Algebraic curves and compact Riemann surfaces

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Throughout, "curve" means smooth projective irreducible curve. In addition, the proofs here won't be totally rigorous, it's more about the connections between the two courses.

Theorem 0.1. Compact Riemann surfaces and algebraic curves (with the Euclidean topology) are the same thing.

Proof. It is easy to see that any algebraic curve is a Riemann surface, by the implicit function theorem on each affine patch. Furthermore, a curve is a closed subset of $\mathbb{P}^n$, which is compact.

The converse is much harder and so omitted. □

1 Local parameter, Laurent series and valuation

Let V be a curve, $P \in V$. We would like to consider the ring $\mathcal{O}_{V,P}$ of functions which are regular at P. Let

$$\mathfrak{m}_{V,P} = \{f \in \mathcal{O}_{V,P} \mid f(P) = 0\}$$

be the maximal ideal of functions vanishing at P. We know that $\mathfrak{m}_{V,P}$ is principal, generated by some π_P, which we call a local parameter.

However, $f \in \mathcal{O}_{V,P}$ also gives us a meromorphic function $f : V \to \mathbb{P}^1$, which is holomorphic at P. Therefore, after taking a chart, we have a Taylor expansion

$$f(z) = z^{m_f(P)} g(z)$$

where g is holomorphic and nonzero on a neighbourhood of 0.

Analogy: The local parameter π_P is just z, from the Taylor expansion.

In fact, if we choose appropriate charts on $\mathbb{P}^1$, then we have that

$$f(z) = z^{m_f(P)}$$

Analogy: The two definitions of the valuation of f at P are the same, they are just the order of the zero, or pole of f at P.

2 Degree and ramification

Now let $f : V \to W$ be a morphism of curves, which by analogy, is an analytic function between compact Riemann surfaces.

By standard Riemann surface theory, if $P \in V$, $Q = f(P) \in W$, then we have charts around P, Q respectively, such that locally, we have

$$f(z) = z^{m_f(P)}$$

The ramification degree in algebraic geometrty is

$$e_P = v_P(f^* \pi_Q)$$

where π_Q is a local parameter at Q. But recall that $f^* \pi_Q = \pi_Q \circ f$. Therefore, if we have local coordinates w near Q and z near P, then what we have is just $w = f(z)$.

Analogy: The two definitions of the ramification degree agree.

Therefore, it should be unsurprising that we have what is called the finiteness theorem in Algebraic geometry, and the valency theorem for Riemann surfaces, which says

$$\deg(f) = \sum_{P \in f^{-1}(Q)} e_P$$

for all $Q \in V$.

Also with this analogy, the algebraic geometry statement that the set of P such that $v_P(f) \neq 0$ is finite follows immediately from compactness and complex analysis. In addition, in both cases we have that a non-constant morphism or analytic map $f : V \to W$ is surjective.

3 Differentials and divisors

With what we have so far, and the fact that all $\omega \in \Omega_{V,P}$ can be written as $\omega = f \mathrm{d}\pi_P$, where $f \in \mathcal{O}_{V,P}$, it should be unsurprising that we have

Analogy: $\mathrm{d}\pi_P$ is the same as the (complex) differential 1-form $\mathrm{d}z$.

4 Riemann-Roch and Riemann-Hurwitz

We don't have the Riemann-Roch theorem for Riemann surfaces in the Part II course, but a very similar statement holds if we replace all occurences of "rational function" by "meromorphic function".

Therefore, we should expect that Riemann-Hurwitz holds in both cases, that is,

$$2g(V) - 2 = \deg(f)(2g(W) - 2) + \sum_{P \in V} (e_P - 1)$$

holds for both $f : V \to W$ analytic, and also $f : V \to W$ morphism of curves.