopy equivalent, via $f : X \to Y$ and $g : Y \to X$. Then f_* and g_* are isomorphisms.

Proof. By symmetry we only need to show g_* is an isomorphism. Fix $x_0 \in X$, and let $y_0 = f(x_0), x_1 = g(y_0), y_1 = f(x_1)$. Then we have maps

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{f_*} \pi_1(Y, y_1)$$

Say id $\sim g \circ f$ via H, then $g_* \circ f_* = u_\# \circ (\mathrm{id}_X)_*$ where $u = H(x_0, t)$ is a path from x_0 to x_1 by the previous lemma. But $u_\#$ is an isomorphism, so g_* is surjective. Similarly $f \circ g \sim$ id implies that g_* is injective. ☐

Corollary 1.39. If X is contractible, then $\pi_1(X, x_0) = 1$.

2 Covering spaces

2.1 Definitions and lifting

Definition 2.1 (evenly covered set)

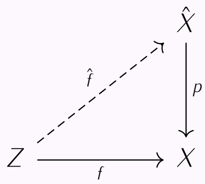
Suppose $p : \hat{X} \to X$ is continuous, we say that $U \subseteq X$ is evenly covered if $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$, where $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

Definition 2.2 (covering map)

A map $p : \hat{X} \to X$ is a covering map if for all $x \in X$, there exists an open neighbourhood U_x which is evenly covered. In this case, we call $\hat{X}$ a covering space for X.

Definition 2.3 (lift)

Suppose $p : \hat{X} \to X$ is a covering map, $f : Z \to X$ continuous. Then we say that $\hat{f} : Z \to \hat{X}$ is a lift of f if $p \circ \hat{f} = f$, that is,



commutes.

Lemma 2.4 (Lebesgue covering). Suppose X is a compact metric space, $\{U_\alpha\}_\alpha$ is an open cover of X. Then there exists $\delta > 0$ such that for all $x \in X$, $B_\alpha(x) \subseteq U_\alpha$ for some α.

Proof. Given $x \in X$, let $\alpha(x)$ and $\delta(x) > 0$ be such that $B_{2\delta(x)}(x) \subseteq U_{\alpha(x)}$. Then $\left\{B_{\delta(x)}\right\}_{x \in X}$ is an open cover of X. Therefore, by compactness there exists a finite subcover $\left\{B_{\delta(x_i)}(x_i)\right\}_{i=1}^n$. Let $\delta = \min\{\delta(x_1), \dots, \delta(x_n)\}$. Then for all $y \in X$, $y \in B_{\delta(x_i)}(x_i)$ for all i. Then

$$B_\delta(y) \subseteq B_{2\delta(x_i)}(x_i) \subseteq U_{\alpha(x_i)}$$

□

Notation 2.5. We say a path γ with $\gamma(0) = x_0$ has the (unique) lifting property if for all $\hat{x}_0 \in p^{-1}(x_0)$, there exists a (unique) lift $\hat{\gamma}$ of γ with $\hat{\gamma}(0) = \hat{x}_0$.

Lemma 2.6. If $f : Z \to U$, Z connected, $\text{im}(f) \subseteq U$, where U is evenly covered, then γ has the unique lifting property.

Proof. Since U is evenly covered, $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$. Then $\hat{x} \in V_{\alpha_0}$ for some α_0. Then $p' = (p|_{V_{\alpha_0}})^{-1} : U \to \hat{X}$ is continuous, with $p'(x_0) = \hat{x}_0$, so $\hat{f} = p' \circ f$ is a lift of f.

For uniqueness, notice that $p^{-1}(U) = U_{\alpha_0} \sqcup (\bigsqcup_{\alpha \neq \alpha_0} V_\alpha)$, which disconnects $p^{-1}(U)$, and as $\text{im}(f)$ is connected, $\text{im}(\hat{f}) \subseteq V_{\alpha_0}$. But p' above is a homeomorphism, so $\hat{\gamma}$ is unique. □

Lemma 2.7. Suppose $\gamma : [a, b] \to X$ with $a' \in [a, b]$, if $\gamma|_{[a,a']}$ has the ULP at a and $\gamma|_{[a',b]}$ has the ULP at a', then γ has the ULP at a.

Proof. We have a lift $\hat{\gamma}_1 : [a, a'] \to \hat{X}$ of $\gamma|_{[a,a']}$ at a, and a lift $\hat{\gamma}_2 : [a', b] \to \hat{X}$ of $\gamma|_{[a',b]}$ at a', such that $\hat{\gamma}_1(a') = \hat{\gamma}_2(a')$. So $\hat{\gamma}_1 \hat{\gamma}_2$ is a lift of γ at a.

For uniqueness, suppose $\hat{\eta}$ is any other lift. Then $\hat{\eta}|_{[a,a']}$ is a lift of $\gamma|_{[a',a]}$, so $\hat{\eta}|_{[a,a']} = \hat{\gamma}_1$. This means that $\hat{\eta}(a') = \hat{\gamma}(a')$, so $\hat{\eta}|_{[a',a]}$ is a lift of $\gamma|_{[a',b]}$, which means that $\hat{\eta}|_{[a',b]} = \hat{\eta}|_{[a',b]}$, so $\hat{\eta} = \hat{\gamma}_1 \hat{\gamma}_2$. ☐

Theorem 2.8 (path lifting)**.** Any $\gamma : I \to X$ has the ULP.

Proof. $p : \hat{X} \to X$ is a covering map, so every $x \in X$ has an evenly covered neighbourhood U_x. Then $\{U_x \mid x \in X\}$, so $\left\{\gamma^{-1}(U_x) \mid x \in X\right\}$ is an open cover of I. Thus, by the Lebesgue covering lemma, there exists $\delta > 0$ such that $B_\delta(t) \subseteq \gamma^{-1}(U_{x(t)})$ for any t.

Choose n such that $1/n < \delta$, $a_i = i/n \in I$. Then $[a_i, a_{i+1}] \subseteq B_\delta(a_i)$, so $\gamma([a_i, a_{i+1}]) \subseteq U_{x_i}$, where $a_i = \gamma(a_i)$. As U_{x_i} is evenly covered, $\gamma|_{[a_i,a_{i+1}]}$ has the ULP at a_i. By induction and the previous lemma, γ has the ULP. ☐

Theorem 2.9 (homotopy lifting)**.** Suppose $p : \hat{X} \to X$ is a covering map, $H : I \times I \to X$ is a homotopy, then H has the lifting property at $(0, 0)$.

Proof. Suppose $\{U_x \mid x \in X\}$ is an open cover of X by evenly covered neighbourhoods. Since I^2 is compact, by the Lebesgue covering lemma, there exists $\delta > 0$ such that $B_\delta(v) \subseteq H^{-1}(U_{H(v)})$ for each $v \in I^2$.

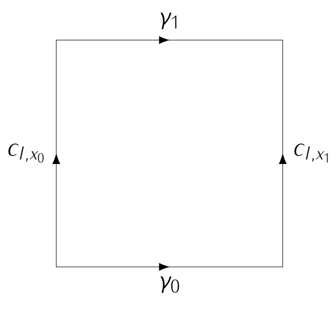
Choose n such that $\sqrt{2}/n < \delta$. Then divide I^2 into squares with side lengths $1/n$. Enumerate them $A_1, A_2, \ldots, A_{n^2}$, starting from the bottom left and going right then up. Label the bottom left corner of A_i as v_i. Now note that $H(A_i) \subseteq H(B_\delta(v_i)) \subseteq U_{x_i}$ is evenly covered. Thus, H_{A_i} has the ULP at v_i, as I^2 is connected. Let $B_k = \bigcup_{i=1}^k A_i$.

We will prove by induction that $H|_{B_k}$ has LP at $(0, 0)$. For $k = 1$,$B_1 = A_1$, so we are done. Now suppose $H|_{B_k}$ has a lift $\hat{H} : B_k \to X$ with $\hat{H}_k(0, 0) = \hat{x}$. Now as $H|_{A_k}$ has the lifting property at v_{k+1}. So choose a lift $\hat{h}_k : A_{k+1} \to \hat{X}$ with $\hat{h}_k(v_{k+1}) = \hat{H}^k(v_{k+1})$.

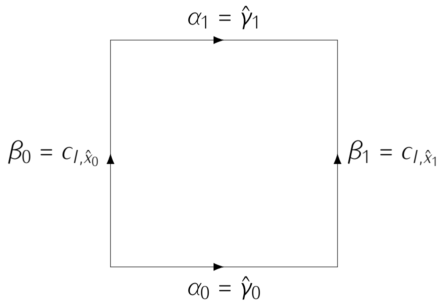
Now note that $B_k \cap A_{k+1}$ is either one or two edges of A_{k+1}, both coming from v_{k+1}. By uniqueness of path lifting, $\hat{H}_k|_{A_{k+1} \cap B_k} = \hat{h}_k|_{A_{k+1} \cap B_k}$, so by the gluing lemma we have a well defined lift $\hat{H}_{k+1}$ of H on B_{k+1}. ☐

Proposition 2.10. Suppose $\gamma_0, \gamma_1 \in \Omega(X, x_0, x_1)$, $\gamma_0 \sim_e \gamma_1$. Suppose $\hat{\gamma}_i$ is a lift of $\hat{X}$ with $\hat{\gamma}_i(0) = \hat{x}_0$. Then $\hat{\gamma}_0 \sim_e \hat{\gamma}_1$. In particular, $\hat{\gamma}_0(1) = \hat{\gamma}_1(1)$.

Proof. Suppose $H : I^2 \to X$ is a homotopy between γ_0 and γ_1.



By homotopy lifting, we have a lift $\hat{H} : I^2 \to \hat{X}$ with $\hat{H}(0, 0) = \hat{x}_0$. Let $\alpha_i(t) = \hat{H}(t, i)$ and $\beta_i(t) = \hat{H}(i, t)$. By uniqueness of path lifting.



That is, $\hat{\gamma}_0 \sim_e \hat{\gamma}_1$ via $\hat{H}$. □

Corollary 2.11. $p_* : \pi_1(\hat{X}, \hat{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof.

$$\begin{aligned} p_*[\gamma_0] = p_*[\gamma_1] &\implies p \circ \gamma_0 \sim_e p \circ \gamma_1 \\ &\implies \widehat{p \circ \gamma_0} \sim_e \widehat{p \circ \gamma_1} \\ &\implies \gamma_0 \sim_e \gamma_1 \end{aligned}$$

□

2.2 Universal covers

Definition 2.12 (universal cover)
A covering $p : \hat{X} \to X$ is called a universal cover of X if $\hat{X}$ is simply connected.

Suppose $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ is a covering map, then we have a map $\varepsilon_p : \Omega(X, x_0) \to p^{-1}(x_0)$, given by

$$\varepsilon_p(\gamma) = \hat{\gamma}(1)$$

where $\hat{\gamma}$ is the lift of γ at $\hat{x}_0$. Furthermore, if $\gamma_0 \sim_e \gamma_1$, then $\varepsilon_p(\gamma_0) = \varepsilon_p(\gamma_1)$, so we have a map $\varepsilon_p : \pi_1(X, x_0) \to p^{-1}(x_0)$.

Proposition 2.13. If $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ is a universal cover, then $\varepsilon_p : \pi_1(X, x_0) \to p^{-1}(x_0)$ is a bijection.

Proof. Suppose $\varepsilon_p[\gamma_0] = \varepsilon_p[\gamma_1] = \hat{x}_1$. Then $\hat{\gamma}_0, \hat{\gamma}_1 \in \Omega(\hat{X}, \hat{x}_0, \hat{x}_1)$, and as $\hat{X}$ is simply connected, $\hat{\gamma}_0 \sim_e \hat{\gamma}_1$. So $\gamma_0 = p \circ \hat{\gamma}_0 \sim_e p \circ \hat{\gamma}_1 = \gamma_1$. That is, $[\gamma_0] = [\gamma_1]$. So ε_p is injective.

Now given $\hat{x} \in p^{-1}(x_0)$, as $\hat{X}$ is path connected, there exists a path $\eta \in \Omega(\hat{X}, \hat{x}_0, \hat{x})$. $p(\hat{x}) = x_0$, so if $\gamma = p \circ \eta$, then $\gamma \in \Omega(X, x_0)$. Then $\eta = \hat{\gamma}$ by uniqueness of lifts. So $\varepsilon_p[\gamma] = \hat{x}$. That is, ε_p is surjective. □

Theorem 2.14. $\pi_1(S^1, 1) = \mathbb{Z}$

Proof. $p : \mathbb{R} \to S^1$, $p(t) = e^{2\pi i t}$ is a universal cover. Then $p^{-1}(1) = \mathbb{Z}$, so $\varepsilon_p : \pi_1(S^1, 1) \to \mathbb{Z}$ is a bijection. Suffices to show this is a homomorphism. Given $n \in \mathbb{Z}$, define $\phi_n : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x + n$. Then $p \circ \phi_n = p$, so if $\gamma \in \Omega(S^1, 1)$, $\hat{\gamma}$ lift of γ to $\mathbb{R}$, with $\hat{\gamma}(0) = 0$. Then

$$p \circ (\phi_n \circ \hat{\gamma}) = p \circ \hat{\gamma} = \gamma$$

so $\phi_n \circ \hat{\gamma}$ is the lift of γ with $\phi_n \circ \hat{\gamma}(0) = n$. Suppose $\varepsilon_p[\gamma] = n$ and $\varepsilon_p[\gamma'] = n'$, then $\hat{\gamma}(1) = n$ and $\hat{\gamma}'(1) = n'$. So $\phi_n \circ \gamma'$ is a lift of $\gamma'(1)$ starting at n. So $\widehat{\gamma\gamma'} = \hat{\gamma}(\phi_n \circ \hat{\gamma}')$. Thus,

$$\varepsilon_p([\gamma][\gamma']) = \overline{\gamma\gamma'}(1) = \phi_n \circ \hat{\gamma}'(1) = n + n'$$

So ε_p is a homomorphism. □

Proposition 2.15. Suppose $z \in S^1$, $u, v \in \Omega(S^1, z, 1)$. Then $u_\# = v_\#$.

Proof. Consider the composition $v_\#^{-1} \circ u_\# = (v^{-1} \circ u)_\#$. Then

$$(v^{-1} \circ u)_\#[\gamma] = [vu^{-1}\gamma uv^{-1}] = [\eta][\gamma][\eta]^{-1} = [\gamma]$$

where $\eta = vu^{-1}$ and as $\pi_1(S^1, 1) = \mathbb{Z}$ abelian. So $v_\# = u_\#$. □

Definition 2.16 (degree)
The degree of a map $f : S^1 \to S^1$, is defined to be $\deg(f) = u_\# \circ f_*(1) \in \mathbb{Z}$.

Proposition 2.17.

(i) $\deg(z \mapsto z^n) = n$,

(ii) $g \sim g'$ if and only if $\deg(g) = \deg(g')$.

(iii) g extends to D^2 if and only if $\deg(g) = 0$.

Definition 2.18 (wedge product)
Suppose $(X_i, x_i)_i$ pointed spaces, then the wedge product is

$$\bigvee_i (X_i, x_i) = \frac{\bigsqcup_i X_i}{\{x_i\}}$$

2.3 Covering transformations

Definition 2.19 (locally path connected)
A space X is locally path connected if for every $U \subseteq X$ open, $x \in U$, there exists an open $V \subseteq U$ such that V is path connected and $x \in V$.

Proposition 2.20 (simply connected lifting)**.** Suppose Z is simply connected and locally path connected. Given $f : (Z, z_0) \to (X, x_0)$, f has a unique lift to $\hat{f} : (Z, z_0) \to (\hat{X}, \hat{x}_0)$.

Proof. Suppose $\hat{f} : (Z, z_0) \to (\hat{X}, \hat{x}_0)$ is a lift of f. Given $z \in Z$, choose $\gamma \in \Omega(Z, z_0, z)$, as Z is path connected. Then $\hat{f} \circ \gamma$ is a lift of $f \circ \gamma$, since $p \circ (\hat{f} \circ \gamma) = (p \circ \hat{f}) \circ \gamma = f \circ \gamma$, and $\hat{f} \circ \gamma(0) = \hat{f}(z_0) = \hat{x}_0$, so $\hat{f} \circ \gamma = \widehat{f \circ \gamma}$, as path lifting is unique. So $\hat{f}(z) = \hat{f}(\gamma(1)) = \hat{f} \circ \gamma(1) = \widehat{f \circ \gamma}(1)$, so $\hat{f}$ is unique.

Now if $\gamma_0, \gamma_1 \in \Omega(Z, z_0, z)$, then $\gamma_0 \sim_e \gamma_1$ as Z is simply connected. So $f \circ \gamma_0 \sim_e g \circ \gamma_1$, i.e. $\widehat{f \circ \gamma_0}(1) = \widehat{f \circ \gamma_1}(1)$. So we can define $\hat{f}(z) = \widehat{f \circ \gamma_0}(1)$, where $\gamma \in \Omega(Z, z_0, z)$ arbitrary. Then

$$p(\hat{f}(z)) = p \circ \widehat{f \circ \gamma}(1) = f \circ \gamma(1) = f(z)$$

So $\hat{f}$ is a lift. If $z = z_0$, choosing $\gamma = c_{I, z_0}$, we find that $f \circ \gamma = c_{I, x_0}$ and $\widehat{f \circ \gamma} = c_{I, \hat{x}_0}$, i.e. $\hat{f}(z_0) = \hat{x}_0$. Finally, we need to show that $\hat{f}$ is continuous. Given $U \subseteq \hat{X}$ open neighbourhood of $\hat{f}(z)$, we want to find V open neighbourhood of z such that $\hat{f}(V) \subseteq U$.

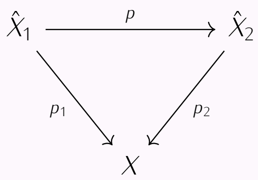
Step 1: Find $U' \subseteq U$, $\hat{f}(z) \in U'$ such that $p(U')$ is open and evenly covered. Since p is a covering map, there exists $W \subseteq X$ open, $f(z) \in W$ which is evenly covered. So $p^{-1}(W) = \bigsqcup_\alpha X_\alpha$, and $p(\hat{f}(z)) = f(z)$, so $\hat{f}(z) \in W_{\alpha_0}$ for some α_0. Then $W_{\alpha_0} \subseteq \hat{X}$ open, and let $U' = U \cap W_{\alpha_0}$. Then $\hat{f}(z) \in U'$. Furthermore, as $p|_{W_{\alpha_0}} : W_{\alpha_0} \to W$ is a homeomorphism. So $p(U')$ is open and evenly covered.

Step 2: $f : Z \to X$ is continuous, so we can find $V' \subseteq Z$ open such that $z \in V'$ with $f(V') \subseteq p(U')$. As Z is locally path connected, there exists $C \subseteq V'$ open, $z \in V$ such that V is path connected.

Step 3: We show $\hat{f}(V) \subseteq U$. Given $z' \in V$, choose $\gamma' \in \Omega(V, z, z')$, then $\operatorname{im}(f \circ \gamma') \subseteq f(V) \subseteq p(U')$ is evenly covered. So $\tilde{\gamma}' = p^{-1} \circ f \circ \gamma'$ are lifts of γ' with $\tilde{\gamma}'(0) = p^{-1}(f(z)) = \hat{f}(z)$. Then $\gamma\gamma' \in \Omega(Z, z_0, z')$ and $\widetilde{f \circ (\gamma\gamma')} = (f \circ \hat{\gamma})\tilde{\gamma}'$. So $\hat{f}(z') = \widetilde{f \circ (\gamma\gamma')}(1) = \tilde{\gamma}'(1) = p^{-1}(f(\gamma'(1))) \in U'$. □

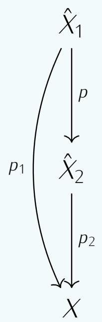
Definition 2.21 (covering transformation)

Suppose $p_i : \hat{X}_i \to X$ are covering maps, a covering transformation $p : (p_1, \hat{X}_1) \to (p_2, \hat{X}_2)$ is a map $p : \hat{X}_1 \to \hat{X}_2$ such that



commutes. Equivalently, p is a lift of p_1 to $\hat{X}_2$.

Lemma 2.22. If X is locally path connected, $p : (p_1, \hat{X}_1) \to (p_2, \hat{X}_2)$ is a covering transformation, then $p : \hat{X}_1 \to \hat{X}_2$ is a covering map. That is, we have a tower of covering maps



Proof. Given $\hat{x}_2 \in \hat{X}_2$, we want to find an evenly covered neighbourhood U of $\hat{x}_2$. Let $x = p_2(\hat{x}_2)$, and as p_1, p_2 are covering maps, we have V_1, V_2 evenly covered neighbourhoods of x by p_1, p_2 respectively. Then $V = V_1 \cap V_2$ is an open evenly covered neighbourhood for p_1 and p_2. As X is locally path connected, we can assume without loss of generality that V is a path connected open neighbourhood of x which is evenly covered by p_1 and p_2. Then

$$p_1^{-1}(V) = \bigsqcup_{\alpha} A_\alpha \quad \text{and} \quad p_2^{-1}(V) = \bigsqcup_{\beta} V_\beta$$

where each $A_\alpha \simeq V \simeq V_\beta$ are path connected. Fix α, and let $x_\alpha = p_1|_{A_\alpha}^{-1}(x)$. Then $p_2(p_1(x_\alpha)) = p_1(x_\alpha) = x$, so $p(x_\alpha) = x_\beta$ for some β, where $x_\beta = p_2|_{B_\beta}^{-1}(x)$. V_α, V_β are path connected, so $p(V_\alpha) \subseteq V_\beta$ since the image of a path connected space is path connected and each V_β is a path component of $p_2^{-1}(V)$.

Then $p|_{A_\alpha} : A_\alpha \to B_\beta$ satisfies $p_2|_{B_\beta} \circ p|_{A_\alpha} = p_1|_{A_\alpha}$. That is, $p|_{A_\alpha} = p_2|_{B_\beta}^{-1} \circ p_1|_{A_\alpha}$ is a homeomorphism. So

$$p^{-1}(B_\beta) = \bigsqcup_{\alpha \text{ s.t. } p(x_\alpha) = x_\beta} A_\alpha$$

and $p|_{A_\alpha} : A_\alpha \to B_\beta$ is a homeomorphism. So the $\{V_\beta\}$ are evenly covered, i.e. p is a covering map. □

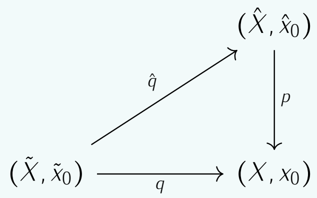
Lemma 2.23. If $p : \hat{Y} \to Y$ is a bijective covering map, then p is a homeomorphism.

Proof. Let $p^{-1} : Y \to \hat{Y}$ be the inverse of p. Since Y has an open cover $\{U_y\}$ so that U_y is evenly covered. But p is a bijection, then $p^{-1}|_{U_y} : U_y \to p^{-1}(U_y)$ is a homeomorphism. SO p^{-1} is continuous. □

Definition 2.24 (covering isomorphism)
$p : (p_1, \hat{X}_1) \to (p_2, \hat{X}_2)$ is a covering isomorphism if p is a covering transformation and p is a homeomorphism.

From now on, suppose X is locally path connected, $q : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a universal cover, $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ is a covering map.

Proposition 2.25. There exists a unique covering transformation $\hat{q}$ making the following diagram commute.



Proof. $\tilde{X}$ is locally path connected as X is locally path connected, and $\tilde{X}$ is simply connected, so existence and uniqueness is just simply connected lifting. □

Corollary 2.26. If $\hat{X}$ is also a universal cover, then $\hat{q}$ is a covering isomorphism.

Proof. $\tilde{X}$ is simply connected, so $\hat{q} : \tilde{X} \to \hat{X}$ is a universal cover. This means that there is a bijection $q^{-1}(\hat{x}) \leftrightarrow \pi_1(\hat{X}, \hat{x}) = 1$, so $\hat{q}$ is a bijection. □

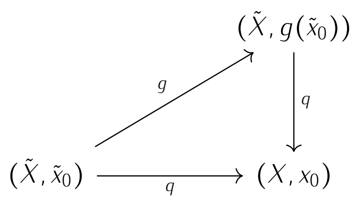
Definition 2.27 (deck group)
The deck group is

$$G_D(p) = \{\text{covering automorphisms } g : (p, \hat{X}) \to (p, \hat{X})\}$$

which is a group under composition, and it acts on $\hat{X}$ by the left, so $g \cdot \hat{x} = g(\hat{x})$.

Theorem 2.28. $G_D(q) \simeq \pi_1(X, x_0)$

Proof. First of all, we show that there is a bijection $G_D(q) \to q^{-1}(x_0)$, given by $g \mapsto g(\tilde{x}_0)$. Injectivity follows from uniqueness, and surjectivity follows from existence in the proposition, i.e.



So we have bijections $\varepsilon_q : \pi_q(X, x_0) \to q^{-1}(x_0)$ and $G_D(q) \to q^{-1}(x_0)$. Composing these bijections, we get

$$[\gamma][\gamma'] \mapsto \varepsilon_q([\gamma\gamma']) = \widetilde{\gamma\gamma'}(1) = \tilde{\gamma}(g_{\tilde{\gamma}(1)} \circ \tilde{\gamma}')$$

where $g_{\tilde{\gamma}(1)}$ is the unique element of $G_D(q)$ sending $\tilde{x}_0 \mapsto \tilde{\gamma}(1)$, since $g_{\tilde{\gamma}(1)} \circ \tilde{\gamma}'$ is a lift of γ' starting at $\tilde{\gamma}(1)$. So we have that

$$\widetilde{\gamma\gamma'}(1) = (g_{\tilde{\gamma}(1)} \circ \tilde{\gamma}')(1) = g_{\tilde{\gamma}(1)}(\tilde{\gamma}'(1)) = g_{\tilde{\gamma}(1)}(g_{\tilde{\gamma}'(1)}(\tilde{x}_0))$$

So we have that the composition is $[\gamma][\gamma'] \mapsto g_{\tilde{\gamma}(1)} \circ g_{\tilde{\gamma}'(1)}$, which is a homomorphism. □

2.4 Galois correspondence

Proposition 2.29 (towers of covering maps)**.** Let $G = G_D(q) = \pi_1(X, x_0)$, $H \leq G$ is a subgroup. Then we have a tower of covering maps

$$\tilde{X} \xrightarrow[\pi_H]{} X_H \xrightarrow[p_H]{} X$$

where

$$X_H = H \backslash \tilde{X} = \frac{\tilde{X}}{h \cdot x \sim x}$$

is the orbit space, π_H is the quotient map, and $p_H : X_H \to X$ is the projection map $p_H(H \cdot x) = q(x)$. In particular, if $H = G$, then p_G is a bijective covering map, i.e. $X \simeq G \backslash \tilde{X}$.

Proof. First we show that p_H is well defined. Given $\tilde{x} \in \tilde{X}$, $q(h(\tilde{x})) = q(\tilde{x})$, so the output is independent of the choice of $\tilde{x}$.

Now given $x \in X$, choose a neighbourhood U of x which is evenly covered by q. Then

$$q^{-1}(U) = \bigsqcup_{\alpha} V_\alpha = \bigsqcup_{g \in G_D(q)} g \cdot V$$

where $\tilde{x}_0 \in V$. Furthermore,

$$p_H^{-1}(U) = \pi_H\left(\pi_H^{-1}\left(p_H^{-1}(U)\right)\right) = \pi_H(q^{-1}(U)) = \bigsqcup_{Hg \in H \backslash G} \pi_H(g \cdot V)$$

where we used the fact that π_H is surjective. To justify the fact that this is a disjoint union, we need to show that if $\pi_H(g \cdot V) \cap \pi_H(k \cdot V) \neq \emptyset$, then in fact $g = hk$ for some $h \in H$. To see this, suppose $t \in \pi_H(g \cdot V) \cap \pi_H(k \cdot V)$. Then there exists $x, y \in V$ such that $[g \cdot x] = t = [k \cdot y]$, that is, $g \cdot x = h \cdot k \cdot y$ for some $h \in H$. Now $z = g \cdot x \in g \cdot V$ and $z = h \cdot k \cdot y \in hk \cdot V$, so as we have a disjoint union, we must have that $g = hk$. Thus, U is an evenly covered neighbourhood of x by p_H.

Finally, the preimage of each $\pi_H(g \cdot V)$ is given by

$$\pi_H^{-1}(\pi_H(g \cdot V)) = \bigsqcup_{hg \in Hg} hg \cdot V$$

so π_H is also a covering map.

In the case where $H = G$, we can see that $Gg = G$, so there is only one preimage for every point. Thus p_G is a bijection. □

Suppose $q : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a universal cover, and suppose $\tilde{X}$ is locally path connected. So X is path connected and locally path connected. Define

- $\mathcal{S}(X, x_0) = \{H \leq \pi_1(X, x_0)\}$
- $$\mathcal{C}(X, x_0) = \left\{(p, \hat{X}, \hat{x}_0) \mid p : (\hat{X}, \hat{x}_0) \to (X, x_0) \text{ covering map, } \hat{X} \text{ path connected}\right\} / \sim$$

where $(p, \hat{X}, \hat{x}_0) \sim (p', \hat{X}', \hat{x}_0')$ if there is a covering isomorphism between them. Define

- $\alpha : \mathcal{S}(X, x_0) \to \mathcal{C}(X, x_0)$ by
$$\alpha(H) = (p_H, X_H, \pi_H(\tilde{x}_0) = x_{0,H})$$
- $\beta : \mathcal{C}(X, x_0) \to \mathcal{S}(X, x_0)$ by
$$\beta(p, \hat{X}, \hat{x}_0) = p_*(\pi_1(\hat{X}, \hat{x}_0))$$

Theorem 2.30. α and β are inverse bijections.

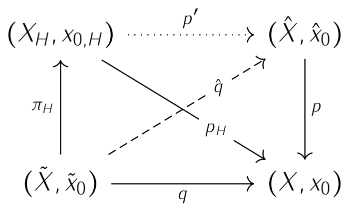
Proof. $\beta(\alpha(H)) = p_{H*}(\pi_1(X_H, x_{0,H}))$, and we have isomorphisms

$$H \xrightarrow[\gamma \mapsto [\pi_H \circ \tilde{\gamma}]]{\sim} \pi_1(X_H, x_{0,H}) \xrightarrow[[\eta] \mapsto [p_H \circ \eta]]{\sim} p_{H*}(\pi_1(X_H, x_{0,H}))$$

where $\tilde{\gamma} : I \to \tilde{X}$ is the lift of γ at $\tilde{\gamma}(0) = \tilde{x}_0$. These compose to give

$$[\gamma] \mapsto [p_H \circ \pi_H \circ \tilde{\gamma}] = [q \circ \tilde{\gamma}] = [\gamma]$$

So $\beta(\alpha(H)) = H$. Conversely, $\alpha(\beta(p, \hat{X}, \hat{x}_0)) = (p_H, X_H, x_{0,H})$ where $H = p_*(\pi_1(\hat{X}, \hat{x}_0))$. Consider the following diagram



As $\tilde{X}$ is simply connected and locally path connected, we have a lift $\hat{q}$ of q. **We want to show that there exists p' such that the diagram commutes.** To see this, given $h \in H$, $h = [p \circ \gamma]$ for some $\gamma \in \Omega(\hat{X}, \hat{x}_0)$. Then $\hat{q}(\tilde{x}) = \widetilde{q \circ \eta}(1)$ where $\eta \in \Omega(\tilde{X}, \tilde{x}_0, \tilde{x})$. Let $\delta \in \Omega(\tilde{X}, \tilde{x}_0, h \cdot \tilde{x}_0)$ and $\nu = \delta(h \circ \eta)$, we get that $q \circ \nu = (q \circ \delta)(q \circ h \circ \eta)$. But $h \in H \leq G_D(q)$, so $q \circ h = q$. Furthermore, $q \circ \delta \sim_e p \circ \gamma$.

To see this, consider the lift $\widetilde{p \circ \gamma} : I \to \tilde{X}$ at $\tilde{x}_0$. By the definition of the isomorphism $G_D(q) \leftrightarrow \pi_1(X, x_0) \geq H \ni h$, we have that $\widetilde{p \circ \gamma}(1) = h \cdot \tilde{x}_0$. So $\widetilde{p \circ \gamma} \in \Omega(\tilde{X}, \tilde{x}_0, h \cdot \tilde{x}_0)$. As $\tilde{X}$ is simply connected, $\widetilde{p \circ \gamma} \sim_e \delta$, so $q \circ \delta \sim_e p \circ \gamma$.

With all of this, we gat that $q \circ \nu \sim_e (p \circ \gamma)(q \circ \eta)$, so in particular, $\widetilde{q \circ \nu} \sim_e \gamma(\widetilde{q \circ \delta})$. Therefore,

$$\hat{q}(h \cdot \tilde{x}) = \widetilde{q \circ \nu}(1) = \widetilde{q \circ \eta}(1) = \hat{q}(\tilde{x})$$

So $\hat{q}$ factors as in the diagram. In this case, $\hat{X}$ is connected, so p' is surjective. Suffices to show p' is injective. Suppose $\hat{q}(\tilde{x}) = \hat{q}(\tilde{y}) = \hat{x}$, then let $\gamma \in \Omega(\tilde{X}, \tilde{x}, \tilde{y})$. Then $\hat{q} \circ \gamma \in \Omega(\hat{X}, \hat{x})$ and $[q \circ \gamma] = [p \circ (\hat{q} \circ \gamma)] \in H \leq G_D(q)$[1] sends $\tilde{x}$ to $\tilde{y}$, so $\pi_H(\tilde{x}) = \pi_H(\tilde{y})$. Thus, p' is a covering isomorphism, and $\alpha \circ \beta = \mathrm{id}$. ☐

Corollary 2.31.

$$[\pi_1(X, x_0) : H] = |p^{-1}(x_0)|$$

Definition 2.32 (normal covering)

$p : \hat{X} \to X$ is a normal cover if $G_D(p)$ acts transitively on $p^{-1}(x_0)$.

Proposition 2.33. The universal cover is always a normal cover.

Proposition 2.34. Conjugation corresponds to a change in base point. That is, gHg^{-1} corresponds to $(p_H, X_H, \hat{\gamma}(1))$, where $g = [\gamma]$ and $\hat{\gamma}$ is a lift of γ in X_H starting at $x_{0,H}$.

[1]There is a mild abuse of notation here, since what we have here is $p_*(\pi_1(\hat{X}, \hat{x}))$, but this is *isomorphic* to H.

Corollary 2.35. $H \trianglelefteq G$ is a normal subgroup if and only if $p_H : X_H \to X$ is a normal covering. If so, then

$$G_D(p_H) \simeq \frac{G}{H}$$

Proof. If H is normal, then $gHg^{-1} = H$, and so $(p_H, X_H, x_{0,H})$ and $(p_H, X_H, \hat{\gamma}(1))$ are isomorphic. That is, we have an element ϕ of $G_D(p_H)$ such that $\phi(x_{0,H}) = \hat{\gamma}(1)$. The converse holds by applying the inverse bijection and the transitive action.

Now define a homomorphism $\phi : G \to G_D(p_H)$ by sending $[\gamma]$ to the element $\tau \in G_D(p_H)$, defined by $\tau(x_{0,H}) = \hat{\gamma}(1)$, where $\hat{\gamma}$ is the lift of γ by p_H starting at $x_{0,H}$. This defines a group homomorphism, which is surjective (as covering transformations which agree at one point are the same). Moreoever, $\ker(\phi) = H$. The result follows by the isomorphism theorem. □

3 Seifart van Kampen

3.1 Free groups

Definition 3.1 (free group)

A free group on a generating set S is a group F_S and a subset $S \subseteq F_S$ such that whenever G is a group, $\phi : S \to G$ is a function, there exists a unique homomorphism $\Phi : F_S \to G$ with $\Phi|_S = \phi$.

Lemma 3.2. If F_S, F_T are free groups, $\phi : S \to T$ is a bijection, then $\Phi : F_S \to F_T$ is an isomorphism.

Proof. Let $\psi = \phi^{-1}$, then as F_T is free, there exists a homomorphism $\Psi : F_T \to F_S$, such that $\Psi \circ \Phi(s) = s$ for all $s \in S$, so by uniqueness $\Psi \circ \Phi = \mathrm{id}_{F_S}$. Similarly, $\Phi \circ \Psi = \mathrm{id}_{F_T}$. □

Corollary 3.3. If F_S, F'_S are free groups on S, then $F_S \simeq F'_S$, so the isomorphism class onyl depends on $|S|$.

Notation 3.4. Write $F_n = F_{\{a_1,\ldots,a_n\}}$.

Definition 3.5 (normal closure)

Define the normal closure of $S \subseteq G$ by

$\langle\langle S \rangle\rangle = \bigcap_{H \trianglelefteq G \text{ s.t. } S \subseteq H} H$

i.e. it is the smallest normal subgroup of G containing S.

Definition 3.6 (presentation)

Given a set S, $R \subseteq F_S$, define the presentation

$$\langle S \mid R \rangle = \frac{F_S}{\langle\langle R \rangle\rangle}$$

Proposition 3.7. Any group G can be written as a presentation.

Proof. Let $S = G$, $R = \ker(\Phi : F_G \to G)$. □

Proposition 3.8. Given $\langle S \mid R \rangle$, $w \in F_S$, then

$$\langle S \mid R \rangle \cong \langle S \cup \{\alpha\} \mid R \cup \{\alpha w^{-1}\} \rangle$$

Proof. We have natural homomorphisms $\Phi : \langle S \mid R \rangle \to \langle S \cup \{\alpha\} \mid R \cup \{\alpha w^{-1}\} \rangle$ induced by $S \hookrightarrow S \cup \{\alpha\}$, and Ψ induced by $\mathrm{id} : S \to S$ and $\alpha \mapsto w$. □

Similarly,

(i) If $w \in R$, we can replace $w \to sws^{-1}$ for some $s \in S$.

(ii) If $w_1, w_2 \in R$, we can replace $w_1 \to w_1 w_2$, and keep w_2 as is.

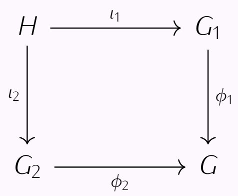
3.2 Amalgamated free products

Suppose we have $\iota_1 : H \to G_1$ and $\iota_2 : H \to G_2$ homomorphisms.

Definition 3.9

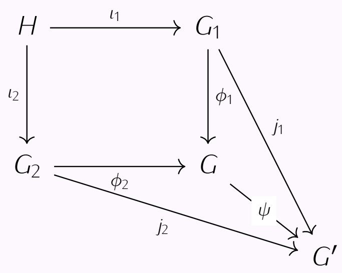
A group G is an amalgamated free product of G_1 and G_2 along H if

(i) There are homomorphisms $\phi_i : G_i \to G$ such that



commutes, and

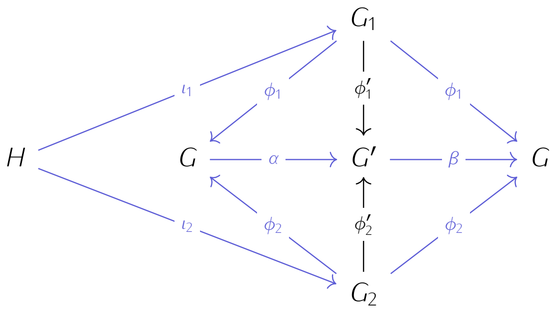
(ii) if $j_i : G_i \to G'$ are homomorphisms, then there exists a unique $\psi : G \to G'$ such that



commutes.

Proposition 3.10 (uniqueness of amalgamated free products)**.** If G and G' are both amalgamated free products of G_1 and G_2 along H, then $G \cong G'$.

Proof. We defined G via a universal property, so we have homomorphisms $\alpha : G \to G'$ and $\beta : G' \to G$.



Applying uniqueness in (ii) to the purple diagram, we get that $\beta \circ \alpha = \mathrm{id}_G$. Swapping G and G' we get that $\alpha \circ \beta = \mathrm{id}_{G'}$. Thus, α and β are inverses of each other, and $G \cong G'$. □

Proposition 3.11 (exisrence of amalgamated free products)**.**

Proof. Choose presentations $G_i = \langle S_i \mid R_i \rangle$ and $H = \langle T \mid W \rangle$. Then define

$$G = \langle S_1 \cup S_2 \cup T \mid R_1 \cup R_2 \cup \left\{ t^{-1}\iota_j(t) \mid t \in T, j = 1, 2 \right\} \rangle$$

and $\phi_i : G_i \to G$ by $s \in S_i \mapsto s \in S_i$. Now given $j_i : G_i \to G'$, define $\Psi : G \to G'$ by

$$\begin{aligned} s \in S_1 &\mapsto j_1(s) \\ s \in S_2 &\mapsto j_2(s) \\ t \in T &\mapsto j_1 \circ \iota_1(t) = j_2 \circ \iota_2(t) \end{aligned}$$

□

3.3 Seifart van Kampen

Theorem 3.12. Suppose $U_1, U_2 \subseteq X$ open, $X = U_1 \cup U_2$, where $U_1, U_2, U_1 \cap U_2$ are path connected, and $x_0 \in U_1 \cap U_2$. Let $j_l : U_i \hookrightarrow X$ be the inclusion maps, and $j_{i*} : \pi_1(U_i, x_0) \to \pi_1(X, x_0)$ be the induced map on fundamental groups. Then $\pi_1(X, x_0)$ is generated by $\mathrm{im}(j_{1*}) \cup \mathrm{im}(j_{2*})$.

Proof. If $\gamma \in \Omega(X, x_0)$, then $\gamma^{-1}(U_1), \gamma^{-1}(U_2)$ is an open cover of I. By the Lebesgue covering lemma, we can find $N \in \mathbb{N}$ such that for each j, $[j/N, (j+1)/N] \subseteq \gamma^{-1}(U_i)$ for some i. Label $[j/N, (j+1)/N]$ as 1 or 2 accordingly (if it is in both, wlog choose 1).

Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be the points j/N where the label changes, $I_i = [t_{i-1}, t_i]$ and $\gamma_i = \gamma|_{I_i}$. Then each $\gamma(t_i) \in U_1 \cap U_2$ and $\gamma(I_i) \subseteq U_{i \bmod 2}$[2] and $\gamma = \gamma_1 \cdots \gamma_k$.

Choose $\eta_1, \ldots, \eta_{k-1}$, where each $\eta_i \in \Omega(U_1 \cap U_2, \gamma(t_i), x_0)$, which we can as $U_1 \cap U_2$ is path connected. Then

$$\gamma \sim_e \underbrace{\gamma_1 \eta_1}_{\delta_1} \underbrace{\eta_1^{-1} \gamma_e \eta_2}_{\delta_2} \cdots \underbrace{\eta_{k-2}^{-1} \gamma_{k-1} \eta_{k-1}}_{\delta_{k-1}} \underbrace{\eta_{k-1}^{-1} \gamma_k}_{\delta_k}$$

Each $\delta_i \in \Omega(U_{i \bmod 2}, x_0)$, so $[\delta_i] \in \mathrm{im}(j_{(i \bmod 2)*})$. Then $[\gamma] = [\delta_1] \ldots [\delta_k]$ is a product of elements in $\mathrm{im}(j_{1*}) \cup \mathrm{im}(j_{2*})$. □

Theorem 3.13 (Seifart van Kampen)**.** Suppose $X = U_1 \cup U_2$, $U_i \subseteq X$ open, $U_1, U_2, U_1 \cap U_2$ are all path connected, $x_0 \in U_1 \cap U_2$. We have a diagram of inclusions

[2] We can assume without loss of generality that $\gamma(I_1) \subseteq U_1$.

$$\begin{array}{ccc} (U_1 \cap U_2, x_0) & \xhookrightarrow{\iota_1} & (U_1, x_0) \\ \Big\downarrow \iota_2 & & \Big\downarrow j_1 \\ (U_2, x_0) & \xhookrightarrow{j_2} & (X, x_0) \end{array}$$

which induces (by functoriality)

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \xrightarrow{\iota_{1*}} & \pi_1(U_1, x_0) \\ \iota_{2*} \Big\downarrow & & \Big\downarrow j_{1*} \\ \pi_1(U_2, x_0) & \xrightarrow[j_{2*}]{} & \pi_1(X, x_0) \end{array}$$

and we have that $\pi_1(X, x_0) = G_1 *_H G_2$, where $G_i = \pi_1(U_i, x_0)$ and $H = \pi_1(U_1 \cap U_2, x_0)$.

Proof. Non-examinable, so omitted. □

4 Simplicial complexes

4.1 Definitions

Definition 4.1 (n-simplex)
The n-simplex is

$$\Delta^n = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \sum_i x_i = 1 \right\}$$

with the subspace topology from $\mathbb{R}^{n+1}$.

Definition 4.2 (face)
For $I \subseteq \{0, \ldots, n\}$, the I-th face of Δ^n is

$$e_I = \{x \in \Delta^n \mid x_i = 0 \text{ for } i \notin I\}$$

Let $F(\Delta^n) = \{e_I \mid I \subseteq \{0, \ldots, n\}\}$ be the set of all faces of Δ^n.

Notation 4.3. If $I = \{i_0, \ldots, i_k\}$ with $i_0 < i_1 < \cdots < i_k$, we write

$$e_{i_0 \ldots i_k} := e_I$$

Proposition 4.4.

(i) $e_I \subseteq \Delta^n$ is closed, $e_I \simeq \Delta^{|I|-1}$,

(ii) $e_I \subseteq e_J$ if and only if $I \subseteq J$,

(iii) $e_I \cap e_J = e_{I \cap J}$.

Definition 4.5 (affine linear map)

A map $|f|: \Delta^n \to \mathbb{R}^m$ is affine linear if it is the restriction of a linear map $\mathbb{R}^{n+1} \to \mathbb{R}^m$. Equivalently,

$$|f|\left(\sum_i x_i e_i\right) = \sum_i x_i |f|(e_i)$$

Definition 4.6 (simplicial map)

A map $|f|: \Delta^n \to \Delta^m$ is simplicial if it takes vertices of Δ^n to vertices of Δ^m, i.e. a map $\hat{f}: \{0, \dots, n\} \to \{0, \dots, m\}$ with $|f|(e_i) = e_{\hat{f}(i)}$. That is, $|f|(e_l) = e_{\hat{f}(l)}$.

Definition 4.7 (affine linearly independent)

$v_0, \dots, v_n \in \mathbb{R}^N$ are affine linearly independent if whenever $\sum_i t_i v_i = 0$ and $\sum_i t_i = 0$, then $t_i = 0$ for all i.

Proposition 4.8. The following are equivalent.

(i) $v_0, \dots, v_n$ are affine linearly independent,

(ii) whenever $\sum_i t_i v_i = \sum_i t_i' v_i$, with $\sum_i t_i = \sum_i t_i'$, then $t_i = t_i'$ for all i,

(iii) the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent,

(iv) the map $|f|: \Delta^n \to \mathbb{R}^N$ given by $|f|(e_i) = v_i$ is injective.

Definition 4.9 (Euclidean simplex)

If $v_0, \dots, v_n$ are affine linearly independent, write $[v_0, \dots, v_n] = \operatorname{im}(|f|) = \{\sum_i x_i v_i \mid \sum_i x_i = 1, x_i \geq 0\}$ for the Euclidean simplex with vertices $v_0, \dots, v_n$.

Proposition 4.10. $|f|: \Delta^n \to [v_0, \dots, v_n]$ is a homeomorphism.

Proof. By the topological inverse function theorem, as Δ^n is compact and $[v_0, \dots, v_n]$ is Hausdorff. □

Lemma 4.11. For $X \subseteq \mathbb{R}^n$, let

$$Z(X) = \left\{x \in X \mid \text{if } x = \sum_i t_i x_i, t_i > 0, \sum_i t_i = 1, x_i \in X \text{ then } x_i = x \text{ for some } i\right\}$$

be the set of points in X which is not contained in the interior of any simplex contained in X. Then $Z([v_0, \dots, v_n]) = \{v_0, \dots, v_n\}$.

Proof. $\subseteq$ is clear from the definition of a simplex, so we show that $v_k \in Z([v_0, \dots, v_n])$. Suppose $v_k = \sum t_i x_i$, $t_i > 0$ and $\sum_i t_i = 1$. Then

$$x_i = \sum_{j=0}^{n} s_{ij} v_j$$

as $x_j \in [v_0, \dots, v_n]$. So $v_k = \sum_j \sum_i s_{ij} v_j$. Since the v_j are affine linearly independent, and $\sum_j \sum_i t_i s_{ij} = 1$, we must have that $\sum_i t_i s_{ij} = 0$ for $j! = 0$. But $t_i > 0$ and $s_{ij} \geq 0$, so the only case is when $s_{ij} = 0$ for $j \neq k$. So $x_j = v_k$ for all j. □

Corollary 4.12. $[v_0, \dots, v_n] = [v_0', \dots, v_n']$ if and only if $\{v_0, \dots, v_n\} = \{v_0', \dots, v_n'\}$.

Proof. $\Longleftarrow$ is obvious. $\Longrightarrow$ follows from applying Z to both sides.. □

Definition 4.13
Let $\mathcal{S}(\mathbb{R}^n)$ be the set of Euclidean simplices $\sigma \subseteq \mathbb{R}^n$.

4.2 Simplicial complexes

4.2.1 Abstract simplicial complexes

Definition 4.14 (abstract simplicial complex)
An abstract simplicial complex in Δ^n is a subset $K \subseteq F(\Delta^n)$ such that if $e_J \in K$ and $I \subseteq J$, $e_I \in K$.

Definition 4.15 (polyhedron)
If K is an abstract simplicial complex, its polyhedron is $|K| = \bigcup_{e_I \in K} e_I \subseteq \Delta^n$

Definition 4.16 (skeleton, vertex set)
For $-1 \le r \le n$, the r-skeleton of an abstract simplicial complex K is

$$K_r = \{e_I \in K \mid |I| \le r + 1\}$$

The vertex set of K is $V(K) = |K_0|$.

Definition 4.17 (dimension)
The dimension of an abstract simplicial complex is

$$\dim(K) = \max\{\dim(e_I) = |I| - 1 \mid e_I \in K\}$$

Definition 4.18 (simplicial map)
If K, L are abstract simplicial complexes in Δ^n, Δ^m respectively, a simplicial map $f : K \to L$ is a map $f : K \to L$ such that there exists a simplicial map $|f| : \Delta^n \to \Delta^m$ such that $f(e_I) = |f|(e_I)$ for all $e_I \in K$.

Equivalently, there is a map $\hat{f} : \{0, \dots, n\} \to \{0, \dots, m\}$ such that $f(e_I) = e_{\hat{f}(I)}$ for all $e_I \in K$, and $e_I \in K$ implies that $f(e_I) \in L$.

Definition 4.19 (simplicial isomorphism)
A simplicial map $f : K \to L$ is a simplicial isomorphism if it is a bijection.

4.2.2 Euclidean simplicial complexes

Definition 4.20 (Euclidean simplicial complex)
$K \subseteq \mathcal{S}(\mathbb{R}^n)$ is a Euclidean simplicial complex if

(i) K is finite,

(ii) If $\sigma \in K$ and $\tau \in F(\sigma)$, then $\tau \in K$, where

$$F([v_0, \ldots, v_n]) = \{[v_{i_0}, \ldots, v_{i_k} \mid I = i_0 \ldots i_k \subseteq \{0, \ldots, n\}]\}$$

is the set of faces of $[v_0, \ldots, v_n]$,

(iii) If $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2 \in F(\sigma_1) \cap F(\sigma_2)$. In particular, $\sigma_1 \cap \sigma_2 \in K$.

We define the polyhedron and r-skeleton as for abstract simplicial complexes.

Proposition 4.21. Suppose $|\phi| : \Delta^n \to \mathbb{R}^n$ is affine linear, K is an ASC in Δ^n and $|\phi||_{|K|}$ is injective. Then $L = \phi(K) = \{|\phi|(e_I) \mid e_I \in J\}$ is an ESC.

In this case, we say that $L = \phi(K)$ is a realisation of K.

Proof. Property (i) folloes form the fact that since $F(\Delta^n)$ is finite, K must be finite, so L is finite. For (ii), if $\sigma \in \phi(K)$, $\sigma = |\phi|(e_I)$ where $e_I \in K$, and so if τ is a fact of σ, then $\tau = |\phi|(e_J)$ for some $J \subseteq I$. As K is an ASC, $e_J \in K$. So $\tau \in \phi(K) = L$.

Finally, for (ii), suppose $\sigma = |\phi|(e_I)$ and $\tau = |\phi|(e_J)$. Then $\sigma \cap \tau = |\phi|(e_I) \cap |\phi|(e_J) = |\phi|(e_I \cap e_J)$ by injectivity. But $e_{I \cap J} = e_I \cap e_J \in K$ as K is down-directed, and so $\sigma \cap \tau = |\phi|(e_{I \cap J}) \in F(\sigma) \cap F(\tau)$. □

Proposition 4.22. If $L \subseteq \mathbb{R}^n$ is an ESC, then $L = \phi(K)$ for some ASC K, and $|\phi| : K \to L$ is a homeomorphism. That is, every ESC is the realisation of an ASC, and any two such K are isomorphic.

Proof. Let $V(L) = |L_0| = \{v_0, \ldots, v_n\}$. Define $K = \{e_I \in \Delta^n \mid [v_{i_0}, \ldots, v_{i_k}] \in L\}$, and define $|\phi| : \Delta^n \to \mathbb{R}^n$ by $|\phi|(e_i) = v_i$.

We want to show that $|\phi||_{|K|}$ is injective. If $\sigma = [v_{i_0}, \ldots, v_{i_k}] \in K$, then $v_{i_0}, \ldots, v_{i_k}$ are affine linearly independent, so $|\phi||_{e_I}$ is injective.

Now suppose $|\phi|(p) = |\phi|(q) = x \in \mathbb{R}^n$, where $p \in e_I \in L$ and $q \in e_J \in L$. Then $x \in |\phi|(e_I) \cap |\phi|(e_J)$, which is the intersection of simplices in L, so $x \in |\phi|(e_{I'})$ for some $I' \subseteq I \cap J$. Since $|\phi||_{e_I}$ and $|\phi||_{e_J}$ are injective, we must have that $p, q \in e_{I'}$. But $|\phi||_{e_{I'}}$ is also injective, so $p = q$. □

Definition 4.23

Suppose L_1, L_2 are ESCs. Then $f : L_1 \to L_2$ is a simplicial map if there are realisations $\phi_i : K_i \to L_i$ and a simplicial map $F : K_1 \to K_2$ such that

$$\begin{array}{ccc} K_1 & \xrightarrow{F} & K_2 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ L_1 & \xrightarrow[f]{} & L_2 \end{array}$$

commutes.

4.3 Barycentric subdivision

Definition 4.24 (boundary and interior)

Let σ be an n-dimensional Euclidean simplex, $F(\sigma)$ be the set of faces of σ, which is an Euclidean simplicial complex with $|F(\sigma)| = \sigma$. Define $\oint\sigma = F(\sigma)_{n-1} = F(\sigma) \setminus \sigma$, which is a Euclidean simplicial complex. Then we can define

$$\partial\sigma = |\partial\!\!\!/\sigma| \quad \text{and} \quad \sigma^o = \sigma \setminus \partial\sigma$$

4.3.1 Cones

Definition 4.25 (independent)
Let $X \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}^n$. We say that p is independent of X if for each $x \in X$, the ray $\overline{px}$ from p to x has $px \cap X = \{x\}$.

Definition 4.26 (cone)
If p is independent of X, define the cone

$$C_p(X) = \{tp + (1-t)x \mid t \in [0,1], x \in X\}$$

Proposition 4.27. If $X = [v_0, \dots, v_n]$, then p is independent to X if and only if $v_0, \dots, v_n, p$ are affine linearly independent. In this case, $C_p(X) = [v_0, \dots, v_n, p]$.

Definition 4.28 (cone of an ESC)
If K is an ESC in $\mathbb{R}^n$, and p is independent of $|K|$, then define the cone of K,

$$C_p(K) = K \cup \left\{[v_0, \dots, v_j, p] \mid [v_0, \dots, v_j] \in K\right\}$$

Lemma 4.29. If p is independent of $|K|$, then $C_p(K)$ is an ESC, and $\left|C_p(K)\right| = C_p(|K|)$.

Definition 4.30 (barycentre)
If $\sigma = [v_0, \dots, v_n]$ is a simplex, define its barycentre to be

$$b_\sigma = \frac{1}{n+1}\sum_{i=0}^{n} v_i$$

Lemma 4.31. b_σ is independent of $\partial\sigma$, and $C_{b_\sigma}(\partial\sigma) = \sigma$.

Theorem 4.32 (barycentric subdivision)**.** There exists maps $\beta : \mathcal{S}(\mathbb{R}^n) \to \{\text{ESCs in } \mathbb{R}^n\}$ and $B : \{\text{ESCs in } \mathbb{R}^n\} \to \{\text{ESCs in } \mathbb{R}^n\}$ such that $|\beta(\sigma)| = \sigma$ and $|B(K)| = |K|$.

Proof. We will define β, B for simplices/ESCs of dimension $\le n$ inductively. More precisely, we have:

Proposition 4.33 ($p_1(n)$)**.** If $\sigma \in \mathcal{S}(\mathbb{R}^N)$ is an n -simplex, then $\beta(\sigma)$ is an ESC of dimension n, with polyhedron $|\beta(\sigma)| = \sigma$. Moreover, if τ is a face of σ, $\sigma' \in \beta(\sigma)$, then $\sigma' \cap \tau \in \beta(\tau)$.

Proposition 4.34 ($p_2(n)$). If K is an n-dimensional ESC, then $B(K)$ is an n-dimensional ESC, with polyhedron $|B(K)| = |K|$.

Base case: $p_1(-1)$ and $p_2(-1)$. In this case, $\sigma = \varnothing$, so $\beta(\sigma) = \varnothing$, $K = \{\varnothing\}$, so $B(K) = K$.
Inductive case (i): $p_2(n-1) \implies p_1(n)$. Define

$$\beta(\sigma) = C_{\beta_\sigma}(B(\partial\!\!\!/\sigma))$$

As $\partial\!\!\!/\sigma$ is an ESC of dimension $n-1$, $B(\partial\!\!\!/\sigma)$ is an ESC of dimension $n-1$ as well, by $p_2(n-1)$, and $|B(\partial\sigma)| = |\partial\!\!\!/\sigma| = \partial\sigma$. Since b_σ is independent of $\partial\sigma = |B(\partial\!\!\!/\sigma)|$, we have that C_{b_σ} is an ESC with polyhedron $|C_{b_\sigma}|(B(\partial\!\!\!/\sigma)) = C_{b_\sigma}(\partial\sigma) = \sigma$. The statement about faces follows from:

Lemma 4.35. If $\sigma \in C_p(K)$, then $\sigma \cap |K| \in K$.

Inductive case (ii): $p_1(n) \implies p_2(n)$. Define

$$B(K) = \bigcup_{\sigma \in K} \beta(\sigma)$$

Then we need to check that this is an ESC. (i) Finiteness is obvious. (ii) If $\sigma \in B(K)$, then $\sigma \in \beta(\sigma')$ for some $\sigma' \in K$, so if τ is a face of σ, then $\tau \in \beta(\sigma')$ since $\beta(\sigma')$ is an ESC. So $\tau \in B(K)$.

Finally for (iii), suppose $\sigma_1, \sigma_2 \in B(K)$, where $\sigma_i \in \beta(\sigma_i')$, $\sigma_i' \in K$. Then $\sigma_1 \cap \sigma_2 \subseteq \sigma_1' \cap \sigma_2'$ since $|\beta(\sigma_i')| = \sigma_i'$. Let $\tau = \sigma_1' \cap \sigma_2' \in K$. Then $\sigma_1 \cap \tau, \sigma_2 \cap \tau \in \beta(\tau)$ by $p_1(n)$, and as $\beta(\tau)$ is an ESC, $\sigma_1 \cap \sigma_2 = \sigma_1 \cap \sigma_2 \cap \tau = (\sigma_1 \cap \tau) \cap (\sigma_2 \cap \tau) \in \beta(\tau)$ as $\beta(\tau)$ is an ESC. But $\beta(\tau) \subseteq B(K)$, so $\sigma_1 \cap \sigma_2 \in B(K)$.

Therefore$B(K)$ is an ESC with $|B(K)| = \bigcup_{\sigma \in K} |\beta(\sigma)| = \bigcup_{\sigma \in K} \sigma = |K|$. □

4.4 Simplicial approximation

4.4.1 Mesh

Lemma 4.36. If $\sigma \in \mathcal{S}(\mathbb{R}^n)$, $x, w \in \sigma$, then $\|w - x\| \leq \max_{v \in V(\sigma)} \|v - w\|$.

Proof. Write $x = \sum x_i v_i$, $w = \sum x_i w$, then

$$\|w - x\| = \left\|\sum x_i(v_i - w)\right\| \leq \sum x_i \|v_i - w\| \leq \sum x_i \max_i \|v_i - w\| = \max_i \|v_i - w\|$$

□

So $\|x - w\| \leq \max_{v, v' \in V(\sigma)} \|v - v'\|$.

Definition 4.37 (mesh)
The mesh of a simplex σ is

$$\mu(\sigma) = \max_{v, v' \in V(\sigma)} \|v - v'\| = \max_{x, w \in \sigma} \|x - w\|$$

The mesh of an ESC is

$$\mu(K) = \max_{\sigma \in K} \mu(\sigma)$$

Lemma 4.38. If b_σ is the barycentre of $\sigma = [v_0, \ldots, v_n]$, then

$$\max_{x \in \sigma} \|b_\sigma - x\| \leq \frac{n}{n+1} \mu(\sigma)$$

Proof. As $\|b_\sigma - x\| \leq \max_{v\in V(\sigma)} \|b_\sigma - v\|$, suffices to prove the result for a vertex $v \in V(\sigma)$. Now

$$\|b_\sigma - v_i\| = \frac{1}{n+1}\left\|\sum_{i\neq j} v_j - nv_i\right\| \leq \frac{1}{n+1}\sum_{i\neq j}\|v_j - v_i\| \leq \frac{n}{n+1}\mu(\sigma)$$

□

Corollary 4.39. Let σ be an Euclidean simplex, $\dim(\sigma) = n$. Then $\mu(\beta(\sigma)) \leq \frac{n}{n+1}\mu(\sigma)$. If K is an ESC with $\dim(K) = n$, then $\mu(B(K)) \leq \frac{n}{n+1}\mu(K)$.

Proof. Let $\tau \in \beta(\sigma)$. Suppose $\tau \in B(\partial\sigma)$. Then $\mu(\tau) \leq \frac{n-1}{n}\mu(B(\partial\sigma)) \leq \frac{n}{n+1}\mu(\sigma)$ by induction on n. Otherwise, $\tau = [v_0, \dots, v_k, b_\sigma]$, where $[v_0, \dots, v_k] \in B(\partial\sigma)$, then $\|v_i - v_j\| \leq \frac{n-1}{n}\mu(\sigma)$ by induction on n and $\|v_i - b_\sigma\| \leq \frac{n}{n+1}\mu(\sigma)$ by the lemma. As $\frac{n-1}{n} \leq \frac{n}{n+1}$, we are done. □

4.4.2 Simplicial approximation

Lemma 4.40.

(i) if $x \in \Delta^n$, then there is a unique $I \subseteq \{0, \dots, n\}$ such that $x \in e_I^o$,

(ii) if $x \in e_I^o$, then $x \in e_J$ if and only if $e_I \subseteq e_J$ if and only if $I \subseteq J$.

(iii) if K is an ASC in Δ^n, $x \in e_I^o$ and $x \in |K|$, then $e_I \in K$.

Proof. For (i), take $I = \{i \mid x_i > 0\}$. Then (ii) follows from (i). For (iii), as $x \in |K|$, $x \in e_J$ for some $e_J \in K$. By (ii), $e_I \subseteq e_J$. Since K is an ESC, $e_I \in K$. □

Corollary 4.41. Suppose K is an ESC, if $x \in |K|$, then there is a unique $\sigma \in K$ with $x \in \sigma^o$.

Proof. Let $\phi : L \to K$ be a realisation of K, where L is an ASC. Let $x' = |\phi|^{-1}(x) \in |K|$. Then by (i) of the lemma, there exists a unique I such that $x \in e_I^o$, so $e_I \in L$ by (iii). So $\phi(e_I) = \sigma$ os the unique $\sigma \in K$ with $x \in \sigma^o$. □

Definition 4.42 (star)
If K is an ESC, $v \in V(K)$, then the star of K at v is

$$\mathrm{St}_K(v) = \bigcup_{\sigma\in K \text{ s.t. } v\in\sigma} \sigma^o$$

Lemma 4.43.

(i) If $x \in |K|$, $x \in \sigma^o$, then $x \in \mathrm{St}_K(v) \iff v \in V(\sigma)$.

(ii)

$$\mathrm{St}_K(v) = |K| \setminus \bigcup_{\sigma\in K \text{ s.t. } v\notin V(\sigma)} \sigma^o = |K| \setminus \bigcup_{\sigma\in K \text{ s.t. } v\notin V(\sigma)} \sigma$$

(iii) $\{\mathrm{St}_K(v) \mid v \in V(K)\}$ is an open cover of $|K|$.

Proof. (i) follows form the fact that if $x \in |K|$, then $x \in \sigma^o$ for a unique $\sigma \in K$. (ii) the first equality follows form (i), the second follows from the fact that if $\tau \in F(\sigma)$, $v \notin V(\sigma)$ then $v \notin V(\tau)$. (iii) from (ii), we have that $\mathrm{St}_K(v)$ is the complement of a finite union of closed sets, so $\mathrm{St}_K(v)$ is open. If $x \in |K|$, then $x \in \sigma^o$ for some $\sigma \in K$. If $v \in V(\sigma)$, then $x \in \mathrm{St}_K(v)$. □

Definition 4.44 (simplicial approximation)

Suppose K, L are ESCs, $f : |K| \to |L|$ continuous, $\hat{g} : V(K) \to V(L)$ is called a simplicial approximation to f if for all $v \in V(K)$,

$$f(\mathrm{St}_K(v)) \subseteq \mathrm{St}_L(\hat{g}(v))$$

Theorem 4.45. Let $\phi : K' \to K$ be a realisation of K, and define $g' : |K'| \to \mathbb{R}^m$, where $L \subseteq \mathbb{R}^m$ to be the affine linear map given by $|g'|(v) = \hat{g}(\phi(v))$ for all $v \in V(K)$. Let $|g| = |g'| \circ |\phi|^{-1}$. Then $|g|$ defines a simplicial map $g : K \to L$ and $|g| \sim f$.

Proof. Let $\sigma \in K$, we want to show $|g|(\sigma) \in L$. Let $x \in \sigma^o$ be an arbitrary point in the interior. Then $f(x) \in |L|$, so $f(x) \in \tau^o$ with $\tau \in L$. So $x \in \bigcap_{v \in V(\sigma)} \mathrm{St}_K(v)$, so

$$f(x) \in \bigcap_{v \in V(\sigma)} f(\mathrm{St}_K(v)) \subseteq \bigcap_{v \in V(\sigma)} \mathrm{St}_L(g(v))$$

as $\hat{g}$ is a simplicial approximation to f. Now if $v \in V(\sigma)$, $f(x) \in \tau^o$ and $f(x) \in \mathrm{St}_L(g(v))$, so $g(v) \in \tau$ by part (i) of the lemma. So every vertex of $|g|(\sigma)$ is a vertex of τ, and so $|g|(\sigma)$ is a face of $\tau \in L$, so $|g|(\sigma) \in L$ as required. So $g : K \to L$ is simplicial.

For the homotopy, define $H : |K| \times I \to \mathbb{R}^m$ by $H(x,t) = t|g|(x) + (1-t)f(x)$. This is a homotopy in $\mathbb{R}^m$, so we need to show that it is a homotopy in $|L|$.

Suppose $x \in \sigma^o$ and $f(x) \in \tau^o$ as before. Then $x = \sum_{v_i \in V(\sigma)} x_i v_i$, so $|g|(x) = \sum_{v_i \in V(\sigma)} x_i |g|(v_i) \in \tau$ since $|g|(v_i) \in \tau$ for all i. Since τ is convex and $|g|(x), f(x) \in \tau$, we must have that $H(x,t) \in \tau$ for $t \in [0,1]$. So $H : |K| \times I \to |L|$, which is the required homotopy. □

Theorem 4.46 (simplicial approximation). Let K, L be Euclidean simplicial complexes, $f : |K| \to |L|$ be a continuous map. Then there exists $r > 0$ and a simplicial map $g : B^r(K) \to L$ such that $|g| \sim f$.

Proof. We have an open cover $\{\mathrm{St}_L(v) \mid v \in V(L)\}$ of $|L|$. $f : |K| \to |L|$ is continuous, so $\left\{f^{-1}(\mathrm{St}_L(v)) \mid v \in V(L)\right\}$ is an open cover of $|K|$. $|K|$ is a compact metric space, so we can apply the Lebesgue covering lemma to find $\delta > 0$ and a function $|K| \to V(L)$ sending each $x \in |K|$ to a vertex $v_x \in V(L)$ such that $B_\delta(x) \subseteq f^{-1}(\mathrm{St}_L(v_x))$. Let r be such that $\mu(B^r(K)) < \delta$, and let $K' = B^r(K)$. If $\sigma \in K'$ and $x \in V(\sigma)$, then $\sigma \subseteq B_\delta(x)$ as $\mu(K') < \delta$. If $x in V(K')$, Then

$$\mathrm{St}_{K'}(x) = \bigcup_{\sigma \text{ s.t. } x \in V(\sigma)} \sigma^o \subseteq \bigcup_{\sigma \text{ s.t. } x \in V(\sigma) \sigma \subseteq B_\delta(x)} \sigma \subseteq B_\delta(x)$$

Hence $f(\mathrm{St}_{K'}(x)) \subseteq f(B_\delta(x)) \subseteq \mathrm{St}_L(v_x)$, so the function $\hat{f} : V(K') \to V(L)$ given by $\hat{g}(x) = v_x$ is a simplicial approximation of f. So by the previous theorem, $\hat{g}$ determines a simplicial map $g : K' \to L$ with $|g| \sim f$. □

5 Simplicial homology

5.1 Chain complexes

Definition 5.1 (chain complex)

A (finitely generated) chain complex $(C_\bullet, d)$ is

(i) (finitely generated) free ablelian groups $(C_i)_{i \in \mathbb{Z}}$ (where finitely many C_i are nonzero),

(ii) group homomorphisms $d_i : C_i \to C_{i-1}$,

(iii) such that $d^2 = 0$, i.e. $d_i \circ d_{i+1} = 0$ for all i.

Notation 5.2. We write $C_* = \bigoplus_i C_I$ and $d : C_* \to C_*$ given by $d = \bigoplus_i d_i$.

Definition 5.3 (reduced chain complex of simplex)
The reduced chain complex of Δ^n is

$$\tilde{C}_i(\Delta^n) = \langle e_I \mid |I| = i+1 \rangle$$

with differential

$$de_I = \sum_{j=0}^{k} (-1)^j e_{I_{\hat{j}}}$$

where if $I = i_0 \dots i_k$, with $i_0 < i_1 < \dots < i_K$, then $I_{\hat{j}} = I \setminus \{i_j\}$.

Proposition 5.4. $d^2 = 0$ in $\tilde{C}_*(\Delta^n)$.

Proof. The e_I are a basisis for $\tilde{C}_*(\Delta^n)$, so suffices to show that $d^2(e_I) = 0$. But

$$d^2(e_I) = \sum_{j<k} c_{jk} e_{I_{\widehat{jk}}}$$

where $I_{\widehat{jk}} = I \setminus \{i_j, i_k\}$ and

$$c_{jk} = (-1)^j(-1)^{k-1} + (-1)^k(-1)^j = 0$$

which corresponds to removing j then k, and removing k then j respectively. □

Definition 5.5 (chain complex of a simplex)
The chain complex of Δ^n is

$$C_i(\Delta^n) = \begin{cases} C_i(\Delta^n) & \text{if } i \neq -1 \\ 0 & \text{if} i = -1 \end{cases}$$

and differential as in the reduced case, except $d_0 = 0$.

Proposition 5.6. $d^2 = 0$ in $C_i(\Delta^n)$.

Definition 5.7 (reduced chain complex of an abstract simplicial complex)
If K is an ASC in Δ^n, then we get a new chain complex with

$$\tilde{C}_i(K) = \langle e_I \mid e_I \in K, |I| = i+1 \rangle \leq \tilde{C}_i(\Delta^n)$$

Since K is an ASC, $de_I \in K$ for any $e_I \in K$, so we get $d : \tilde{C}_i(K) \to \tilde{C}_{i-1}(K)$.

We can define the chain complex of an ASC in the same way. Note both of these are chain complexes as they are subcomplexes of the chain complexes of Δ^n, and so we must have that $d^2 = 0$.

Definition 5.8 (cycle, closed, boundary, exact)
Given a chain complex (C_*, d), we say x is a cycle, or x is closed if $dx = 0$. We say x is a boundary, or x is exact if $x = dy$ for some y.

We write $Z_k(C_*) = \ker(d_k)$ for the subgroup of cycles, and $B_k(C_*) = \mathrm{im}(d_{k+1})$ for the subgroup of boundaries.

Remark 5.9. All boundaries are cycles as $d^2 = 0$.

Definition 5.10 (homology group of a chain complex)
Let (C_*, d) be a chain complex, then its k-th homology group is

$$H_k(C_*) = \frac{Z_k(C_*)}{B_k(C_*)}$$

Definition 5.11 ((reduced) homology group of a simplicial complex)
If K is an ASC in Δ^n, define the i-th reduced homology group of K to be

$$\tilde{H}_i(K) = H_i(\tilde{C}_*(K))$$

and the i-th homology group of K is

$$H_i(K) = H_i(C_*(K))$$

5.2 Chain maps and chain homotopies

5.3 Chain maps

Definition 5.12 (chain map)
Suppose (C_*, d) and (C'_*, d') are chain complexes. A chain map $f : C_* \to C'_*$ is

(i) For each i, a group homomorphism $f_i : C_i \to C'_i$,

(ii) such that the following diagram commutes.

$$\begin{array}{ccc} C_i & \xrightarrow{d_i} & C_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} \\ C'_i & \xrightarrow[d'_i]{} & C'_{i-1} \end{array}$$

Equivalently, if $f = \bigoplus_i f_i$, then $fd = d'f$.

Proposition 5.13 (functoriality)**.** Homology is functorial, that is, given a chain map $f : C_* \to C'_*$, we have an induced map $f_* : H_i(C_*) \to H_i(C'_*)$, given by $f_*[x] = [fx]$.

Proof. The only thing we need to check is that f_* is well defined. But $f(\ker(d)) \subseteq \ker(d')$ and $f(\mathrm{im}(d)) \subseteq \mathrm{im}(d')$. So f_* is well defined. ☐

5.3.1 Chain maps from simplicial maps

So far, we have simplices e_I, where $I = i_0 \dots i_k$ with $i_0 < \cdots < i_k$. We will now drop the assumption that $i_0 < \cdots < i_k$.

Note: The definitions in this section are different to the ones in the notes, which involve crossings of links and so on. This should be simpler and equivalent.

Definition 5.14 (orientation)
Let $I = (i_0, \dots, i_k) \in \{0, \dots, n\}^{k+1}$, with $i_0, \dots, i_k$ distinct. Then we define the orientation of I to be

$$S(I) = \varepsilon(f)$$

where $f \in S_n$ is the permutation sending $(i_0, \dots, i_k)$ to $(i'_0, \dots, i'_k)$, where $i'_0, \dots, i'_k$ are $i_0, \dots, i_k$ in increasing order.

Definition 5.15 (oriented simplex)
Let $I = (i_0, \dots, i_k)$ be as above, and $I' = i'_0 \dots i'_k$. Then we define the oriented simplex

$$e_I = S(I) e_{I'}$$

Remark 5.16. Note in the definition above we required the i_j distinct. If not, we define $e_I = 0$.

Proposition 5.17.

$$d(e_I) = \sum_{j=0}^{k} (-1)^j (e_{I_{\hat{j}}})$$

where $I_{\hat{j}}$ is obtainted by omitting the j-th entry of I.

Proof. The only thing we need to do here is to keep track of the signs, which follows from the definition of orientations. □

Definition 5.18 (induced chain map from simplicial map)
If $f : K \to L$ is a simplicial map, define $f_\# : C_*(K) \to C_*(L)$ by $f_\#(e_I) = e_{\hat{f}(I)}$.

5.3.2 Chain homotopies

Definition 5.19 (chain homotopy)
If $f, g : (C, d) \to (C', d')$ are chain maps, a chain homotopy from f to g is a map $h : C_* \to C'_{*+1}$ such that $d'h + hd = f - g$.

Lemma 5.20. If $f_0 \sim f_1$, then $f_{0*} = f_{1*} : H_*(C) \to H_*(C')$.

Proof. Suppose $x \in Z_*(C)$. Then $dx = 0$, so

$$f_{1*}[x] - f_{0*}[x] = [(f_1 - f_0)x] = [(d'h + hd)x] = [d'(hx)] = 0$$

□

Definition 5.21 (contractible)
We say that a chain complex (C, d) is contractible if $\mathrm{id}_C \sim 0_C$.

Lemma 5.22. If (C,d) is contractible, then $H_*(C) = 0$.

Proof. If $[x] \in H_*(C)$, $[x] = \mathrm{id}_*[x] = 0_*[x] = [0] = 0$. □

5.4 Homology groups of spheres

Definition 5.23 (cone of an ASC)
If K is an ASC in Δ^n, $e_0 \in K$, the cone $C_{e_0}(K)$ is

$$C_{e_0}(K) = K \cup \{e_{0I} \mid e_I \in K\}$$

Proposition 5.24. $C_{e_0}(K)$ is an ASC, and if $L = |K|$ is a realisation of K, then

$$|C_{e_0}(K)| = C_p(L)$$

for some p independent of L.

Proposition 5.25. Define $\hat{\Delta}^n = \left\{e_I \in \Delta^{n+1} \mid 0 \notin i\right\} \simeq \Delta^n$, then $\Delta^{n+1} = C_{e_0}(\hat{\Delta}^n)$.

Proposition 5.26. $\tilde{C}(C_{e_0}(K))$ is contractible.

Proof. Define $h : \tilde{C}_j(C_{e_0}(K)) \to \tilde{C}_{j+1}(C_{e_0}(K))$ by

$$h(e_I) = \begin{cases} 0 & \text{if } 0 \in I \\ e_{0I} & \text{if } 0 \notin I \end{cases}$$

If $0 \in I$, then $dh(e_I) = 0$, and

$$hd(e_I) = h\left(\sum_{j=0}^{k}(-1)^j e_{I_j}\right) = h\left(e_{I \setminus \{0\}} + \sum e_{I'}\right) = e_I$$

where I' are such that $0 \in I'$. On the other hand, if $0 \notin e_I$, then

$$dh(e_I) = d(e_{0I}) = e_I + \sum(-1)^{j+1} e_{0I_j} = e_I - hd(e_I)$$

In either case, $(dh + hd)(e_I) = e_I$. □

Corollary 5.27.

(i) $\tilde{H}(C_{e_0}(K)) = 0$,

(ii) $H_i(C_{e_0}(K)) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$

Proof. (i) is immediate from $\tilde{C}(C_{e_0}(K)$ being contractible. For (ii), when $i \neq 0, -1$ the homology and reduced homology groups are the same, so we only need to check these two cases.

Now $\tilde{H}_{-1}(C_{e_0}(K)) = 0$, so $d_0 : \tilde{C}_0 \to \tilde{C}_{-1} \simeq \mathbb{Z}$ is surjective. Then we have that

$$\mathbb{Z} \simeq \mathrm{im}(\tilde{d}_0) \simeq \frac{\tilde{C}_0}{\ker(\tilde{d}_0)} \simeq \frac{\tilde{C}_0}{\mathrm{im}(\tilde{d}_1)} = \frac{C_0}{\mathrm{im}(d_1)} = \frac{\ker(d_0)}{\mathrm{im}(d_1)} = H_0(C_{e_0}(K))$$

□

Corollary 5.28.

$$H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Proof. $S^n = \Delta^n \setminus \{e_{0\ldots n}\}$. □

5.5 Exact sequences and snake lemma

Definition 5.29

The sequence

$$\cdots \longrightarrow A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1} \longrightarrow \cdots$$

is exact at A_k if $\ker(f_k) = \operatorname{im}(f_{k+1})$. It is exact if it is exact at every A_k.

Definition 5.30 (short exact sequence)

A short exact sequence of is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Definition 5.31 (SES of chain complexes)

A SES of chain complexes is

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

are chain maps f, g such that $\operatorname{im}(f) = \ker(g)$, f injective and g surjective.

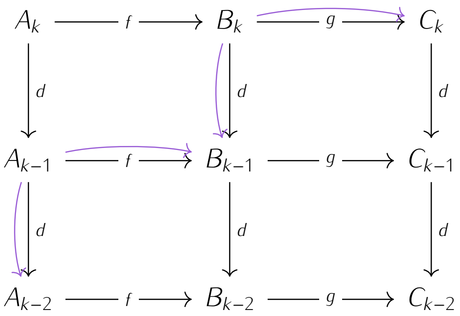
Lemma 5.32 (snake lemma)**.** If

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a SES of chain complexes, then we have a LES

$$\begin{array}{ccccc} & & & & \cdots \\ & & & \overset{\partial}{\swarrow} & \\ H_k(A) & \xrightarrow{f_*} & H_k(B) & \xrightarrow{g_*} & H_k(C) \\ & & \overset{\partial}{\swarrow} & & \\ H_{k-1}(A) & \xrightarrow{f_*} & H_{k-1}(B) & \xrightarrow{g_*} & H_{k-1}(C) \\ & & & \overset{\partial}{\swarrow} & \\ & & \cdots & & \end{array}$$

definition of ∂.



Given $c \in Z_k(C)$, we define $\partial[c]$ by

(i) As g is surjective, we have $b \in B_k$ such that $g(b) = c$.

(ii) $g(db) = d(gb) = dc = 0$, so $db \in \ker(g) = \operatorname{im}(f)$. Say $db = fa$ for some $a \in A_{k-1}$.

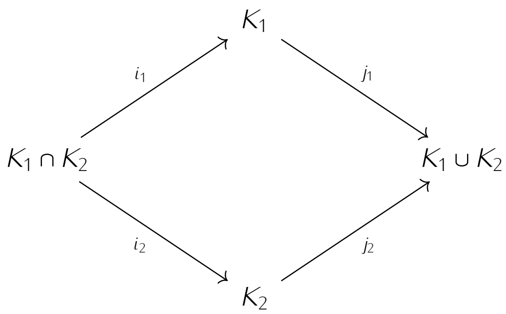
(iii) $f(da) = df(a) = d(db) = 0$. As f is injective, $da = 0$. So $a \in Z_{k-1}(A)$.

(iv) $\partial[c] = [a] \in H_{k-1}(A)$.

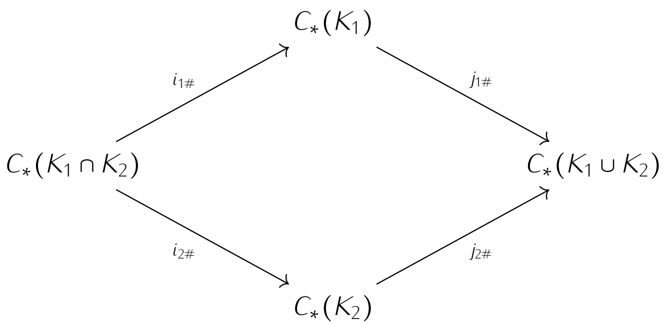
□

5.5.1 Mayer-Vietoris

If K_1, K_2 are ASCs in Δ^n, then $K_1 \cap K_2$ and $K_1 \cup K_2$ are both ASCs in Δ^n, and we have a squeare of simplicial maps given by inclusion.



By functoriality, we have a square at the level of chain complexes,



Proposition 5.33. Define maps $i : C_*(K_1 \cap K_2) \to C_*(K_1) \oplus C_*(K_2)$ and $j : C_*(K_1) \oplus C_*(K_2) \to C_*(K_1 \cup K_2)$ by

$$i(x) = (i_{1\#}(x), i_{2\#}(x)) \quad \text{and} \quad j(x, y) = j_{1\#}(x) - j_{2\#}(x)$$

Then

$$0 \longrightarrow C_*(K_1 \cap K_2) \xrightarrow{\ i\ } C_*(K_1) \oplus C_*(K_2) \xrightarrow{\ j\ } C_*(K_1 \cup K_2) \longrightarrow 0$$

is a SES of chain complexes.

Proof. It is easy to check exactness at each group. □

Theorem 5.34 (Mayer-Vietoris)**.** We have a long exact sequence in homology

$$\cdots \longrightarrow H_i(K_1 \cap K_2) \xrightarrow{\ i_*\ } H_i(K_1) \oplus H_i(K_2) \xrightarrow{\ j_*\ } H_i(K_1 \cup K_2) \xrightarrow{\ \partial\ } H_{i-1}(K_1 \cap K_2) \xrightarrow{\ i_*\ } H_{i-1}(K_1) \oplus H_{i-1}(K_2) \xrightarrow{\ j_*\ } H_{i-1}(K_1 \cup K_2) \longrightarrow \cdots$$

Proof. Snake lemma. □

Corollary 5.35. $H_i(K \sqcup L) = H_i(K) \oplus H_i(L)$ for all i.

5.6 Euler characteristic and Lefschetz fixed point theorem

Let (C, d) be a chain complex over $\mathbb{Q}$, so $H_*(C)$ is a $\mathbb{Q}$-vector space. $f : \mathbb{C} \to \mathbb{C}$ is a chain map, which induces $f_* : H_*(C) \to H_*(C)$. Both f, f_* are linear endomorphisms of a vector space.

Definition 5.36 (Lefschetz number)
The Lefschetz number of f is

$$L(f) = \sum_k (-1)^k \operatorname{tr}(f_k)$$

, and the Lefschetz number of f_* is

$$L(f_*) = \sum_k (-1)^k \operatorname{tr}(f_{k*})$$

.

Proposition 5.37. $L(f) = L(f_*)$.

Proof. Let $U_k = \operatorname{im}(d_{k+1}) \leq \ker(d_k) \leq C_k$. Then $\ker(d_k) = U_k \oplus V_k$, and $C_k = U_k \oplus V_k \oplus U_k'$. With this, $d : U_k' \to U_{k-1}$ is an isomorphism. With respect to this decomposition, we have that

$$d_k = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also, as $f(\operatorname{im}(d_{k+1})) \leq \operatorname{im}(d_{k+1})$ and $f(\ker(d_k)) \leq \ker(d_k)$, we have that

$$f_k = \begin{pmatrix} A_k & X_K & * \\ 0 & B_k & * \\ 0 & 0 & A_k' \end{pmatrix}$$

And $df = fd$ implies $A_{k-1} = A'_k$, so

$$H_k(C) = \frac{\ker(d_k)}{\operatorname{im}(d_{k+1})} = \frac{U_k \oplus V_k}{U_k} \simeq U_k$$

and by considering the matrix of the induced map on a quotient, we find that $f_* : H_k(C) \to H_k(C)$ is given by

$$f_*[v] = [B_k v]$$

Therefore,

$$L(f) = \sum (-1)^k \operatorname{tr}(f_k) = \sum (-1)^k (\operatorname{tr}(A_k) + \operatorname{tr}(B_k) + \operatorname{tr}(A_{k-1})) = \sum (-1)^k \operatorname{tr}(B_k) = L(f_*)$$

□

Definition 5.38 (Euler characteristic)
If K is a simplicial complex, $C = C_*(K)$, then define the Euler characteristic of K to be

$$\chi(K) = \chi(C) = L(\mathrm{id}_C) = \sum (-1)^k \dim(H_k(K)) = \sum (-1)^k \dim(C_k(K))$$

Proposition 5.39. The Euler characteristic is a topological invariant, depending only on $|K|$.

Theorem 5.40 (Lefschetz fixed point theorem)**.** Suppose $F : |K| \to |K|$ is continuous, $L(F) \neq 0$. Then F has a fixed point.

Proof. We prove the contrapositive. Suppose F has no fixed point. Then as $|K|$ is compact, there exists $\varepsilon > 0$ such that $\|F(x) - x\| \geq \varepsilon$ for all $x \in |K|$. If $f : B^{r+n}K \to B^rK$ is a simplicial approximation of F, then $F_*(\sigma)$ does not contain σ for any $\sigma \in B^{r+n}K$ if the mesh is $< \varepsilon$. So $L(F) = L(f) = 0$. □

6 Homology of triangulable spaces

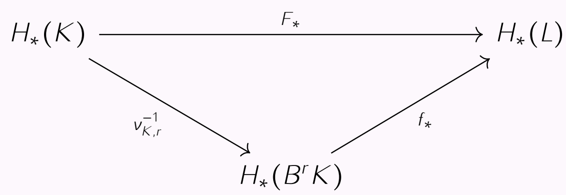
Theorem 6.1. If $f_0, f_1 : K \to L$ are simplicial approximations to $F : |K| \to |L|$, with $f_{0\#} \sim f_{1\#}$. Then $f_{0*} \sim f_{1*}$.

Theorem 6.2. There is an isomorphism $v_K : H_*(BK) \to H_*(K)$ such that $v_* = f_*$, where $f : BK \to K$ is any simplicial approximation to the identity map on $|BK| = |K|$. That is, we have

$$H_*(B^kK) \xrightarrow{v} H_*(B^{k-1}K) \xrightarrow{v} \cdots \xrightarrow{v} H_*(K)$$
$$H_*(B^kK) \xrightarrow{v_{K,r}} H_*(K)$$

Definition 6.3 (induced map on homology of a continuous map)
Suppose $F : |K| \to |L|$ is continuous, let $f : B^rK \to L$ be a simplicial approximation to F. Then define

$F_* : H_*(K) \to H_*(L)$ by



Theorem 6.4 (functoriality).

(i) F_* is well defined, that is, it does not depend on the choice of f,

(ii) $(\mathrm{id}_K)_* = \mathrm{id}_{H_*(K)}$,

(iii) $(F \circ G)_* = F_* \circ G_*$,

Theorem 6.5 (homotopy invariance). If $F_0, F_1 : |K| \to |L|$, with $F_0 \sim F_1$, then $F_{0*} \sim F_{1*}$.

Proposition 6.6 (homotopy invariance of homology). If $|K| \sim |L|$, then $H_*(K) \simeq H_*(L)$.

Proof. Suppose $F : |K| \to |L|$, $G : |L| \to |K|$ are such that $F \circ G \sim \mathrm{id}_{|L|}$ and $G \circ F \sim \mathrm{id}_{|K|}$. Then

$$F_* \circ G_* = (F \circ G)_* = \mathrm{id}_{|L|*} = \mathrm{id}_{H_*(L)}$$

and vice versa. □

Definition 6.7 (triangulable)
A topological space X is triangulable if there exists an ASC K with $|K| \simeq X$.

Proposition 6.8. If X is triangulable, then there is a well defined homology group $H_*(X) = H_*(K)$.

Proposition 6.9. If $|K|$ is path connected, then $H_0(K) \simeq \mathbb{Z}$.

Proof. As $|K|$ is path connected, if we define the maps $F_i : \Delta^0 \to |K|$, $F_i(e_0) = e_i$, then $F_i \sim F_j$. As $F_{i*}[e_0] = [e_i]$, $[e_i] = [e_j]$ for all i, j. □

Corollary 6.10.

$$H_0(K) = \mathbb{Z}^{\text{number of path components of } |K|}$$

6.1 Brouwer

Proposition 6.11. There is no retraction $r : D^n \to S^{n-1}$.

Proof. $S^{n-1} \hookrightarrow D^n \to S^{n-1}$, so if r is a retraction, $r \circ \iota = \text{id}$, so $(r \circ \iota)_* = r_* \circ \iota_* = \text{id}_{H_*(S^{n-1})}$. But this means we get

$$\begin{array}{ccccc} & & \overset{\text{id}}{\frown} & & \\ H_{n-1}(S^{n-1}) & \xrightarrow{\iota_*} & H_{n-1}(D^n) & \xrightarrow{r_*} & H_{n-1}(S^{n-1}) \\ \Vert & & \Vert & & \Vert \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

which is clearly false. □

Theorem 6.12 (Brouwer)**.** If $F : D^n \to D^n$ is continuous, then F has a fixed point.

Proof. Suppose not. Then consider the ray $\overline{f(x)x}$ for each $x \in D^n$. The intersection of this ray with $\partial D^n = S^{n-1}$ defines a retraction. Contradiction. □

6.2 Homology of surfaces

Theorem 6.13. If Σ_g is a genus g compact orientable surface, then

$$H_*(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 6.14. If S_r is the r-th compact non-orientable surface, obtained by gluing Möbius bands, then

$$H_*(S_r) = \begin{cases} \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2 & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Both proofs involve an inductive construction of the surfaces, and using the Mayer-Vietoris sequence. Note that the proofs given in lectures weren't the nicest, so we will also give a different proof afterwards.

6.2.1 Proof of the orientable case

Let K_1 be a triangulation of T^2, and $K_1' = K_1 \setminus \sigma$, where σ is a 2-simplex. Then $\partial K_1' = \partial\sigma = S^1$. Inductively, define

$$K_g = K_{g-1}' \cup_{S^1} K_1' \quad \text{and} \quad K_g' = K_g \setminus \sigma$$

where $\sigma \in K_g$ is a 2-simplex.

Proposition 6.15 ($p_1(g)$)**.**

$$H_*(K_g) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } * = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 6.16 ($p_2(g)$).

$$H_*(K_g') = \begin{cases} \mathbb{Z}^2 g & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Base case: $p_1(1)$. That is, the homology of the torus. This is an easy Mayer-Vietoris computation, where we take the square for T^2 and cut it into two vertical rectangles.

Inductive case (i): $p_1(g) \implies p_2(g)$. Using Mayer-Vietoris with

$$K_g = K_g' \cup \Delta^2$$

gives the required result.

Inductive case (ii): $p_2(g) \implies p_1(g+1)$. We will use the Mayer-Vietoris sequence with

$$K_{g+1} = K_g' \cup K_1'$$

gives the required result.

6.2.2 Proof of the non-orientable case

We will only go through the construction of the surfaces. The induction is the same as in the orientable case.

Let L_1 be a triangulation of $\mathbb{RP}^2$,

$$L_{r+1} = L_r' \cup_{S^1} L_1' \quad \text{and} \quad L_r' = L_r \setminus \sigma$$

as before. The inductive hypotheses are:

Proposition 6.17 ($q_1(r)$).

$$H_*(L_r) = \begin{cases} \mathbb{Z}^{r-1} \oplus \mathbb{Z}/2 & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 6.18 ($q_2(r)$).

$$H_*(L_r') = \begin{cases} \mathbb{Z}^r & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$$

6.2.3 Proof: Gluing a 2-cell onto a wedge of circles

Consider the standard gluing pattern of the (non-)orientable surfaces, i.e.

$$\Sigma_g = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$$

$$S_r = a_0 a_0 a_1 a_1 \cdots a_r a_r$$

Therefore, by Mayer Vietoris, if we glue a 2 cell onto a wedge of circles, we get

$$0 \longrightarrow H_2(\Sigma_g) \longrightarrow \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}^{2g} \longrightarrow H_1(\Sigma_g) \longrightarrow 0$$

where the map ϕ is given by

$$\phi(1) = [a_1] + [b_1] - [a_1] - [b_1] + \cdots + [a_g] + [b_g] - [a_g] - [b_g] = 0$$

which breaks the LES into two SES, and gives the required result.

In the non-orientable case, we have

$$0 \longrightarrow H_2(S_r) \longrightarrow \mathbb{Z} \xrightarrow{\psi} \mathbb{Z}^{g+1} \longrightarrow H_1(S_r) \longrightarrow 0$$

where ψ is given by

$$\psi(1) = 2([a_0] + [a_1] + \cdots + [a_g])$$

Then $H_2(S_r) = \ker(\psi) = 0$, and $H_1(S_r) = \operatorname{coker}(\psi) = \mathbb{Z}^r \oplus (\mathbb{Z}/2\mathbb{Z})$ by Smith normal form. Note the indexing here is off by one compared to the previous section, i.e. $S_r = L_{r+1}$.