Analysis of functions

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1 Measure theory and integration

1.1 L^p spaces

Definition 1.1 (convolution)

For $f, g : \mathbb{R}^n \to \mathbb{R}$, the convolution of f, g is $f * g : \mathbb{R}^n \to \mathbb{R}$, defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y$$

whenever the integral exists.

Proposition 1.2.

(i) $f * g = g * f$,

(ii) $(f * g) * h = f * (g * h)$,

(iii) and if $f(x-y)g(y)$ is $\mathrm{d}y\mathrm{d}x$ measurable, then

$$\int_{\mathbb{R}^n} (f * g)(x)\mathrm{d}x = \left(\int_{\mathbb{R}^n} f(x)\mathrm{d}x\right)\left(\int_{\mathbb{R}^n} g(y)\mathrm{d}y\right)$$

Definition 1.3 ($L^p_{\text{loc.}}(\mathbb{R}^n)$)
For $f : \mathbb{R}^n \to \mathbb{R}$, $f \in L^p_{\text{loc.}}(\mathbb{R}^n)$ if $f 1_K \in L^p(\mathbb{R}^n)$ for all compact $K \subseteq \mathbb{R}^n$.

Proposition 1.4. Let $f \in L^1_{\text{loc.}}(\mathbb{R}^n)$, $g \in C^k_c(\mathbb{R}^n)$, then $f * g \in C^k(\mathbb{R}^n)$, with

$$\nabla^\alpha(f * g) = f * (\nabla^\alpha g)$$

for all $|\alpha| \leq k$.

Proof. **Case** $k = 0$. Define the translation

$$T_z f(x) = f(x - z)$$

then we have the following properties.

- $T_z(f * g) = f * T_z g$, which follows by the definition of convolution.
- $T_z g \to g$ pointwise as $z \to 0$, as g is continuous,
- $|T_z g(x)| \leq \|g\|_{L^\infty} 1_{B_R(0)}(x)$, for $\|z\| \leq 1$, $\|x\| + 1 < R$ and $\text{supp}(g) \subseteq B_R(0)$, which follows from g being compactly supported.

With all of this, we get that

$$|f(y) T_z g(x - y)| \leq \|g\|_{L^\infty} |f(y)| 1_{B_R(0)}(x - y)$$

which is an integrable function of y as $f \in L^1_{\text{loc.}}$. Thus, by the dominated convergence therem, we have that

$$T_z(f * g)(x) = (f * T_z g)(x) = \int_{\mathbb{R}^n} f(y) T_z g(x - y)\mathrm{d}y \to \int_{\mathbb{R}^n} f(y) g(x - y)\mathrm{d}y = (f * g)(x)$$

So $f * g \in C^0$.
Case $k = 1$. Define the difference quotient

$$\Delta_i^h g(x) = \frac{g(x + he_i) - g(x)}{h}$$

Then by the definition of the partial derivatives, we have that for all $x \in \mathbb{R}^n$,

$$\Delta_i^h g(x) \to \nabla_i g(x)$$

as $h \to 0$. Fix x, h. Then by the mean value theorem, there exists t with $|t| < |h|$ such that

$$\Delta_i^h g(x) = \nabla_i g(x + te_i)$$

which means that

$$\left|\Delta_i^h g(x)\right| \leq \|\nabla_i g(x)\|_{L^\infty} 1_{B_R(0)}(x)$$

as before. Applying the dominated convergence theorem, we get that

$$\Delta_i^h(f * g) = f * (\Delta_i^h) \to f * \nabla_i g$$

which means that $f * g \in C^1$, with derivative given by the above.

Case $k > 1$. In this case, suffices to notice that by the $k = 1$ case, we have for $|\alpha| \le k-1$,

$$\nabla^\alpha \nabla_j (f * g) = \nabla^\alpha (f * \nabla_j g)$$

But $f \in L^1_{\mathrm{loc.}}$ and $\nabla_j g \in C_c^{k-1}$, so by the $k-1$ case, we have that

$$\nabla^\alpha (f * \nabla_j g) = f * (\nabla^\alpha \nabla_i g)$$

□

Theorem 1.5 (Minkowski's inequality)**.** Let $p < \infty$, $f : \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}$ be Borel. Then

$$\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x,y) \mathrm{d}x \right|^p \mathrm{d}y \right)^{1/p} \le \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x,y)|^p \right)^{1/p} \mathrm{d}x$$

Proposition 1.6. Let $p < \infty$, $g \in L^p(\mathbb{R}^n)$, then

$$\|T_z g - g\|_{L^p} \to 0$$

as $z \to 0$.

Proof. If $g = 1_R$, where R is a rectanle, then the result is clearly true. Similarly, the result is clearly true for a finite union of (disjoint) rectangles. Now let B be a Borel set with $|B| < \infty$. Then for all $\varepsilon > 0$, there exists a finite union of rectangles $R = R_1 \cup \cdots \cup R_m$, such that

$$\|1_B - 1_R\|_{L^p} = |R \Delta B|^{1/p} < \varepsilon$$

Which means that

$$\|T_z 1_B - 1_B\|_p \le \underbrace{\|T_z 1_B - T_z 1_R\|_p}_{=\|1_B - 1_R\| < \varepsilon \text{ by translation invariance}} + \underbrace{\|T_z 1_R - 1_R\|_p}_{<\varepsilon \text{ for } z \text{ small}} + \underbrace{\|1_R - 1_B\|_p}_{<\varepsilon} < 3\varepsilon \text{ for } z \text{ small}$$

Thus the result holds for $g = 1_B$, and hence the result holds for simple functions. Finally, for $g \in L^p$, then there exists $\tilde{g}$ simple such that $\|g - \tilde{g}\|_p < \varepsilon$. Thus

$$\|T_z g - g\|_p \le \|T_z g - T_z \tilde{g}\|_p + \|T_z \tilde{g} - \tilde{g}\|_p + \|g - \tilde{g}\|_p < 3\varepsilon$$

for z small enough. □

Definition 1.7 (smooth molifier)

$\varphi \in C_c(\mathbb{R}^n)$ is called a smooth molifier if $\varphi \ge 0$, and $\int \varphi \mathrm{d}x = 1$.

Theorem 1.8. Smooth molifiers exist. Furthermore, if φ is a smooth mollifier, set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then for $g \in L^p(\mathbb{R}^n)$, $p < \infty$, we have that

$$\varphi_\varepsilon * g \in C^\infty(\mathbb{R}^n) \quad \text{and} \quad \varphi_\varepsilon * g \to g \text{ in } L^p \text{ as } \varepsilon \to 0$$

Proof. First, notice that we have

$$\begin{aligned}|\varphi_\varepsilon * g(x) - g(x)| &= \left|\int_{\mathbb{R}^n} \varphi_\varepsilon(y)\,(g(x-y) - g(x))\mathrm{d}y)\right| \\ &= \left|\int_{\mathbb{R}^n} \varphi(z)\,(g(x-\varepsilon z) - g(z))\,\mathrm{d}z\right| \\ &\le \int_{\mathbb{R}^n} \varphi(z)|T_{\varepsilon z} g(x) - g(x)|\mathrm{d}z\end{aligned}$$

Using the above, and Minkowski's integral inequality, we get that

$$\begin{aligned}\|\varphi_\varepsilon * g - g\|_{L^p} &= \left(\int_{\mathbb{R}^n} |\varphi_\varepsilon * g - g|^p \mathrm{d}x\right)^{1/p} \\ &\le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \varphi(z)^p |T_{\varepsilon z} g(x) - g(x)|^p \mathrm{d}x\right)^{1/p} \mathrm{d}z \\ &= \int_{\mathbb{R}^n} \varphi(z)\|T_{\varepsilon z} g - g\|_{L^p}\mathrm{d}z & \to 0 \text{ as } \varepsilon \to 0\end{aligned}$$

by the dominated convergence theorem. □

Corollary 1.9. $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $p < \infty$.

Proof. The theorem implies that $C^\infty \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. But since $\left\|f - f1_{B_R(0)}\right\|_{L^p} \to 0$ as $R \to \infty$ by the dominated convergence theorem. Furthermore, f, g are compactly supported, then so is $f * g$. Hence we have that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. □

1.2 Lebesgue differentiation theorem

Notation 1.10 (average). Define the average of a function f on a measurable set A with finite measure by

$$⨍_A f\mathrm{d}x = \frac{1}{|A|}\int_A f\mathrm{d}x$$

Definition 1.11 (Hardy-Littlewood maximal function)
For $f \in L^1(\mathbb{R}^n)$, the Hardy-Littlewood maximal function $Mf : \mathbb{R}^n \to [0, \infty]$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{B_r(x)}\int_{B_r(x)} |f(y)|\mathrm{d}y = \sup_{r>0} ⨍_{B_r(x)} |f(y)|\mathrm{d}y$$

Proposition 1.12. For $f \in L^1(\mathbb{R}^n)$, then Mf is a Borel function which is finite a.e. and

$$|\{Mf > \lambda\}| \le \frac{3^n}{\lambda}\|f\|_{L^1}$$

Proof. Define $A_\lambda = \{Mf > \lambda\}$. For $x \in A_\lambda$, there exists $r_x > 0$ such that

$$⨍_{B_{r_x}(x)} |f(y)|\mathrm{d}y > \lambda$$

To show that Mf is measurable, suffices to show $A_\lambda = (Mf)^{-1}((\lambda, \infty])$ is open. Equivalently, A_λ^C is closed. Let (x_k) be a sequence in A_λ^C, and say $x_k \to x$. Now suppose if $x \in A_\lambda$. Then by the dominated convergence theorem, we have that

$$⨍_{B_{r_x}(x_k)} |f(y)|\mathrm{d}y \to ⨍_{B_{r_x}(x)} |f(y)|\mathrm{d}y$$

Since $x_k \notin A_\lambda$ for all k, the left hand side is $\leq \lambda$ for all k. Hence in the limit, the right hand side is $\leq \lambda$. Contradiction as $x \in A_\lambda$. Thus $x \in A_\lambda^C$, so A_λ^C is closed.

For the inequality, by the inner regularity of the Lebesgue measure, suffices to prove it for $K \subseteq A_\lambda$ compact. Now $\{B_{r_x}(x) \mid x \in K\}$ is an open cover of K, so by compactness we have a finite subcover. Write $B_i = B_{r_i}(x_i)$, and we have $K \subseteq B_1 \cup \cdots \cup B_N$.

By Wiener's covering lemma, there exists a finite subcollection B_{i_k} which are pairwise disjoint, and with

$$\bigcup_i B_i \subseteq \bigcup_k 3B_{i_k}$$

where $3B_{i_k} = B_{3r_{i_k}}(x_{i_k})$. This means that we have that

$$|K| \leq \left|\bigcup_i B_i\right| \leq 3^n \sum_k |B_{i_k}| \leq \frac{3^n}{\lambda} \sum_k \int_{B_{i_k}} |f(y)| \mathrm{d}y \leq \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(y)| \mathrm{d}y = \frac{3^n}{\lambda} \|f\|_{L^1}$$

□

Theorem 1.13 (Lebesgue differentiation theorem)**.** Let $f \in L^1(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \fint_{B_r(x)} |f(y) - f(x)| \mathrm{d}y = 0$$

for almost every x. The x such that the limit exists are called Lebesgue points of f.

Proof. Let

$$A_\lambda = \left\{ x \in \mathbb{R}^n \mid \limsup_{r \to 0} \fint_{B_r(x)} |f(y) - f(x)| \mathrm{d}y > 2\lambda \right\}$$

Suffices to show that $|A_\lambda| = 0$ for all $\lambda > 0$, since the set of non-Lebesgue points are $\bigcup_n A_{1/n}$, which would then have measure zero by countable subadditivity.

Given $\varepsilon > 0$, take $g \in C_c^\infty(\mathbb{R}^n)$ with $\|f - g\|_{L^1} < \varepsilon$. Then

$$\fint_{B_r(x)} |f(y) - f(x)| \mathrm{d}y \leq \underbrace{\fint_{B_r(x)} |f(y) - g(y)| \mathrm{d}y}_{\leq M(f-g)(x)} + |f(x) - g(x)| + \underbrace{\fint_{B_r(x)} |g(x) - g(y)| \mathrm{d}y}_{\to 0 \text{ as } r \to 0 \text{ since } g \in C_c^\infty}$$

So we have that

$$\limsup_{r \to 0} \fint_{B_r(x)} |f(y) - f(x)| \mathrm{d}y \leq M(f-g)(x) + |f(x) - g(x)|$$

If $x \in A_\lambda$, then we must have that $M(f-g)(x) > \lambda$ or $|f(x) - g(x)| > \lambda$. But we have that

$$|\{M(f-g) > \lambda\}| \leq \frac{3^n}{\lambda} \|f - g\|_{L^1}$$

and by Markov's inequality, we have that

$$|\{|f - g| > \lambda\}| \leq \frac{\|f - g\|_{L^1}}{\lambda}$$

Hence we have that

$$|A_\lambda| \leq \frac{3^n + 1}{\lambda} \|f - g\|_{L^1} < \frac{3^n + 1}{\lambda} \varepsilon$$

But $\varepsilon > 0$ was arbitrary, so we are done. □

1.3 Egorov, Lusin

Theorem 1.14 (Egorov)**.** Let $E \subseteq \mathbb{R}^n$ measurable, $|E| < \infty$, $f_k : E \to \mathbb{R}$ measurable such that $f_k \to f$ a.e. Then for any $\varepsilon > 0$, there exists a closed subset $A_\varepsilon \subseteq E$ such that $|E \setminus A_\varepsilon| < \varepsilon$ and $f_k \to f$ uniformly on A_ε.

Proof. Without loss of generality, we may assume $f_k(x) \to f(x)$ on all of E. Then define

$$E_{k,n} = \left\{ x \in E \;\middle|\; \left|f_j(x) - f(x)\right| < \frac{1}{n} \text{ for all } j > k \right\}$$

Then $E_{k,n} \subseteq E_{k+1,n}$ for all k, and as the f_k converges pointwise, $\bigcup_k E_{k,n} = E$. So $|E_{k,n}| \nearrow |E|$ as $k \to \infty$. Let k_n be such that $|E \setminus E_{k_n,n}| < 2^{-n}$. Then set

$$A_N = \bigcap_{n \geq N} E_{k_n,n}$$

We have that

$$|E \setminus A_N| \leq \sum_{n \geq N} |E \setminus E_{k_n,n}| \leq 2^{-N+1} \to 0 \text{ as } N \to \infty$$

Therefore suffices to show that $f_j \to f$ uniformly on A_N. For $x \in A_N$, and any $n \geq N$, $x \in E_{k_n,n}$, so we have that

$$\left|f_j(x) - f(x)\right| < \frac{1}{n} \text{ for all } j > k_n$$

So we must have that

$$\limsup_{j \to \infty} \sup_{x \in A_N} \left|f_j(x) - f(x)\right| \leq \frac{1}{n}$$

But $n \geq N$ was arbitrary, so we are done. □

Theorem 1.15 (Lusin)**.** Let $f : E \to \mathbb{R}$ be Borel, where $E \subseteq \mathbb{R}^n$, with $|E| < \infty$. Then for every $\varepsilon > 0$, there exists $F_\varepsilon \subseteq E$ closed, such that $|E \setminus F_\varepsilon| < \varepsilon$, and $f|_{F_\varepsilon}$ is continuous.

Remark 1.16. This does not imply f itself is continuous at all $x \in F_\varepsilon$, since F_ε has the subspace topology.

Proof. **Step 1: The statement is true if f is simple**[1]**.** Let $f = \sum_{m=1}^{M} a_m 1_{A_m}$, where the A_m are disjoint sets of finite measure. Without loss of generality, we can assume $\bigcup_m A_m = E$.

Fix $\varepsilon > 0$. By the inner regularity of the Lebesgure measure, there exists compact sets $K_m \subseteq A_m$ with $|A_m \setminus K_m| < \varepsilon/M$. Then define

$$F_\varepsilon = \bigcup_m K_m$$

which means that $|E \setminus F_\varepsilon| < \varepsilon$. Since f is constant on each K_m, K_m compact and $d(K_m, K_\ell) > 0$ for all $m \neq \ell$. So f is continuous on F_ε.

Step 2: The result for measurable f. First take simple functions $f_n \to f$ a.e.. By step 1, we have $C_n \subseteq E$ be such that $|C_n| < 2^{-n}$, $f_n|_{E \setminus C_n}$ continuous. Fix $\varepsilon > 0$. By Egorov, there exists a set A_ε such that $f_n \to f$ uniformly on A_ε and $|E \setminus A_\varepsilon| < \varepsilon$. Set

$$F_{\varepsilon,N} = A_\varepsilon \setminus \bigcup_{m \geq N} C_m$$

Then $|E \setminus F_{\varepsilon,N}| < 2\varepsilon$ if N is sufficiently large. Since for $n \geq N$, $f_n|_{F_{\varepsilon,N}}$ are all continuous, and $f_n \to f$ uniformly on $F_{\varepsilon,N}$, $f : F_{\varepsilon,N}$ is continuous. Finally, $F_{\varepsilon,N}$ is not necessarily closed, but by the regularity of the Lebesgue measure we can choose $F_\varepsilon \subseteq F_{\varepsilon,N}$ closed, with $|F_{\varepsilon,N} \setminus F_\varepsilon| < \varepsilon$. So

$$|E \setminus F_\varepsilon| < 3\varepsilon$$

with $f|_{F_\varepsilon}$ continuous, and F_ε closed. □

[1] In this course, simple does not necessarily mean nonnegative. We allow negative scalars.

2 Banach and Hilbert spaces

Notation 2.1 (Inner product)**.** In this course, for the inner product (of two functions in L^2), we use the convention that it is conjugate linear in the first argument, and linear in the second argument. That is,

$$\langle f, g\rangle_{L^2} = \int_E \overline{f} g \mathrm{d}\mu$$

2.1 Radon-Nikodym

Definition 2.2 (absolutely continuous)

Let $(E, \mathcal{E})$ be a measurable space, μ, ν measures on $(E, \mathcal{E})$. Then ν is absolutely continuous with respect to μ, written $\nu \ll \mu$ if for all $A \in \mathcal{E}$ with $\mu(A) = 0$, $\nu(A) = 0$.

Definition 2.3 (mutually singular)

Let $(E, \mathcal{E})$ be a measurable space, μ, ν measures on $(E, \mathcal{E})$. Then μ and ν are mutually singular, written $\mu \perp \nu$ if there exists $B \in \mathcal{E}$ such that $\mu(B) = \nu(B^C) = 0$.

Theorem 2.4 (Radon-Nikodym)**.** Let μ, ν be finite measures on $(E, \mathcal{E})$, with $\nu \ll \mu$. Then there exists $\omega \in L^1(E, \mathcal{E}, \mu)$ such that for all $A \in \mathcal{E}$,

$$\nu(A) = \int_A \omega \mathrm{d}\mu$$

or equivalently, for all $h : E \to [0, \infty]$ Borel,

$$\int_E h \mathrm{d}\nu = \int_E h\omega \mathrm{d}\mu$$

Proof. **Step 1: Riesz representation.** Set $\alpha = \mu + 2\nu$, $\beta = 2\mu + \nu$. Then α and β are finite measures. Define the functional

$$\Lambda(f) = \int_E f \mathrm{d}\beta$$

Then

$$|\Lambda(f)| \leq \int_E |f| \mathrm{d}\beta \leq 2 \int_E |f| \mathrm{d}\alpha \leq 2\sqrt{\alpha(E)} \|f\|_{L^2(\alpha)}$$

So $\Lambda : L^2(\alpha) \to \mathbb{R}$ is a bounded linear map. Hence by the Riesz representation theorem, there exists $g \in L^2(\alpha)$ such that $\Lambda(f) = \langle g, f\rangle_{L^2(\alpha)}$ for all $f \in L^2(\alpha)$. This means that

$$\Lambda(f) = \int f \mathrm{d}\beta = \int g f \mathrm{d}\alpha \iff \int_E f(2-g) \mathrm{d}\nu = \int_E f(2g-1) \mathrm{d}\mu \qquad (*)$$

Step 2: $g \geq 1/2$ **μ-a.e. and ν-a.e.** Set

$$f = 1_{A_j} \quad \text{where} \quad A_j = \left\{ x \in E \,\middle|\, g(x) < \frac{1}{2} - \frac{1}{j} \right\}$$

This gives us that

$$\int_E f(2g-1) \mathrm{d}\nu \leq \frac{-2}{j} \nu(A_j) \quad \text{and} \quad \int_E f(2-g) \mathrm{d}\mu \geq \frac{3}{2} \mu(A_j)$$

Which means that $\nu(A_j) = \mu(A_j) = 0$ for all j, so $g \geq \frac{1}{2}$ μ-a.e. and ν-a.e.
Step 3: $\mu(\{g = 1/2\}) = 0$**.** Set $f = 1_Z$, where $Z = \{g = 1/2\}$, to get that

$$\frac{3}{2}\mu(Z) = 0 \implies \mu(Z) = 0$$

Step 4: $g \leq 2$ μ-a.e. and ν-a.e. Set

$$A_j = \left\{ x \in E \;\middle|\; g(x) > 2 + \frac{1}{j} \right\}$$

and proceed as in step 2.

Step 5: Defining ω. By the monotone convergence theorem, we can extend (*) to all $f \geq 0$. Now given h as in the statement, set

$$f(x) = \frac{h(x)}{2g(x) - 1} \quad \text{and} \quad \omega(x) = \frac{2 - g(x)}{2g(x) - 1}$$

where $f(x) < \infty$ μ-a.e. by step 3. Then we have that

$$\int_E h \mathrm{d}\nu = \int_E f(2g - 1) \mathrm{d}\nu = \int_E f(2 - g) \mathrm{d}\mu = \int_E h\omega \mathrm{d}\mu$$

In particular, setting $h = 1$ and using the fact that $\mu(E) < \infty$, $\omega \in L^1(\mu)$. □

2.2 Dual of L^p

Notation 2.5. Let $q \in [1, \infty]$, then for every $g \in L^q(\mathbb{R}^n)$, define the function $\Lambda_g \in L^p(\mathbb{R}^n) \to \mathbb{F}$ by

$$\Lambda_g(f) = \int fg \mathrm{d}x$$

Notation 2.6. We denote $X' = \mathcal{B}(X, \mathbb{F})$ for the dual of X.

Proposition 2.7. $\Lambda_g \in (L^p(\mathbb{R}^n))'$ and $\|\Lambda_g\|_{(L^p)'} = \|g\|_{L^q}$.

Proof. By Hölder's inequality, we have that $|\Lambda_g(f)| \leq \|f\|_{L^p} \|g\|_{L^q}$, so $\Lambda_g \in (L^p(\mathbb{R}^n))'$. Furthermore, equality holds since we have that

$$\|g\|_{L^q} = \int |g(x)| h(x) \mathrm{d}x \quad \text{where} \quad h(x) = \frac{|g|^{q-1}}{\|f\|_{L^q}^{q-1}}$$

and $h \in L^p$, with $\|h\|_{L^p} = 1$. □

Corollary 2.8. The map $J : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)'$ given by $g \mapsto \Lambda_g$ is a linear isometry.

Definition 2.9 (positive opertor)
$\Lambda \in L^p(\mathbb{R}^n)'$ is positive if $\Lambda(f) \geq 0$ for all $f \in L^p(\mathbb{R}^n)$ with $f \geq 0$ a.e.

Lemma 2.10. Let $\Lambda \in L^p(\mathbb{R}^n)'$ be positive. Then there exists $g \in L^q(\mathbb{R}^n) \geq 0$ such that $\Lambda = \Lambda_g$. That is,

$$\Lambda(f) = \int fg \mathrm{d}x$$

for all $f \in L^p(\mathbb{R}^n)$.

Proof. Let $\mathrm{d}\mu = e^{-|x|^2} \mathrm{d}x$ be the Gaussian measure, then $\mu(\mathbb{R}^n) = 1 < \infty$. Define

$$\nu(A) = \Lambda(e^{-|x|^2/p} 1_A)$$

To check that ν is a finite measure, clerly $\nu(\emptyset) = 0$ and $\nu(\mathbb{R}^n) < \infty$. Let $(A_k) \subseteq \mathcal{B}(\mathbb{R}^n)$ be disjoint, and set $B_m = \bigcup_{k=1}^m A_k$. Then

$$|\nu(B_\infty) - \nu(B_m)| \leq \|\Lambda\| \left\| e^{-|x|^2} \right\|_{L^p} \leq \|\Lambda\| \mu(B_\infty - B_m)^{1/p} \to 0$$

as $m \to \infty$, so ν is countably additive. Furthermore, if $\mu(A) = 0$, then $|\nu(A)| \leq \|\Lambda\| \mu(A)^{1/p} = 0$, so $\nu \ll \mu$. Therefore, by Radon-Nikodym, we have $\omega \in L^1(\mu)$ nonnegative such that

$$\nu(A) = \int_A \omega \mathrm{d}\mu = \int_A \omega e^{-|x|^2} \mathrm{d}x$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$. Now let $\tilde{f}$ be simple, and $f = e^{-|x|^2/p} \tilde{f}$. Then by linearity of Λ, we have that

$$\Lambda(f) = \int \tilde{f} \mathrm{d}\nu = \int \tilde{f} \omega e^{-|x|^2} \mathrm{d}x = \int f \tilde{\omega} \mathrm{d}x \quad \text{where} \quad \tilde{\omega} = \omega e^{-|x|^2/q}$$

Thus, $\Lambda(f) = \int f \tilde{\omega} \mathrm{d}x$ for all f of the above form, which are dense in L^p, so we are done. Remains to show that $\tilde{\omega} \in L^q$. But

$$\|\tilde{\omega}\|_{L^q} = \sup \left\{ \int |f \tilde{\omega}| \mathrm{d}x \;\middle|\; \|f\|_{L^p} \leq 1 \right\} \leq \|\Lambda\| < \infty$$

In fact, equality holds by Hölder. □

Theorem 2.11. If $p < \infty$ then J is surjective. That is, $J : L^q(\mathbb{R}^n) \simeq L^p(\mathbb{R}^n)'$.

Proof of real valued case. Given $\Lambda \in L^p(\mathbb{R}^n)'$ real valued, there exists $\Lambda_\pm$ bounded positive such that $\Lambda = \Lambda_+ - \Lambda_-$. The result then follows by the lemma. □

Proof of the complex valued case. If $\Lambda \in L^p(\mathbb{R}^n)'$ is complex valued, then $\Lambda_r(f) = \mathfrak{R}(\Lambda(f))$ and $\Lambda_i(f) = \mathfrak{I}(\Lambda(f))$ are $\mathbb{R}$-linear, such that

$$\Lambda(f_r + f_i) = \Lambda_r(f_r) - \Lambda_i(f_i) + i\Lambda_r(f_i) + i\Lambda_i(f_r)$$

and the result then follows from the real valued case. □

2.3 Riesz-Markov

Theorem 2.12 (Riesz-Markov)**.** Given $\Lambda : C_c(\mathbb{R}^n) \to \mathbb{R}$ positive bounded linear, there exists a unique finite Borel measure μ on $\mathbb{R}^n$ such that

$$\Lambda(f) = \int_{\mathbb{R}^n} f \mathrm{d}\mu$$

for all $f \in C_c(\mathbb{R}^n)$.

Definition 2.13 (signed measure)

A signed measure is the difference of two mutually singular positive finite measures.

Corollary 2.14. The dual space of $C_c(\mathbb{R}^n)$ with the L^1 norm is the space of signed measures on $\mathbb{R}^n$.

2.4 Hahn-Banach

Definition 2.15 (sublinear)

Let X be a real vector space, then $p : X \to \mathbb{R}$ is sublinear if

(i) $p(x+y) \leq p(x) + p(y)$,

(ii) $p(tx) = tp(x)$ for all $x \in X, t > 0$.

Lemma 2.16 (codimension 1 case of Hahn-Banach)**.** Let X be a real vector space, $p : X \to \mathbb{R}$ sublinear, $M \subseteq X$ is a subspace. Suppose $\ell : M \to \mathbb{R}$ is linear, with $\ell(y) \leq p(y)$ for all $y \in M$. Then for $x \in X \setminus M$, let $\tilde{M} = M \oplus \operatorname{span}\{x\}$. Then there is an extension $\tilde{\ell} : \tilde{M} \to \mathbb{R}$ linear, such that $\tilde{\ell}|_M = \ell$, and $\ell(z) \leq p(z)$ for all $z \in \tilde{M}$.

Proof. If $z \in \tilde{M}$, then there exists unique $y \in M$ and $\lambda \in \mathbb{R}$ such that $z = y + \lambda x$. Therefore, by linearity, suffices to define $a = \tilde{\ell}(x)$. Let

$$a = \sup\{\ell(y) - p(y-x) \mid y \in M\}$$

As $0 \in M$, we only need to check that it is bounded above. For $y, z \in M$, we have that

$$\ell(y) + \ell(z) = \ell(y = z) \leq p(y+z) \leq p(y-x) + p(z+x)$$

Hence $\ell(y) - p(y-x) \leq p(z+x) - \ell(z)$, so the sup is well defined, with $\ell(y) - a \leq p(y-x)$ for all $y \in M$. In addition, we have that

$$\ell(z) + a \leq p(z+x) + p(y-x) - \ell(y) + a$$

for all $y, z \in M$. Taking the infimum over all y, we find that

$$\ell(z) + a \leq p(z+x) - \sup_y\{\ell(y) - p(y-x)\} + a = p(z+x)$$

Therefore, we find that

$$\tilde{\ell}(y+\lambda x) = \ell(y) + \lambda a = \begin{cases} \lambda\left(\ell\left(\frac{1}{\lambda}y\right) + a\right) \leq \lambda p\left(\frac{1}{\lambda}y + x\right) = p(y+\lambda x) & \text{if } \lambda > 0 \\ |\lambda|\left(\ell\left(\frac{1}{|\lambda|}y\right) - a\right) \leq |\lambda| p\left(\frac{1}{|\lambda|}y - x\right) = p(y+\lambda x) & \text{if } \lambda < 0 \end{cases}$$

Hence $\tilde{\ell}(z) \leq p(z)$ for all $z \in \tilde{M}$. □

Corollary 2.17. If $\dim(X/M) < \infty$, then any $\ell : M \to \mathbb{R}$ with $\ell(y) \leq p(y)$ for all $y \in M$ can be extended to $\tilde{\ell} : X \to \mathbb{R}$ with $\ell(x) \leq p(x)$ for all $x \in X$.

Theorem 2.18 (Hahn-Banach)**.** Let X be a real vector space, $p : X \to \mathbb{R}$ be sublinear, $M \subseteq X$. Then for any $\ell : M \to \mathbb{R}$ linear with $\ell(y) \leq p(y)$ for all $y \in M$, there exists $\tilde{\ell} : X \to \mathbb{R}$ linear such that $\tilde{\ell}|_M = \ell$, and $\tilde{\ell}(x) \leq p(x)$ for all $x \in X$.

Proof. Define

$$\mathcal{S} = \left\{(N, \tilde{\ell}) \mid M \leq N \leq X, \tilde{\ell} : N \to \mathbb{R} \text{ linear}, \tilde{\ell}(x) \leq p(x) \text{ for all } x \in N, \tilde{\ell}|_M = \ell\right\}$$

Define the ordering $(N_1, \tilde{\ell}_1) \leq (N_2, \tilde{\ell}_2)$ if $N_1 \leq N_2$, with $\tilde{\ell}_2|_{N_1} = \tilde{\ell}_1$. For every chain $T \subseteq S$, we have an upper bound

$$N_T = \bigcup_{(N,\tilde{\ell}) \in T} N \quad \text{and} \quad \tilde{\ell}_T = \tilde{\ell}(x)$$

where $x \in N, (N, \tilde{\ell}) \in T$. Then $\tilde{\ell}_T$ is well defined as T is a chain. Hence by Zorn's lemma, $\mathcal{S}$ has a maximal element $(N, \tilde{\ell})$. By maximality and the codimension 1 case, $N = X$. □

Corollary 2.19. Let X be a normed vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $M \leq X$ subspace. For every bounded linear functional $\Lambda : M \to \mathbb{F}$, there is a bounded linear $\tilde{\Lambda} : X \to \mathbb{F}$ such that

$$\|\tilde{\Lambda}\| = \|\Lambda\| \quad \text{and} \quad \tilde{\Lambda}|_M = \Lambda$$

Proof. If $\mathbb{F} = \mathbb{R}$, $p(x) = \|\Lambda\|\|x\|$ is sublinear, and the result follows from Hahn-Banach.

If $\mathbb{F} = \mathbb{C}$, then $\Lambda(x) = \ell(x) - i\ell(ix)$, with $\ell(x) = \mathfrak{R}(\Lambda(x))$ real linear. Since $|\Lambda(x)| = \ell(e^{i\theta}x)$ for some θ, we have that

$$\sup_{x \in N, \|x\| \leq 1} \|\Lambda(x)\| = \sup_{x \in N, \|x\| \leq 1} \ell(x)$$

for any subspace N of X. Apply Hahn-Banach to ℓ and define $\tilde{\Lambda}$ as above. ☐

Corollary 2.20. If X is a normed vector space, $x \in X$, then there exists $\Lambda_x \in X'$ such that $\|\Lambda_x\| = 1$, $\Lambda_x(x) = \|x\|$. Λ_x is called a support functional for X.

Proof. Define $\ell(tx) = t\|x\|$, and extend by Hahn-Banach. ☐

Corollary 2.21. Let X be a normed vector space, $x \in X$, then $x = 0$ if and only if $\Lambda(x) = 0$ for all $\Lambda \in X'$.

Corollary 2.22. Let X be a normed vector space, $x, y \in X$ distinct, then there exists $\Lambda \in X'$ such that $\Lambda(x) \neq \Lambda(y)$.

Corollary 2.23. The map $\Phi : X \to X''$, with $\Phi(x) = \hat{x}$, $\hat{x}(\Lambda) = \Lambda(x)$ is an isometry.

Definition 2.24 (reflexive)
X is reflexive if Φ is surjective, i.e. $X'' = X$.

Theorem 2.25 (Geometric Hahn-Banach)**.** Let $A, B \subseteq X$ be disjoint nonempty convex subsets of a Banach space X over $\mathbb{R}$ or $\mathbb{C}$. Then

(i) if A is open, then there exists $\Lambda \in X'$ and $\gamma \in \mathbb{R}$ such that

$$\mathfrak{R}(\Lambda(y)) < \gamma \leq \mathfrak{R}(\Lambda(y))$$

for all $x \in A, y \in B$. Furthermore, if B is also open, we can make $\gamma < \mathfrak{R}(\Lambda(y))$ strict.

(ii) if A is compact and B is closed, then there exists $\Lambda \in X'$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\mathfrak{R}(\Lambda(x)) < \gamma_1 < \gamma_2 < \mathfrak{R}(\Lambda(y))$$

for all $x \in A, y \in B$.

Proof. We can assume without loss of generality that X is a real vector space, since we can apply the argument to the real part of a linear functional.

(i) Choose $a_0 \in A, b_0 \in B$, and set $x_0 = b_0 - a_0$ and

$$C = A - B + x_0 = \{a - b + x_0 \mid a \in A, b \in B\}$$

Then $0 \in C$, C is convex, $x_0 \notin \mathbb{C}$ as $A \cap B = \varnothing$. Let $p(x) = \inf\left\{t > 0 \mid t^{-1}x \in C\right\}$. Then p is sublinear, with $p(x) \le k\|x\|$ for all $x \in X$, and $p(x) < 1$ if and only if $x \in C$. Let $M = \operatorname{span}\{x_0\}$, and define $\ell : M \to \mathbb{R}$ by $\ell(tx_0) = t$. Then we have that

$$\ell(tx_0) = \begin{cases} t \le tp(x_0) = p(tx_0) & \text{if } t \ge 0 \\ t < 0 \le p(tx_0) & \text{if } t < 0 \end{cases}$$

Hence by Hahn-Banach, we can extend ℓ to $\Lambda : X \to \mathbb{R}$, with $\Lambda(x) \le p(x)$ for all $x \in X$. Moreoever,

$$-k\|x\| \le -p(-x) \le \Lambda(x) \le p(x) \le k\|x\| \implies |\Lambda(x)| \le k\|x\|$$

so $\Lambda \in X'$ with $\|\Lambda\| \le k$. Furthermore, for any $a \in A, b \in B$,

$$\Lambda(a - b + x_0) \le p(a - b + x_0) < 1$$

So $\Lambda(a) - \Lambda(b) + \Lambda(x_0) < 1$. But $\Lambda(x_0) = \ell(x_0) = 1$. So $\Lambda(a) < \Lambda(b)$. Now as $\Lambda \in X'$ is nonzero, Λ is an open map. So $\Lambda(A)$ is open, and $\Lambda(A), \Lambda(B)$ are connected, so $\Lambda(A), \Lambda(B)$ are disjoint intervals in $\mathbb{R}$. The result then follows.

(ii) Since A is compact and B is closed,

$$d = \inf\{\|a - b\| \mid a \in A, b \in b\} > 0$$

Let $V = B_{d/2}(0)$. Then $A + V$ os open, convex and disjoint from B. Apply (i) with $A + V, B$, we have $\Lambda \in X'$ such that $\Lambda(A + V), \Lambda(B)$ are disjoint intervals in $\mathbb{R}$. Finally, note that $\Lambda(A) \subseteq \Lambda(A + V)$ is compact, which gives the required result. □

Corollary 2.26. Let X be a Banach space, $M \le X$ a subspace, $x_0 \notin \overline{M}$. Then there exists $\Lambda \in X'$ such that $\Lambda x_0 = 1$ and $\Lambda(x) = 0$ for all $x \in M$.

Proof. Use (ii) of the above with $A = \{x_0\}$ and $B = \overline{M}$. □

3 Weak topology and compactness

Definition 3.1 (seminorm)

A seminorm p on a vector space X is a map $p : X \to \mathbb{R}$ such that

(i) $p(x + y) \le p(x) + p(y)$,

(ii) $p(\lambda x) = |\lambda| p(x)$,

(iii) $p(x) \ge 0$.

Definition 3.2 (separating family)

A family $\mathscr{P}$ of seminorms is separating if for every $x \ne 0$, there exists $p \in \mathscr{P}$ such that $p(x) \ne 0$.

Definition 3.3 (induced topology)

The topology $\tau_{\mathscr{P}}$ induced by a family $\mathscr{P}$ of separating seminorms has neighbourhood basis of 0 given by

$$\dot{\beta} = \{\text{finite intersections of } V(p, n) = \{x \mid p(x) < 1/n\} \text{ with } p \in \mathscr{P}, n \in \mathbb{N}\}$$

That is, the topology has neighbourhood basis

$$\beta = \left\{x + B \mid x \in X, B \in \dot{\beta}\right\}$$

Proposition 3.4.

(i) $(X, \tau_{\mathscr{P}})$ is a locally convex topological vector space,

(ii) every open set is a union of elements of β,

(iii) each $p \in \mathscr{P}$ is continuous.

(iv) for a sequence (x_n) in X, $x_k \to x$ if and only if $p(x_k - x) \to 0$ for all $p \in \mathscr{P}$,

(v) if $\mathscr{P} = (p_k)_{k \in \mathbb{N}}$ is countable, then the topology is metrisable, with metric given by

$$d_{\mathscr{P}}(x, y) = \sum_k 2^{-k} \frac{p_k(x - y)}{1 + p_k(x - y)}$$

Proof. (ii), (iii) and (iv) follow from definitions, (i) and (v) are omitted. □

Definition 3.5 (Fréchet space)
if $(X, d_{\mathscr{P}})$ is complete, we call X a Fréchet space.

3.1 Strong, weak and weak-$*$ topologies

Let X be a Banach space.

Definition 3.6 (strong topology)
The strong topology τ_s is generated by the seminorms $\mathscr{P}_s = \{\|\cdot\|\}$. That is, it is the norm topology.

Definition 3.7 (weak topology)
The weak topology is generated by $\mathscr{P}_w = \{p_\Lambda \mid \Lambda \in X'\}$ where $p_\Lambda(x) = |\Lambda(x)|$. By Hahn-Banach, $\mathscr{P}_w$ is separating and the induced topology τ_w is called the weak topology.

Definition 3.8 (weak-$*$ topology)
The weak-$*$ topology on X' is generated by the family of seminorms

$$\mathscr{P}_{w*} = \{p_x \mid x \in X\} \quad \text{where} \quad p_x(\Lambda) = |\Lambda(x)|$$

Notation 3.9. We write $x_k \to x$ for convergence in τ_s, $x_k \rightharpoonup x$ for convergence in τ_w and $\Lambda_k \xrightarrow{*} \Lambda$ for convergence in τ_{w*}.

Proposition 3.10 (convergence).

(i) a sequence (x_k) converges to x in τ_s if and only if $\|x_k - x\| \to 0$,

(ii) a sequence (x_k) converges to x in τ_w if and only if $\Lambda(x_k - x) \to 0$ for all $\Lambda \in X'$,

(iii) a sequence (Λ_k) converges to Λ in τ_{w*} if and only if $\Lambda_k(x) \to \Lambda(x)$ for all $x \in X$.

(iv) $\Lambda_k \to \Lambda \implies \Lambda_k \rightharpoonup \Lambda \implies \Lambda_k \overset{*}{\rightharpoonup} \Lambda$,

(v) if X is reflexive, $X'' = X$, then $\tau_w = \tau_{w*}$.

3.2 Banach-Alaoglu

Theorem 3.11 (Banach-Alaoglu)**.** Let X be a separable Banach space, $(\Lambda_j) \subseteq X'$ be a bounded sequence, wlog $\|\Lambda_j\| \leq 1$ for all j. Then there is a subsequence (j_i) and $\Lambda \in X'$ such that $\|\Lambda\| \leq 1$ and

$$\Lambda_{j_i} \overset{*}{\rightharpoonup} \Lambda$$

Proof. **Step 1: Construction** Let $D = \{x_k\}_{k=1}^{\infty} \subseteq X$ be a dense subset. Since $(\Lambda_j(x_1))$ is bounded, by Bolzano-Weierstrass there exists a subsequence $J_1 \subseteq \mathbb{N}$ and $\Lambda(x_1) \in \mathbb{F}$ such that $\Lambda_j(x_1) \to \Lambda(x_1)$ along J_1. Iterating this, we get

$$J_1 \supseteq J_2 \supseteq \dots$$

and $(\Lambda(x_k))_k$ such that $\Lambda_j(x_k) \to \Lambda(x_k)$ along J_ℓ for all $\ell \geq k$. Taking the diagonal subsequence J of the (J_k), i.e. $J = \{j_1 < j_2 < \dots\}$, j_k is the k-th element of J_k, we find that for all k, $\Lambda_j(x_k) \to \Lambda(x_k)$ along J.

Step 2: $\Lambda : D \to \mathbb{R}$ is uniformly continuous, so it extends to $\Lambda : X \to \mathbb{F}$ continuous.

Fix $x, y \in D$ with $\|x - y\| < \varepsilon$. Then there exists $j \in J$ such that $|\Lambda_j(x) - \Lambda(x)| < \varepsilon$ and $|\Lambda_j(y) - \Lambda(y)| < \varepsilon$. Hence

$$|\Lambda(x) - \Lambda(y)| \leq |\Lambda(x) - \Lambda_j(x)| + |\Lambda_j(x) - \Lambda_j(y)| + |\Lambda_j(y) - \Lambda(y)| < 3\varepsilon$$

Step 3: $\Lambda : X \to \mathbb{F}$ is linear. Let $x, y \in X$, $a \in \mathbb{F}$. Set $z = x + ay$. Then for $x', y', z' \in D$, and $j \in \mathbb{N}$, we have that

$$\begin{aligned} |\Lambda(z) - \Lambda(x) - a\Lambda(y)| \leq{}& |\Lambda(z) - \Lambda(z')| + |\Lambda(x) - \Lambda(x')| + |a| \cdot |\Lambda(y) - \Lambda(y')| \\ &+ |\Lambda(z') - \Lambda_j(z')| + |\Lambda(x') - \Lambda_j(x')| + |a| \cdot |\Lambda(y') - \Lambda_j(y')| \\ &+ |\Lambda_j(z' - x' - ay')| \end{aligned}$$

Fix $\varepsilon > 0$. By continuity, if $\|x - x'\|, \|y - y'\|, \|z - z'\|$ sufficiently small, then we can make the first line $< \varepsilon$. Similarly as $\|z' - x' - ay'\| \leq \|x - x'\| + \|y - y'\| + \|z - z'\|$, we can make the last line $< \varepsilon$ as well, since $\|\Lambda_j\| \leq 1$. For the middle term, we can take $j \to \infty$ along J, to get that for $j \in J$ large enough, we can make the middle line $< \varepsilon$ as well. So we get that

$$|\Lambda(z) - \Lambda(x) - a\Lambda(y)| < 3\varepsilon$$

But $\varepsilon > 0$ was arbitrary so we are done.

Step 4: $\|\Lambda\| \leq 1$ and $\Lambda_j \overset{*}{\rightharpoonup} \Lambda$.

For the first one, notice that by density,

$$\|\Lambda\| = \sup_{x \in X, \|x\| \leq 1} |\Lambda(x)| = \sup_{x \in D, \|x\| \leq 1} \leq 1$$

and for the second one, given $\varepsilon > 0$, $x \in X$, choose $x' \in D$ such that $\|x' - x\| < \varepsilon$. Then

$$|\Lambda_j(x) - \Lambda(x)| \leq |\Lambda_j(x - x')| + |\Lambda_j(x') - \Lambda(x')| + |\Lambda(x - x')| < 3\varepsilon$$

for $j \in J$ large enough. □

4 Distributions

4.1 Test functions $\mathscr{D}(U)$ and distributions $\mathscr{D}'(U)$

Definition 4.1 ($C_c^\infty(U)$)
For $U \subseteq \mathbb{R}^n$ open, define

$$C_c^\infty(U) = \{\phi : U \to \mathbb{C} \text{ smooth, with } \operatorname{supp}(\phi) \subseteq U \text{ compact}\}$$

Theorem 4.2. There exists a topology τ on $C_c^\infty(U)$ such that

(i) $(C_c^\infty(U), \tau)$ is a topological vector space,

(ii) a sequence $(phi_j) \subseteq C_c^\infty(U)$ converges to 0 if and only if there exists $K \subseteq U$ comapct, such that

  (a) $\operatorname{supp}(\phi_j) \subseteq K$ for all j,

  (b) for evert multi-index α,

$$\sup_k |\nabla^\alpha \phi_j| \to 0$$

(iii) if Y is a locally convex topological vector space, $\Lambda : C_c^\infty(U) \to Y$ linear, then Λ is continuous if and only if Λ is sequentially continuous.

Definition 4.3 (test functions)
We call $(C_c^\infty(U), \tau)$ the space of test function $\mathscr{D}(U)$.

Definition 4.4 (distributions)
The space of distributions $\mathscr{D}'(U)$ is the dual space of $\mathscr{D}(U)$ with the weak-$*$ topology.

Proposition 4.5 (sequential continuity).

(i) a linear functional $u : \mathscr{D}(U) \to \mathbb{C}$ is in $\mathscr{D}'(U)$ if and only if for all sequences $\phi_j \in \mathscr{D}(U)$ with $\phi_j \to \phi$ in $\mathscr{D}(U)$, we have that $u(\phi_j) \to u(\phi)$ in $\mathbb{C}$,

(ii) a sequence (u_j) in $\mathscr{D}'(U)$ converges to u if and only if $u_j(\phi) \to u(\phi)$ for all $\phi \in \mathscr{D}(U)$,

Definition 4.6 (Dirac delta)
For $x \in U$, the Dirac delta distribution $\delta_x \in \mathscr{D}'(U)$ is defined by $\delta_x(\phi) = \phi(x)$.

Definition 4.7 (embedding of $L^1_{\text{loc.}}$)
For $f \in L^1_{\text{loc.}}(U)$, define $T_f \in \mathscr{D}'(U)$ by

$$T_f(\phi) = \int_U f\phi \mathrm{d}x$$

Proposition 4.8. The map $T : L^1_{\text{loc.}}(U) \to \mathscr{D}'(U)$ is injective.

Proof.

$$Tf = Tg \iff \int (f-g)\phi \mathrm{d}x = 0 \text{ for all } \phi \in C_c^\infty(U) \iff f = g \text{ a.e.}$$

□

Definition 4.9 (multiplication by a function)
If $u \in \mathscr{D}'(U)$ and $\alpha \in C^\infty(U)$, define $\alpha u \in \mathscr{D}'(U)$ by

$$\alpha u(\phi) = u(\alpha\phi)$$

Definition 4.10 (derivative)
If $u \in \mathscr{D}'(U)$, define $\nabla^\alpha u \in \mathscr{D}'(U)$ by

$$\nabla^\alpha u(\phi) = (-1)^{|\alpha|} u(\nabla^\alpha \phi)$$

Proposition 4.11.

(i) $\alpha T_f = T_{\alpha f}$,

(ii) $\nabla^\alpha T_f = T_{\nabla^\alpha f}$.

Proof. Easy to check from the definitions, and by integration by parts for (ii). □

4.2 Compactly supported distributions $\mathscr{E}'(U)$

Consider the space $C^\infty(U) = \{\phi : U \to \mathbb{C} \text{ smooth}\}$. Then we can find a sequence (K_j) of compact subsets of U such that

(i) $K_i \subseteq \mathrm{Int}(K_{i+1})$ for all i,

(ii) $U = \bigcup_i K_i$.

Then for $phi \in C^\infty(U)$, define

$$p_N(\phi) = \sup_{x \in K_N} \sup_{\alpha \leq N} |\nabla^\alpha \phi(x)|$$

Then $\mathscr{P} = \{p_N\}_{N \in \mathbb{N}}$ is a separating family of seminorms.

Definition 4.12 ($\mathscr{E}(U)$)
The space $C^\infty(U)$ with the locally convex topology induced by $\mathscr{P}$ is denoted by $\mathscr{E}(U)$.

Proposition 4.13. $\mathscr{E}(U)$ is a Frćhet space.

Proposition 4.14 (convergence). $(\phi_j) \subseteq \mathscr{E}(U)$ converges to 0 if and only if for all $K \subseteq U$ compact, multi-index α,

$$\sup_{x \in K} \left|\nabla^\alpha \phi_j(x)\right| \to 0$$

Proposition 4.15. The embedding $\mathscr{D}(U) \hookrightarrow \mathscr{E}(U)$ is continuous, and so we have an induced embedding $\mathscr{E}'(U) \hookrightarrow \mathscr{D}'(U)$.

Lemma 4.16. Let $u : \mathscr{E}(U) \to \mathbb{C}$ be linear. Then u is continuous if and only if there is a compact set $K \subseteq \mathbb{R}^n$, $N \in \mathbb{N}$, $C > 0$ such that

$$|u(\phi)| \le C \sup_{x \in K, |\alpha| \le N} |\nabla^\alpha \phi(x)| \qquad (*)$$

for all $\phi \in \mathscr{E}(U)$.

Proof. Since $\mathscr{E}(U)$ is a metric space, $u \in \mathscr{E}'(U)$ if and only if $u(\phi_j) \to 0$ for all sequences $(\phi_j) \subseteq \mathscr{E}(U)$ with $\phi_j \to 0$ in $\mathscr{E}(U)$.

Suppose (*) holds, and let $(\phi_j) \subseteq \mathscr{E}(U)$ be a sequence with $\phi_j \to 0$, which is equivalent to saying for all $\tilde{K} \subseteq U$ compact, $\tilde{N} \in \mathbb{N}$,

$$\sup_{x \in \tilde{K}, |\alpha| \le \tilde{N}} \left|\nabla^\alpha \phi_j(x)\right| \to 0$$

Taking $\tilde{K} = K$ and $\tilde{N} = N$, (*) implies that $u(\phi_j) \to 0$. Conversely, suppose (*) does not hold. Let (K_j) be a sequence of compact subsets of U such that

(i) $K_i \subseteq \mathrm{Int}(K_{i+1})$ for all i,

(ii) $U = \bigcup_i K_i$.

Since (*) does not hold, for each j there exists $\phi_j \in \mathscr{E}(U)$ such that

$$\left| u(\phi_j) \ge j \sup_{x \in K_j, |\alpha| \le j} \left|\nabla^\alpha \phi_j(x)\right| \right|$$

Then $\psi_j = \phi_j / \left|u(\phi_j)\right| \to 0$ in $\mathscr{E}(U)$ since for all $\tilde{K} \subseteq U$ compact, $N \in \mathbb{N}$, there esists $J > \tilde{N}$ such that $\tilde{K} \subseteq K_j$ for all $j \ge J$. Then

$$\sup_{x \in \tilde{K}, |\alpha| \le \tilde{N}} \left|\nabla^\alpha \psi_j(x)\right| \le \frac{1}{j}$$

But $\left|u(\psi_j)\right| = 1$, so $u(\psi_j) \not\to 0$. Hence u cannot be continuous. □

Definition 4.17 (support)

For $u \in \mathscr{D}'(U)$ we say that u has support in a set S if $u(\phi) = 0$ for all $\phi \in C_c^\infty(U \setminus S)$. If we can choose S to be compact, we say that u is compactly supported.

Theorem 4.18.

$$\mathscr{E}'(U) = \{u \in \mathscr{D}'(U) \text{ compactly supported}\}$$

Proof. If $u \in \mathscr{E}'(U)$, then the lemma implies that u has support in a compact set K. Conversely, if $u \in \mathscr{D}'(U)$ has support in a compact set K, define $\tilde{u}(\phi) = u(\chi\phi)$ for all $\phi \in \mathscr{E}(U)$, where $\chi \in C_c^\infty(U)$ is such that $\chi = 1$ on K. In fact, the definition of $\tilde{u}$ is independent of the choice of χ since for any such $\chi, \tilde{\chi}$, $\chi - \tilde{\chi} \in C_c^\infty(U \setminus K)$, so $u(\chi\phi) = u(\tilde{\chi}\phi)$. □

4.3 Tempered distributions $\mathscr{S}'(\mathbb{R}^n)$

Definition 4.19 (rapidly decreasing)

$\phi \in C^\infty(\mathbb{R}^n)$ is rapidly decreasing if

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N \nabla^\alpha \phi(x)\right| < \infty$$

for all N, α.

Definition 4.20 (Schwartz space)
The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is the space of rapidly decreasing functions with the topology generated by the separating family of seminorms

$$p_N(\phi) = \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\le N} \left|(1+|x|)^N \nabla^\alpha \phi(x)\right|$$

Proposition 4.21. $\mathscr{S}(\mathbb{R}^n)$ is a Fréchet space, with

$$\mathscr{D}(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n) \subseteq \mathscr{E}(\mathbb{R}^n)$$

continuously, which induces the inclusions of the dual spaces

$$\mathscr{E}'(\mathbb{R}^n) \subseteq \mathscr{S}'(\mathbb{R}^n) \subseteq \mathscr{D}'(\mathbb{R}^n)$$

Definition 4.22 ({tempered/Schwartz} distribution)
$\mathscr{S}'(\mathbb{R}^n)$ is the space of tempered distributions, or Schwartz distributions.

4.4 Convolution

Notation 4.23 (translation). The translation of a function is given by $\tau_x g(y) = g(y-x)$.

Notation 4.24 (spatial inversion). The spatial inversion of a function is given by $\check{g}(y) = g(-y)$.

Definition 4.25 (convolution of distribution with function)
For $u \in \mathscr{D}'(\mathbb{R}^n), f \in \mathscr{D}(\mathbb{R}^n)$, define $u * \phi(x) = u(\tau_x \check{\phi})$.

Proposition 4.26.

1. $(u_1 + a u_2) * \phi = u_1 * \phi + a u_2 * \phi$,
2. $u * (\phi_1 + a\phi_2) = u * \phi_1 + a u * \phi_2$,
3. $u * \check{\phi}(0) = u(\phi)$,
4. If $f \in L^1_{\text{loc.}}$, $\phi \in \mathscr{D}(\mathbb{R}^n)$, then

$$f * \phi(x) = T_f(\tau_x \check{\phi})$$

Proof. Easy to check. □

Proposition 4.27. Let $u \in \mathscr{D}'(\mathbb{R}^n)$, $\phi \in \mathscr{D}(\mathbb{R}^n)$, then

1. $u * \phi \in \mathbb{C}^\infty(\mathbb{R}^n)$, with
$$\nabla^\alpha(u * \phi) = (\nabla^\alpha u) * \phi = u * (\nabla^\alpha)\phi$$
where the first and third ∇^α are of C^∞ functions, and the second one is of a distribution,
2. if $u \in \mathscr{E}'(\mathbb{R}^n)$, then $u * \phi \in \mathscr{D}(\mathbb{R}^n)$.

Proof. (i) It is easy to check that

$$\frac{1}{h}(u * \phi(x + he_i) - u * \phi(x)) = u\left(\frac{1}{h}(\tau_{x+he_i}\check{\phi} - \tau_x\check{\phi})\right) \to u\left(\tau_x\widetilde{\nabla_i\phi}\right)$$

by the convergence in $\mathscr{D}(\mathbb{R}^n)$ of the argument, and the continuity of u. Hence $\nabla_i(u * \phi)(x) = u(\tau_x\widetilde{\nabla_i\phi})$. By induction, $u * \phi \in C^\infty(\mathbb{R}^n)$, with $\nabla^\alpha(u * \phi) = u * (\nabla^\alpha\phi)$ for all α. Finally, notce that

$$\nabla^\alpha(\tau_x\check{\phi})(y) = \nabla^\alpha_y\phi(x - y) = (-1)^{|\alpha|}\nabla^\alpha\phi(x - y) = (-1)^{|\alpha|}\tau_x\widehat{\nabla^\alpha\phi}(y)$$

and hence $u * (\nabla^\alpha\phi) = \nabla^\alpha u * \phi$.

(ii) By assumption, $u(\phi) = 0$ for all $\phi \in C_c^\infty(\mathbb{R}^n \setminus K)$, where $K \subseteq \mathbb{R}^n$ compact. Hence for all $\phi \in \mathscr{D}(\mathbb{R}^n)$, $\operatorname{supp}(\tau_x\check{\phi}) \cap K = \varnothing$ for all x sufficiently large, so $u * \phi$ has compact support. □

Definition 4.28 (convolution of distributions)
For $u_1 \in \mathscr{D}'(\mathbb{R}^n)$, $u_2 \in \mathscr{E}'(\mathbb{R}^n)$, define $u_1 * u_2$ to be the unique distribution such that

$$(u_1 * u_2) * \phi = u_1 * (u_2 * \phi)$$

for all $\phi \in \mathscr{D}(\mathbb{R}^n)$.

Proposition 4.29. $u * \delta_0 = u$.

Proposition 4.30. Let $u_1 \in \mathscr{D}'(\mathbb{R}^n)$, $u_2 \in \mathscr{E}'(\mathbb{R}^n)$, then

$$\nabla^\alpha(u_1 * u_2) = u_1 * \nabla^\alpha u_2 = \nabla^\alpha u_1 * u_2$$

Proof. Let $\phi \in \mathscr{D}(\mathbb{R}^n)$, then

$$\nabla^\alpha(u_1 * u_2) * \phi = (u_1 * u_2) * \nabla^\alpha\phi = u_1 * (u_2 * \nabla^\alpha\phi) = u_1 * (\nabla\alpha u_2 * \phi)$$

and the other case is similar. □

Definition 4.31 (fundamental solution)
Let $L = \sum_{|\alpha|\le k} a_\alpha\nabla^\alpha$ be a constant coefficient partial differential operator of order k. A fundamental solution of L is a distribution G such that $LG = \delta_0$.

Theorem 4.32. If $G \in \mathscr{D}'(\mathbb{R}^n)$ is a fundamental solution of L and $f \in \mathscr{E}'(\mathbb{R}^n)$, then $u = G * f$ solves $Lu = f$. Moreover, if $f \in \mathscr{D}(\mathbb{R}^n)$, then $u = G * f \in C^\infty(\mathbb{R}^n)$, and u solves $Lu = f$ in the classical sense.

Proof.

$$\begin{aligned} L(G * f) &= \sum_{|\alpha|\leq k} a_\alpha \nabla^\alpha (G * f) \\ &= \sum_{|\alpha|\leq k} a_\alpha \nabla^\alpha G * f \\ &= \left(\sum_{|\alpha|\leq k} a_\alpha \nabla^\alpha G\right) * f \\ &= (LG) * f \\ &= \delta_0 * f \\ &= f \end{aligned}$$

□

5 Fourier transforms

Note that the convention for the Fourier transform in this course is different to the convention in Probability and Measure.

5.1 Fourier transforms of functions

Definition 5.1 (Fourier transform of L^1 functions)
If $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is $\hat{f} = \mathscr{F}(f) : \mathbb{R} \to \mathbb{C}$, defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \mathrm{d}x$$

Lemma 5.2 (Riemann-Lebesgue). If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C(\mathbb{R}^n)$, with

$$\sup_{\xi\in\mathbb{R}^n} \left|\hat{f}(\xi)\right| \leq \|f\|_{L^1} \quad \text{and} \quad \hat{f}(\xi) \to 0 \text{ as } |\xi| \to \infty$$

Proof. Given $\xi_k \to \xi$, for $x \in \mathbb{R}^n$ fixed, $f(x)e^{-i\xi_n\cdot x} \to f(x)e^{-i\xi\cdot x}$, and $\left|f(x)e^{-i\xi_n\cdot x}\right| = |f(x)|$, $f \in L^1$, so by the dominated convergence theorem,

$$\hat{f}(\xi_n) \to \hat{f}(\xi)$$

Hence $\hat{f}$ is continuous. The bound is immediate since

$$\left|\hat{f}(\xi)\right| = \left|\int_{\mathbb{R}^n} f(x)e^{-i\xi\cdot x}\right| \leq \int_{\mathbb{R}^n} |f(x)| = \|f\|_{L^1}$$

To show the decay property, fix $\varepsilon > 0$, and let $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be such that $\|f - f_\varepsilon\| < \varepsilon$. Then by integration by parts, we find

$$\begin{aligned} \hat{f}_\varepsilon(\xi) &= \int_{\mathbb{R}^n} f_\varepsilon(x) e^{-i\xi\cdot x} \mathrm{d}x \\ &= \frac{-1}{|\xi|^2} \int_{\mathbb{R}^n} (\Delta f_\varepsilon)(x) e^{-i\xi\cdot x} \mathrm{d}x \\ &\leq \frac{-1}{|\xi|^2} \|\Delta f_\varepsilon\|_{L^1} \end{aligned}$$

which means that $\limsup_{|\xi|\to\infty} \left|\hat{f}_\varepsilon(\xi)\right| = 0$. Therefore, we have that

$$\left|\hat{f}(\xi)\right| \leq \left|\hat{f}_\varepsilon(\xi)\right| + \left|\hat{f}(xi) - \hat{f}_\varepsilon(\xi)\right| \leq \left|\hat{f}_\varepsilon(\xi)\right| + \|f - f_\varepsilon\|_{L^1} \to \varepsilon$$

as $|\xi| \to \infty$, where we used the inequalty from above to bound the modulus of the Fourier transform by the L^1 norm. As $\varepsilon > 0$ was arbitrary, we have that $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. □

Notation 5.3 (exponential function)**.** We write $e_y(x) = e^{ix\cdot y}$.

Proposition 5.4.

(i) if $f \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\lambda > 0$, and set $f_\lambda(x) = \lambda^{-n} f(x/\lambda)$, then

$$\hat{f}_\lambda(\xi) = \hat{f}(\lambda\xi), \qquad \widehat{e_y f}(\xi) = \tau_y \hat{f}(\xi) \quad \text{and} \quad \widehat{\tau_y f}(\xi) = e_{-y}(\xi)\hat{f}(\xi)$$

(ii) if $f, g \in L^1$, then $f * g \in L^1$ and

$$\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$$

Proof. By change of variables and Fubini. □

Proposition 5.5.

(i) If $f \in C^1(\mathbb{R}^n)$, with $f, \nabla_j f \in L^1(\mathbb{R}^n)$ for all j, then

$$\widehat{\nabla_j f}(\xi) = i\xi_j \hat{f}(\xi)$$

(ii) Suppose $(1 + |x|)f \in L^1(\mathbb{R}^n)$. Then $\hat{f} \in C^1(\mathbb{R}^n)$, and

$$D_j \hat{f}(\xi) = -\widehat{ix_j f}(\xi)$$

Proof. Fix $\varepsilon > 0$. Let $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ be such that

$$\|f - f_\varepsilon\|_{L^1} < \varepsilon \quad \text{and} \quad \|\nabla_k f_\varepsilon - \nabla_k\| < \varepsilon \text{ for all } k$$

Then we have that

$$\widehat{\nabla_j f_\varepsilon}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \nabla_j f_\varepsilon(x) \mathrm{d}x = i\xi_j \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-ix\cdot\xi} f_\varepsilon(x) \mathrm{d}x = i\xi_j \hat{f}_\varepsilon(\xi)$$

Hence we must have that

$$\left|\widehat{\nabla_j f}(\xi) - i\xi_j \hat{f}(\xi)\right| \le \left\|\nabla_j f - \nabla_j f_\varepsilon\right\|_{L^1} + |\varepsilon| \|f - f_\varepsilon\|_{L^1} \le (1 + |\xi|)\varepsilon \to 0$$

as $\varepsilon \to 0$.

(ii) Since $x_j f \in L^1$, $-\widehat{ix_j f} \in C_0$. So we need to show that $\nabla_i \hat{f}$ exists and is equal to $-\widehat{ix_j f}$. But

$$\frac{\hat{f}(\xi + he_j) - \hat{f}(\xi)}{h} = \int_{\mathbb{R}^n} f(x) e^{-i\xi\cdot x} \left(\frac{e^{-ihx_j} - 1}{h}\right) \mathrm{d}x \to \int_{\mathbb{R}^n} -\widehat{ix_j f}(\xi)$$

by the dominated convergence theorem. □

Corollary 5.6. The Fourier transform defines $\mathscr{F} : \mathscr{S}(R^n) \to \mathscr{S}(\mathbb{R}^n)$ continuous.

Proof. Given $f : \mathbb{R}^n \to \mathbb{C}$,

$$\|f\|_{L^1} \le \sup_{x \in \mathbb{R}^n} \left((1 + |x|^{n+1} f(x))\right) \underbrace{\int_{\mathbb{R}^n} \frac{1}{(1 + |z|)^{n+1}} \mathrm{d}z}_{<\infty}$$

Therefore, if $f \in \mathscr{S}(\mathbb{R}^n)<$ then $\nabla^\alpha(x^\beta f(x)) \in L^1$ for any α, β. Hence by the previous proposition,

$$\left|\widehat{\nabla^\alpha(x^\beta f)}(\xi)\right| = \left|\xi^\alpha \nabla^\beta \hat{f}(\xi)\right|$$

In particular, we have that

$$\sup_\xi \left|\xi^\alpha \nabla^\beta \hat{f}(\xi)\right| \leq C \sup_{x \in \mathbb{R}^n, \gamma \leq \alpha} \left((1+|x|)^{|\beta|+n+1} |\nabla^\gamma f(x)|\right) \to 0$$

if $f \to 0$ in $\mathscr{S}(\mathbb{R}^n)$. Therefore, $\hat{f} \to 0 \in \mathscr{S}(\mathbb{R}^n)$. Thus, $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is continuous. □

Theorem 5.7 (Fourier inversion). Let $f \in L^1(\mathbb{R}^n)$, and assume $\hat{f} \in L^1(\mathbb{R}^n)$ as well. Then

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi \cdot x} \mathrm{d}\xi$$

for a.e. $x \in \mathbb{R}^n$. That is, $\mathscr{F}^2(f) = \frac{1}{(2\pi)^n} \check{f}$.

Proof. Let

$$I_\varepsilon(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{ix\cdot\xi} \mathrm{d}\xi$$

Since $\hat{f} \in L^1$, by the dominated convergence theorem,

$$I_\varepsilon(x) \to \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} \mathrm{d}\xi$$

as $\varepsilon \to 0$. On the other hand, we have that

$$\begin{aligned} I_\varepsilon(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) e^{-i\xi\cdot y} \mathrm{d}y \right) e^{-\frac{1}{2}\varepsilon^2|\xi|^2} e^{ix\cdot\xi} \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) \underbrace{\left(\int_{\mathbb{R}^n} e^{\frac{1}{2}\varepsilon^2|\xi|^2} e^{-i(y-x)\cdot\xi} \mathrm{d}\xi \right)}_{=(2\pi)^{n/2}\varepsilon^{-n} e^{-|y-x|^2/2\varepsilon^2}} \mathrm{d}y \\ &= f * \psi_\varepsilon(x) \end{aligned}$$

where $\psi_\varepsilon(x) = \varepsilon^{-n}\psi(x/\varepsilon)$, $\psi(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$. Since ψ is a smooth molifier, $f * \psi_\varepsilon \to f$ in L^1, so we have that

$$f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} \mathrm{d}\xi$$

for a.e. $x \in \mathbb{R}^n$. □

Lemma 5.8. Suppose (f_j) is a sequence in $L^p(\mathbb{R}^n)$, such that $f_j \to f$ uniformly on $\mathbb{R}^n$, and $f_j \to g$ in L^p. Then $f = g$ a.e.

Proof. The uniform limit is also the pointwise limit, so f is measurable. Now for $R > 0$, we have that

$$\left\|f_j - f\right\|^p_{L^p(B_R(0))} = \int_{B_R(0)} \left|f_j - f\right|^p \mathrm{d}x \leq |B_R(0)| \sup_{x \in B_R(0)} \left|f_j(x) - f(x)\right| \to 0$$

as $j \to \infty$. Hence by uniqueness of limits, $f = g$ in $L^2(B_R(0))$, which means that $f = g$ a.e. on $B_R(0)$. But

$$\mathbb{R}^n = \bigcup_{n \in \mathbb{N}} B_n(0)$$

so $f = g$ a.e. on $\mathbb{R}^n$. □

Theorem 5.9 (Parseval-Plancherel). Let $f, g \in L^1 \cap L^2(\mathbb{R}^n)$. Then $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$, with

$$\langle f, g \rangle_{L^2} = \frac{1}{(2\pi)^n} \left\langle \hat{f}, \hat{g} \right\rangle_{L^2}$$

Proof. First of all, we show that this is true for $f, g \in \mathscr{S}(\mathbb{R}^n)$. In this case, $\hat{f}, \hat{g} \in \mathscr{S}(\mathbb{R}^n)$, and we have that

$$\begin{aligned}
\langle f, g \rangle &= \int_{\mathbb{R}^n} \overline{f(x)} g(x) \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \overline{f(x)} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi) e^{i\xi \cdot x} \mathrm{d}\xi \right) \mathrm{d}x \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \overline{f(x) e^{i\xi \cdot x}} \mathrm{d}x \right) \hat{g}(\xi) \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \hat{g}(\xi) \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n} \left\langle \hat{f}, \hat{g} \right\rangle
\end{aligned}$$

Now given $f, g \in L^1 \cap L^2(\mathbb{R}^n)$. Fix $\varepsilon > 0$. By density of $C_c^\infty(\mathbb{R}^n)$, let $(f_j), (g_j)$ be sequences in $C_c^\infty(\mathbb{R}^n) \subseteq \mathscr{S}(\mathbb{R}^n)$ such that

$$\|f - f_j\|_{L^1}, \|f - f_j\|_{L^2}, \|g - g_j\|_{L^1}, \|g - g_j\|_{L^2} < \frac{1}{j}$$

By the Riemann-Lebesgue lemma,

$$\sup_{\xi \in \mathbb{R}^n} \left| \hat{f}(\xi) - \hat{f}_j(\xi) \right| \leq \|f - f_j\|_{L^1} < \frac{1}{j}$$

So $\hat{f}_j \to \hat{f}$ uniformly. Similarly, $\hat{g}_j \to \hat{g}$ uniformly. Furthermore, we have that

$$\left\|\hat{f}_j - \hat{f}_k\right\|_{L^2}^2 = (2\pi)^n \|f_j - f_k\|_{L^2}^2 \to 0$$

as $j, k \to \infty$, $(\hat{f}_j)$ is a Cauchy sequence in L^2, so $\hat{f}_j \to \hat{f}$ in L^2 by the lemma[2]. Similarly, $\hat{g}_j \to \hat{g} \in L^2$. Thus, by continuity, we have that

$$\langle f, g \rangle = \lim_{j \to \infty} \left\langle f_j, g_j \right\rangle = \frac{1}{(2\pi)^n} \lim_{j \to \infty} \left\langle \hat{f}_j, \hat{g}_j \right\rangle = \frac{1}{(2\pi)^n} (\hat{f}, \hat{g})$$

□

Corollary 5.10. $f \mapsto (2\pi)^{-n/2} \hat{f}$ defines an isometry $L^1 \cap L^2 \to L^2$, so by density extends uniquely to a linear map $(2\pi)^{-n/2} \mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Definition 5.11 (Fourier-Plancherel transform)
For $f \in L^2(\mathbb{R}^n)$, write $\hat{f} = \mathscr{F}(f)$ for the Fourier-Plancherel transform of f.

Proposition 5.12.

$$\hat{f} = \lim_{R \to \infty} \widehat{f 1_{B_R(0)}}$$

[2] The limit exists by completeness, the lemma shows that the uniform limit and L^2 limits agree in this case.

5.2 Fourier transforms of tempered distributions

Notation 5.13 (translation of a distribution). For $u \in \mathscr{S}$, we define

$$\tau_x u(\phi) = u(\tau_{-x}\phi)$$

Notation 5.14 (spatial inversion of a distribution). For $u \in \mathscr{S}$, we define

$$\check{u}(\phi) = u(\check{\phi})$$

Definition 5.15 (Fourier transform of a tempered distribution)
Let $u \in \mathscr{S}'(\mathbb{R}^n)$. Define $\hat{u} \in \mathscr{S}'(\mathbb{R}^n)$ by

$$\hat{u}(\phi) = u(\hat{\phi})$$

Proposition 5.16. If $f \in L^1(\mathbb{R}^n)$ then $\widehat{T_f} = T_{\hat{f}}$.

Lemma 5.17. Let $u \in \mathscr{S}'(R^n)$, then

(i) $\widehat{e_\xi u} = \tau_\xi \hat{u}$,

(ii) $\widehat{\tau_x u} = e_{-x}\hat{u}$,

(iii) $\widehat{\nabla^\alpha u} = i^{|\alpha|}\xi^\alpha \hat{u}$,

(iv) $\nabla^\alpha \hat{u} = (-1)^{|\alpha|}\widehat{x^\alpha u}$,

(v) $\hat{\hat{u}} = (2\pi)^n \check{u}$.

Proof. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$, then

$$\widehat{e_\xi u}(\phi) = e_\xi u(\hat{\phi}) = u(e_\xi \hat{\phi}) = u(\widehat{\tau_{-\xi}\phi}) = \hat{u}(\tau_{-\xi}\phi) = \tau_\xi \hat{u}(\phi)$$

So $\widehat{e_\xi u} = \tau \xi u$. The other results follow from the corresponding results for the Fourier transform on $\mathscr{S}$ in a similar way. □

Proposition 5.18. $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is a linear homeomorphism.

Proof. Suppose $u_j \to u$ in $\mathscr{S}'$. That is, $u_j(\phi) \to u(\phi)$ for all $\phi \in \mathscr{S}$. Then for any $\phi \in \mathscr{S}$, we have that

$$\hat{u}_j(\phi) = u_j(\hat{\phi}) \to u(\hat{\phi}) = \hat{u}(\phi)$$

so $\hat{u}_j \to \hat{u}$ in $\mathscr{S}'$. Therefore, $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is continuous. As $\mathscr{F}^4 = (2\pi)^{2n}\,\mathrm{id}$, $\mathscr{F}$ is invertible and has a continuous inverse. □

5.3 Periodic distributions and Fourier series

Definition 5.19 (periodic distribution)
$u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic if for any $g \in \mathbb{Z}^n$, $\tau_g u = u$.

Definition 5.20 (fundamental cell)
The fundamental cell of the lattice $\mathbb{Z}^n$ is

$$q = \left\{ x \in \mathbb{R}^n \;\middle|\; -\frac{1}{2} \le x_i < \frac{1}{2} \right\}$$

We will also need the set $Q = \{x \in \mathbb{R}^n \mid -1 \le x_i < 1\}$.

Lemma 5.21. There exists $\psi \in C_c^\infty(\mathbb{R}^n)$ such that

(i) $\psi \ge 0$,

(ii) $\mathrm{supp}(\psi) \subseteq Q$,

(iii) $\sum_{g \in \mathbb{Z}^n} \tau_g \psi = 1$

Such a ψ is called a periodic partition of unity. Furthermore, suppose ψ, ψ' are both p.p.u.s, then if $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic, then we have that

$$u(\psi) = u(\psi')$$

Proof. Let $\psi_0 \in C_c^\infty(\mathbb{R}^n)$, with $\mathrm{supp}(\psi_0) \subseteq \mathrm{Int}(Q)$, $\psi_0 = 1$ on q, with $\psi_0 \ge 0$. Then set

$$S(x) = \sum_{g \in \mathbb{Z}^n} \psi_0(x - g)$$

Then S is C^∞ as the sum is always finite, and $S(x) \ge 1$ for all $x \in \mathbb{R}^n$. Then

$$\psi(x) = \frac{\psi_0(x)}{S(x)}$$

works. Now suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ periodic, ψ, ψ' are p.p.u.s, then

$$\begin{aligned} u(\psi) &= u\left(\psi \sum_{g \in \mathbb{Z}^n \tau_g \psi'}\right) \\ &= \sum_{g \in \mathbb{Z}^n} u(\psi \cdot \tau_g \psi') \\ &= \sum_{g \in \mathbb{Z}^n} \tau_{-g} u(\tau_{-g} \psi \cdot \psi') \\ &= u\left(\left(\sum_{g \in \mathbb{Z}^n} \tau_{-g} \psi\right) \cdot \psi'\right) \\ &= u(\psi') \end{aligned}$$

□

Corollary 5.22. If ψ is a p.p.u, $f \in L^1_{\mathrm{loc.}}(\mathbb{R}^n)$ periodic, then

$$T_f(\psi) = \int_q f(x) \mathrm{d}x$$

Proof. Choose a sequence ψ_n of p.p.u. such that $\psi_n \to 1_q$ pointwise, with ψ_n bounded. □

Definition 5.23 (average)
if $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic, the average of u over the fundamental cell q is

$$M(u) = u(\psi)$$

for any p.p.u. ψ.

Lemma 5.24. Let $v \in \mathscr{E}'(\mathbb{R}^n)$, then

$$u = \sum_{g\in\mathbb{Z}^n} \tau_g v \tag{*}$$

converges in (the weak-$*$ topology for) $\mathscr{S}'(\mathbb{R}^n)$. Conversely, if $u \in \mathscr{D}'(\mathbb{R}^n)$ is periodic, then there exists v such that (*) holds. Hence every periodic distribution is tempered.

Proof. Let $K = \operatorname{supp}(v)$, then we have seen that there exists $N \in \mathbb{N}, C > 0$ such that

$$|v(\phi)| \le C \sup_{x\in K, |\alpha|\le N} |\nabla^\alpha \phi(x)|$$

for all $\phi \in \mathscr{E}(\mathbb{R}^n)$. Now let $\phi \in \mathscr{S}(\mathbb{R}^n)$, then

$$\left|\tau_g v(\phi)\right| = \left|v(\tau_{-g}\phi)\right| \le C \sup_{x\in K, |\alpha|\le N} |\nabla^\alpha \phi(x+g)|$$

Since K is compact, $K \subseteq B_R(0)$ for some R. Then for all $x \in K$,

$$1 + |g| \le 1 + |x| + |x+g| \le (1+R)(1+|x+g|) \implies 1 \le (1+R)\left(\frac{1+|g+x|}{1+|g|}\right)$$

Hence for any $M \ge 1$, we have that

$$\begin{aligned}\left|\tau_g v(\phi)\right| &\le C\left(\frac{1+R}{1+|g|}\right)^M \sup_{x\in K, \alpha\le N}\left((1+|x+g|)^M |\nabla^\alpha \phi(x+g)|\right)\\ &\le C\left(\frac{1+R}{1+|g|}\right)^M \sup_{x\in\mathbb{R}^n, \alpha\le N}\left((1+|x|)^M |\nabla^\alpha \phi(x)|\right)\end{aligned}$$

In particular, this means that

$$\left|\tau_g v(\phi)\right| \le \frac{C'}{(1+|g|)^{n+1}}$$

for all g. Hence $\sum_{g\in\mathbb{Z}^n} \tau_g v(\phi)$ converges for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, so $\sum_{g\in\mathbb{Z}^n} \tau_g v$ converges in $\mathscr{S}'(\mathbb{R}^n)$.
Conversely, let $u \in \mathscr{D}'(\mathbb{R}^n)$ be periodic, ψ be a p.p.u., then for any $\phi \in \mathscr{D}(\mathbb{R}^n)$,

$$u(\phi) = \left(\sum_{g\in\mathbb{Z}^n} \tau_g\psi\right) u(\phi) = \sum_{g\in\mathbb{Z}^n} u((\tau_g\psi)\phi) = \sum_{g\in\mathbb{Z}^n} u(\psi(\tau_{-g}\phi)) = \sum_{g\in\mathbb{Z}^n} \psi u(\tau_{-g}\phi) = \sum_{g\in\mathbb{Z}^n} \tau_g(\psi u)(\phi)$$

Set $v = \psi u$. Then v has compact support as ψ does. So ψu extends uniquely to $v \in \mathscr{E}'(\mathbb{R}^n)$. (*) holds by the above. □

Lemma 5.25. Suppose $u \in \mathscr{S}'$ satisfies

$$(e_{-g} - 1)u = 0$$

for all $g \in \mathbb{Z}^n$, then

$$u = \sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g}$$

converging in $\mathscr{S}'$, with $c_g \in C$ satisfying the bound

$$|c_g| \leq K(1+|g|)^N$$

for some $K > 0, N \in \mathbb{Z}$.

Proof. Let $\Lambda = 2\pi\mathbb{Z}^n$ be the dual lattice to $\mathbb{Z}^n$, that is,

$$\Lambda = \{2\pi g : g \in \mathbb{Z}^n\}$$

Step 1: $\operatorname{supp}(u) \subseteq \Lambda$. Let $\Lambda_i = \{x \in \mathbb{R}^n \mid x_i \in 2\pi\mathbb{Z}\}$. Suppose $\phi \in \mathscr{D}(\mathbb{R}^n)$ is such that $\operatorname{supp}(\phi) \cap \Lambda_i = \varnothing$. Then let $g = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$ be the i-th standard basis vector, and we get that $(e_{-g} - 1)^{-1}\phi \in \mathscr{S}(\mathbb{R}^n)$, since $\phi(x) = 0$ near $x_1 \in 2\pi\mathbb{Z}$, and $(e_{-g} - 1)(x) = 0$ if and only if $x_1 \in 2\pi\mathbb{Z}$. Hence we have that

$$u(\phi) = (e_{-g} - 1)u\left((e_{-g} - 1)^{-1}\phi\right) = 0$$

Hence we must have that $\operatorname{supp}(u) \subseteq \Lambda_i$. But this holds for all i, so we have that

$$\operatorname{supp}(u) \subseteq \bigcap_i \Lambda_i = \Lambda$$

Step 2: Multiplying by a p.p.u. Let ψ be a p.p.u., and define $\tilde{\psi}(x) = \psi(x/2\pi)$. Then we have that

$$\sum_{g \in \Lambda} \tau_g \tilde{\psi}(x) = 1, \quad \tilde{\psi} \geq 0 \quad \text{and} \quad \operatorname{supp}(\psi) \subseteq \{|x_i| < 2\pi\}$$

Let $v_g = (\tau_{2\pi g}\tilde{\psi})u$, then $\operatorname{supp}(v_g) \subseteq \{2\pi g\}$, with

$$\sum_{g \in \mathbb{Z}^n} v_g = u \quad \text{and} \quad (e_{-k} - 1)v_g = 0$$

Taking $g \in \mathbb{Z}^n$ to be the j-th standard basis vector of $\mathbb{R}^n$, we find that

$$0 = (e^{-ix_j} - 1)v_g = (e^{-i(x_j - 2\pi g_j)} - 1)v_g = (x_j - 2\pi g_j)K(x_j)v_g$$

where $K(x_j)$ is holomorphic, with $K(2\pi g_j) \neq 0$, which follows immediately from Taylor's theorem in Complex Analysis. In this case, we have that $(x_j - 2\pi g_j)v_g = 0$. But as v_g has compact support, it can be extended to $\mathscr{E}'(\mathbb{R}^n)$.

Step 3: Series expansion Let $\phi \in \mathscr{S}(\mathbb{R}^n)$, then by Taylor's theorem, there exists $\phi_j \in C^\infty(\mathbb{R}^n)$ such that

$$\phi(x) = \phi(2\pi g) + \sum_{j=1}^n (x_j - 2\pi g_j)\phi_j(x)$$

Hence we have that

$$v_g(\phi) = v_g(\phi(2\pi g)) + \sum_{j=1}^n (x_j - 2\pi g_j)v_g(\phi_j) = \phi(2\pi g)v_g(1) = \delta_{2\pi g}(\phi)u(\tau_{2\pi g}\tilde{\psi}) = c_g\delta_{2\pi g}(\phi)$$

where $c_g = u(\tau_{2\pi g}\tilde{\psi}) = v_g(1)$.

Step 4: Bounds on the coefficients

Now note that for $u \in \mathscr{S}'$, we have $N, K \in \mathbb{N}$, $C > 0$ such that

$$|u(\phi)| \leq C \sup_{x \in \mathbb{R}^n, |\alpha| \leq K} \left((1+|x|)^N |\nabla^\alpha \phi(x)|\right)$$

for all $\phi \in \mathscr{S}'(\mathbb{R}^n)$. Hence we have that

$$
\begin{aligned}
|c_g| &\le C \sup_{x,|\alpha|\le K} \left((1+|x|)^N |\nabla^\alpha \tilde{\psi}(x-2\pi g)|\right) \\
&\le C \sup_{x,|\alpha|\le K} \left((1+|x+2\pi g|)^N |\nabla^\alpha \tilde{\psi}(x)|\right) \\
&\le C \sup_{x,|\alpha|\le K} \left((1+|x|)^N (1+2\pi|g|)^N |\nabla^\alpha \tilde{\psi}(x)|\right) \\
&\le C'(1+|g|)^N
\end{aligned}
$$

for some $C' > 0$. □

Theorem 5.26. Let $u \in \mathscr{D}'(\mathbb{R}^n)$ be periodic, then

$$u = \sum_{g\in\mathbb{Z}^n} u_g T_{e_{2\pi g}}$$

converges in $\mathscr{S}'(\mathbb{R}^n)$, where $u_g = M(e_{-2\pi g} u) \in \mathbb{C}$ satisfies

$$|u_g| \le C(1+|g|)^N$$

for some $C > 0, N \in \mathbb{N}$.

Proof. Since u is periodic, $u \in \mathscr{S}'(\mathbb{R}^n)$, and its Fourier transform $\hat{u}$ is define. As $\tau_k u = u$ for all $k \in \mathbb{Z}^n$, we get that

$$(e_{-k} - 1)\hat{u} = 0$$

for all $k \in \mathbb{Z}^n$. Hence by the lemma, we gave that

$$\hat{u} = (2\pi)^n \sum_{g\in\mathbb{Z}^n} u_g \delta_{2\pi g}$$

i.e. $c_g = (2\pi)^n u_g$. Applying Fourier invresion, we find that

$$u = \sum_{g\in\mathbb{Z}^n} u_g T_{e_{2\pi g}}$$

□

Remark 5.27. By abuse of notation, we write

$$u = \sum_{g\in\mathbb{Z}^n} u_g e^{2\pi i g\cdot x}$$

and

$$u_g = \int_q u(x) e^{-2\pi i g\cdot x} \mathrm{d}x$$

even though distributions are not functions on $\mathbb{R}^n$.

Definition 5.28 (Fourier coefficients)
The u_g are called the Fourier coefficients of u.

Corollary 5.29 (Poisson summation formula)**.**

$$\sum_{g\in\mathbb{Z}^n} \delta_{x-g} = \sum_{g\in\mathbb{Z}^n} T_{e_{2\pi g}}$$

in $\mathscr{S}'(\mathbb{R}^n)$.

Proof. Let $u = \sum_{g\in\mathbb{Z}^n} \delta_g$. Then u is periodic, so by the theorem, we have that

$$u_k = M(e_{-2\pi k}u) = u(\psi e_{2\pi k}) = \sum_{g\in\mathbb{Z}^n} \psi(g)e^{-2\pi i k\cdot g} = \sum_{g\in\mathbb{Z}^n} \psi(g) = 1$$

where ψ is a p.p.u., and this means that

$$\sum_{g\in\mathbb{Z}^n} \delta_g = \sum_{g\in\mathbb{Z}^n} T_{e_{2\pi g}}$$

□

Theorem 5.30. Let $u \in \mathscr{D}'(\mathbb{R}^n)$ be periodic, with Fourier coefficients $\{u_g\}$. Then

(i) $\nabla^\alpha u \in \mathscr{D}'(\mathbb{R}^n)$ is also periodic, with

$$\nabla^\alpha u = \sum_{g\in\mathbb{Z}^n} (2\pi g)^\alpha u_g T_{e_{2\pi g}}$$

(ii) if $f \in L^1_{\text{loc.}}$ is periodic, $u = T_f$, then $|u_g| \le \|f\|_{L^1(q)}$, and $u_g \to 0$ as $|g| \to \infty$,

(iii) if $f \in \mathbb{C}^{n+1}(\mathbb{R}^n)$ periodic, $u = T_f$, then

$$f(x) = \sum_{g\in\mathbb{Z}^n} u_g e^{2\pi i g\cdot x}$$

converges uniformly,

(iv) if $f, h \in L^2_{\text{loc.}}$ are periodic, with Fourier coefficients $\{f_g\}, \{h_g\}$, then

$$\int_q \overline{f(x)}h(x)\mathrm{d}x = \sum_{g\in\mathbb{Z}^n} \overline{f_g}h_g$$

moreover,

$$f(x) = \sum_{g\in\mathbb{Z}^n} f_g e^{2\pi i g\cdot x}$$

where the series converges in $L^2(q)$.

6 Sobolev spaces

6.1 Sobolev spaces

Definition 6.1 ($W^{k,p}$ spaces)
Let $U \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}$, $p \in [1, \infty]$, then the Sobolev space $W^{k,p}(U)$ is defined by

$$W^{k,p}(U) = \{f \in L^p(U) \mid \forall|\alpha| \le k, \exists f^\alpha \in L^p(U) \text{ s.t. } \nabla^\alpha T_f = T_{f^\alpha} \text{ in } \mathscr{D}'(U)\}$$

Definition 6.2 (weak, distributional derivative)
$\nabla^\alpha f = f^\alpha$ is called the α-th weak, or distributional derivative of f.

Definition 6.3 (Sobolev norm)
For $p < \infty$, the Sobolev norm on $W^{k,p}$ is

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha|\le k} \|\nabla^\alpha f\|_{L^p}^p\right)^{1/p}$$

and for $p = \infty$, the Sobolev norm on $W^{k,\infty}$ is

$$\|f\|_{W^{k,\infty}} = \max|\alpha| \le k \|\nabla^\alpha f\|_{L^\infty}$$

Theorem 6.4. $W^{k,p}(U)$ with the Sobolev norm is a Banch space for $p \in [1, \infty]$. Moreover, when $p = 2$, $W^{k,2}(U)$ is a Hilbert space.

Proposition 6.5.

$$\int_U \nabla^\alpha f \phi \mathrm{d}x = (-1)^{|\alpha|} \int_U f \nabla^\alpha \phi \mathrm{d}x$$

for all $\phi \in C_c^\infty(U)$.

Proof. Follows by the definition of the derivative of a distribution. □

Notation 6.6. For $f \in L^1_{\text{loc.}}(U)$, we say that $\nabla^\alpha f \in L^p(U)$ if and only if there exists $f^\alpha \in L^p(U)$ such that $\nabla^\alpha T_f = T_{f^\alpha}$ in $\mathscr{D}'(U)$.

Definition 6.7 (H^s spaces)
For $s \in \mathbb{R}$, we say that $f \in \mathscr{S}'(\mathbb{R}^n)$ belongs to the Sobolev space $H^s(\mathbb{R}^n)$ if $\hat{f} \in L^2_{\text{loc.}}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} (1 + |\xi|^2)^2 |\hat{f}(\xi)|^2 \mathrm{d}\xi < \infty$$

Proposition 6.8. If $k \in \mathbb{N}$, then $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$.

Proposition 6.9. $H^2(\mathbb{R}^n)$ is a Hilbert space with inner product given by

$$\langle f, g\rangle_{H^s} = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)} \hat{g}(\xi)(1 + |\xi|^2)^s \mathrm{d}\xi$$

6.2 Sobolev embedding and trace

Theorem 6.10 (Sobolev embedding). Let $s > \frac{n}{2} + k$, $f \in H^s(\mathbb{R}^n)$. Then there exists $f^* \in C^k(\mathbb{R}^n)$ such that $f = f^*$ a.e. We'll write $f = f^*$[a] and view $H^s \le C^k$.

[a]Which makes sense since in L^p spaces we only care about functions up to equality a.e.

Proof. First suppose $f \in \mathscr{S}(\mathbb{R}^n)$. Then we have that by the Fourier inversion theorem,

$$\nabla^\alpha f(x) = \frac{i^{|\alpha|}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \xi^\alpha \hat{f}(\xi) \mathrm{d}\xi$$

for all α. Which means that

$$|\nabla^\alpha f(x)| \le \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\xi|^{|\alpha|}|\hat{f}(\xi)| d\xi \le \frac{1}{(2\pi)^n}\left(\int_{\mathbb{R}^n}(1+|\xi|^2)^s|\hat{f}(\xi)|^2 d\xi\right)^{1/2}\left(\int_{\mathbb{R}^n}|\xi|^{2|\alpha|}(1+|\xi|^2)^{-s} d\xi\right)^{1/2}$$

But if $|\alpha| \le k$, then

$$|\xi|^{2|\alpha|}(1+|\xi|^2)^{-s} \le (1+|\xi|^2)^{|\alpha|-s} \le (1+|\xi|^2)^{k-s} < (1+|\xi|^2)^{-n/2}$$

and $C_n = (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{-n/2}d\xi < \infty$, so we have that

$$\sup_{x\in\mathbb{R}^n, |\alpha|\le k} |\nabla^\alpha f(x)| \le C_n \|f\|_{H^s}$$

Now for general $f \in H^s(\mathbb{R}^n)$, let $(f_j) \subseteq \mathscr{S}(\mathbb{R}^n)$ be such that $f_j \to f$ in H^s and $f_j \to f$ a.e. In particular, (f_j) is a Cauchy sequence in H^s, so it is a Cauchy sequence in C^k, and so $f_i \to f^*$ in C^k for some f^* in C^k. But as $f_j \to f$ a.e., we must have that $f = f^*$ a.e. □

Theorem 6.11 (trace). Let $s > 1/2$, then there exists a bounded operator $T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$ such that

$$Tf = f|_{\mathbb{R}^{n-1}\times\{0\}}$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$. Tf is called the trace of f on $\Sigma = \mathbb{R}^{n-1}\times\{0\} \subseteq \mathbb{R}^n$.

6.3 The space $H_0^1(U)$

Let $U \subseteq \mathbb{R}^n$ be open, $f \in C_c^\infty(U)$. Extending f by 0 outside of U, we can view $f \in H^1(\mathbb{R}^n)$. So $C_c^\infty(U) \le H_1(\mathbb{R}^n)$.

Definition 6.12 (H_0^1 space)
Define the Sobolev space $H_0^1(U)$ to be the closure of $C_c^\infty(U) \le H^1(\mathbb{R}^n)$, with the H^1 norm.

Proposition 6.13. $H_0^1(U)$ is a Hilbert space, with inner product

$$\langle u, v\rangle_{H_0^1} = \int_U (\overline{\nabla u}\cdot\nabla v + \overline{u}v)\, dx$$

Proposition 6.14. If $u \in H_0^1(U)$, then $u = 0$ for a.e. $x \notin U$.

Proof. Suffices to show that for all $\phi \in C_c^\infty(\mathrm{Int}(U^c))$, $\int_{\mathbb{R}^n} \phi u dx = 0$. Let $\Lambda_\phi(v) = \int_{\mathbb{R}^n}\phi v dx$. Then $\Lambda_\phi(v) = 0$ for all $v \in C_c^\infty(U)$, and we have that

$$|\Lambda_\phi(v)| \le \|\phi\|_{L^2}\|v\|_{L^2} \le \|\phi\|_{L^2}\|v\|_{H^1}$$

which means that $\Lambda_\phi : H_0^1(U) \to \mathbb{C}$ is continuous, and zero on a dense subspace, so it must be identically zero. □

Proposition 6.15. For ∂U sufficiently lince, any $u \in H_0^1(U)$ vanishes on ∂U in the trace sense.

Proof. $T : H^1(U) \to H^{1/2}(\partial U)$ is bounded, and T is zero on $C_c^\infty(\mathbb{R}^n)$. □

Definition 6.16 ($H^s_{\text{loc.}}$)
For $s > 0$, define the Sobolev space

$$H^s_{\text{loc.}}(U) = \left\{u \in L^2_{\text{loc.}}(U) \mid \chi u \in H^s(\mathbb{R}^n) \text{ for all } \chi \in C^\infty_c(U)\right\}$$

Proposition 6.17. Let U' be open, $\overline{U'} \subseteq U$, then $u \in H^s_{\text{loc.}}(U)$ is in $C^k(U')$ if $s > \frac{n}{2} + k$.

Proof. We can find $\chi \in C^\infty_c(U)$ such that $\chi = 1$ on U'. Hence by the Sobolev embedding theorem, $\chi u \in H^s(\mathbb{R}^n) \leq C^k(\mathbb{R}^n)$. But $\chi = 1$ on U' implies that $u = \chi u \in C^k(U')$. □

6.4 Reillich-Kondrachov

Theorem 6.18. Let $U \subseteq \mathbb{R}^n$ be open and bounded, suppose $(u_j) \subseteq H^1_0(U)$ satisfies $\|u_j\|_{H^1} \leq 1$ for all j, and $u_j \xrightarrow{*} u$ in $L^2(U)$, with $u \in H^1_0(U)$. Then $u_j \to u \in L^2(U)$.

Proof. By Parseval, we have that

$$\|u_j - u\|^2_{L^2} = \frac{1}{(2\pi)^n}\|\hat{u}_j - \hat{u}\|^2_{L^2} = \frac{1}{(2\pi)^n}\int_{|\xi|<R}\left|\hat{u}_j(\xi) - \hat{u}(\xi)\right|^2 \mathrm{d}\xi + \frac{1}{(2\pi)^n}\int_{|\xi|>R}\left|\hat{u}_j(\xi) - \hat{u}(\xi)\right|^2 \mathrm{d}\xi \qquad (*)$$

Fix $\varepsilon > 0$. Then we can bound the second term in (*) by

$$\frac{2}{(2\pi)^n(1+R^2)}\int_{|\xi|>R}(1+|\xi|^2)(\left|\hat{u}_j(\xi)\right|^2 + |\hat{u}(\xi)|^2)\mathrm{d}\xi \leq \frac{2}{(2\pi)^n(1+R^2)}(\|u_j\|^2_{H_1} + \|u\|^2_{H_1}) \leq \frac{4}{(2\pi)^n R^2} < \varepsilon$$

for R large enough. Note that we used

$$\|u\|^2_{H^1} = \langle u, u\rangle_{H^1} = \lim_{j\to\infty}\langle u_j, u\rangle_{H^1} \leq \lim_{j\to\infty}\|u_j\|_{H^1}\|u\|_{H^1} \leq \|u\|_{H^1} \implies \|u\|_{H^1} \leq 1$$

For the first term, since $\hat{u}_j(\xi) = \langle e_\xi, u_j\rangle_{L^2(U)}$, and $e_\xi \in L^2(U)$ since U has finite measure, by $u_j \xrightarrow{*} u$, we get that $\hat{u}_j(\xi) \to \hat{u}(\xi)$ for all $\xi \in \mathbb{R}^n$. Furthermore, we have that

$$\begin{aligned}
\left|\hat{u}_j(\xi) - \hat{u}(\xi)\right|^2 &\leq 2\left(\left|\hat{u}_j(\xi)\right|^2 + |\hat{u}(\xi)|^2\right)\\
&\leq 2\left(\|u_j\|^2_{L^1} + \|u\|^2_{L^1}\right)\\
&\leq 2|U|\left(\|u_j\|^2_{L^2} + \|u\|^2_{L^2}\right)\\
&\leq 4|U|K^2
\end{aligned}$$

where we used the Hölder inequality, and the fact that $\|f\|_{L^2} \leq \|f\|_{H^1}$. Thus, by the dominated convergence theorem, the first term in (*) $\to 0$ as $j \to \infty$. Hence the (*) is $\leq 2\varepsilon$ for j large enough. □

Corollary 6.19. Let $U \subseteq \mathbb{R}^n$ be open and bounded, $(u_j) \subseteq H^1_0(U)$ bounded. Then there is a subsequence (j_k) such that $u_{j_k} \xrightarrow{*} u$ in $H^1_0(U)$, and $u_{j_k} \to u$ in $L^2(U)$.

Corollary 6.20. If $A : L^2(U) \to H^1_0(U)$ is a bounded linear map, then $A : L^2(U) \to L^2(U)$ is compact.

6.5 Application: Elliptic boundary value problems

Elliptic equation on $\mathbb{R}^n$

Proposition 6.21. Suppose $f \in H^s$, then the equation

$$-\Delta u + u = f \tag{*}$$

on $\mathbb{R}^n$ has a unique solution $u \in H^{s+2}$. Furthermore, if $s > n/2 + k$, then $f \in C^k(\mathbb{R}^n)$ and $u \in C^{k+2}(\mathbb{R}^n)$, and the equation holds in the classical sense.

Proof. Taking the Fourier transform of (*), we get

$$(|\xi|^2 + 1)\hat{u}(\xi) = \hat{f}(\xi) \text{ a.e.}$$

So the solution is given by $\hat{u}(\xi) = \dfrac{\hat{f}(\xi)}{1 + |\xi|^2}$ a.e. Fourier inversion shows that (*) has a unique solution. Finally, we have that

$$\|u\|_{H^{s+2}}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s+2} |\hat{u}(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi = \|f\|_{H^s}^2 < \infty$$

□

Dirichlet problem on a bounded domain

In this section, let $U \subseteq \mathbb{R}^n$ be open, and $f \in L^2(U)$. Then consider the Dirichlet problem

$$\begin{cases} -\Delta u + u & = f \text{ in } U \\ u & = 0 \text{ on } \partial U \end{cases} \tag{*}$$

with $u \in H_0^1(U)$, and we wish to find a solution in the distributional sense. That is,

$$\int (-\overline{u}\Delta v + \overline{u} v)\, \mathrm{d}x = \int_U (\overline{\nabla u} \cdot \nabla v + \overline{u} v)\, \mathrm{d}x = \int_U \overline{f} v \mathrm{d}x$$

for all $v \in C_c^\infty(U)$.

Definition 6.22 (weak solution)
$u \in H_0^1(U)$ is a weak solution to (*) if for all $v \in H_0^1(U)$,

$$\langle u, v \rangle_{H^1} = \langle f, v \rangle_{L^2}$$

Proposition 6.23. There exists a unique weak solution u of (*), with $\|u\|_{H^1} \leq \|f\|_{L^2}$. Furthermore, the solution operator $S : L^2(U) \to H_0^1(U)$ is a bounded linear operator. If we consider $H_0^1(U) \leq L^2(U)$, then $S : L^2(U) \to L^2(U)$ is self adjoint.

Proof. Define $\Lambda : H_0^1(U) \to \mathbb{C}$ by $\Lambda(v) = \langle f, v \rangle_{L^2}$. Then Λ is a bounded linear functional. Thus by the Riesz representation theorem, there exists a unique $u \in H_0^1(U)$ such that $\Lambda(v) = \langle u, v \rangle_{H^1}$. Then $\langle u, v \rangle_{H^1} = \langle f, v \rangle_{L^2}$ for all $v \in H_0^1(U)$, so u is the weak solution to (*). The norm bound follows by

$$\|u\|_{H^1}^2 = \langle u, u \rangle_{H^1} = \Lambda(u) = \langle f, u \rangle_{L^2} \leq \|f\|_{L^2} \|u\|_{L^2} \leq \|f\|_{L^2} \|u\|_{H^1}$$

Linearity of S follows from the equations being linear, and uniqueness of the solutions. Furthermore, given $f, g \in L^2(U)$, we have that

$$\langle f, Sg \rangle_{L^2} = \langle Sf, Sg \rangle_{H^1} = \overline{\langle Sg, Sf \rangle_{H^1}} = \overline{\langle g, Sf \rangle_{L^2}} = \langle Sf, g \rangle_{L^2}$$

so $S : L^2(U) \to L^2(U)$ is self adjoint. □

Proposition 6.24. Let $u \in H_0^1(U)$ be the unique solution to (*). If $f \in L^2 \cap H^k_{\text{loc.}}(U)$, then $u \in H_0^1 \cap H^{k+2}_{\text{loc.}}(U)$. Furthermore, if k is large enough, then (*) holds in the classical sense.

Proof. Fix $K \subseteq U$ compact, $\chi \in C_c^\infty(U)$ real valued such that $\chi = 1$ on K. Given $\phi \in \mathscr{S}(\mathbb{R}^n)$, let $v(x) = \chi(x)\phi(x)$, and using the fact that u is a weak solution, we find that

$$\int_U \overline{\nabla u} \cdot \nabla(\chi\phi) + \overline{u}\chi\phi \mathrm{d}x = \int_U \overline{f}\chi\phi \mathrm{d}x$$

Rearranging, we find that

$$\int_U \overline{\nabla(\chi u)} \cdot \nabla\phi + \overline{\nabla u} \cdot (\nabla\chi)\phi - \overline{u}\nabla\phi \cdot \nabla\chi + \overline{u}\chi\phi \mathrm{d}x = \int_U \overline{f}\chi\phi \mathrm{d}x$$

Integrating some of the terms by parts, we find

$$\int_U -\overline{\chi u}\Delta\phi + 2\overline{\nabla u} \cdot (\nabla\chi)\phi + \overline{u}\phi(\Delta\chi) + \overline{u}\chi\phi \mathrm{d}x = \int_U \overline{f}\chi\phi \mathrm{d}x$$

Hence v satisfies

$$\int_{\mathbb{R}^n} \overline{v}(-\Delta\phi + 1)\mathrm{d}x = \int_{\mathbb{R}^n} \overline{g}\phi \mathrm{d}x$$

where

$$g = -2\overline{\nabla u} \cdot \nabla\chi - u\Delta\chi + f\chi \in L^2(\mathbb{R}^n)$$

So v is a solution to $-\Delta v + v = g$. Hence $v \in H^2(\mathbb{R}^n)$. For any $\psi \in C_c^\infty(U)$, we can take $K = \text{supp}(\psi)$, then $\psi u = \psi v \in H^2(\mathbb{R}^n)$, so $u \in H_0^1 \cap H^2_{\text{loc.}}(U)$.

In general, we have that if $f \in L^2 \cap H^k_{\text{loc.}}(U)$, then $g \in H^k(\mathbb{R}^n)$, so $v \in H^{k+2}(\mathbb{R}^n)$ and hence $u \in H_0^1 \cap H^{k+2}_{\text{loc.}}(U)$. Finally, note that by the Sobolev embedding theorem, if $k + 2 \geq \frac{n}{2} + \ell$, then $v \in C^\ell(\mathbb{R}^n)$. But being C^ℓ is a local property, and every point in U has a compact neighbourhood contained in U, so $u \in C^\ell(\mathbb{R}^n)$. ☐

Dirichlet problem with potential

Let $U \subseteq \mathbb{R}^n$ be open and bounded, $V : U \to \mathbb{R}$ smooth and bounded, $f \in L^2(U)$. Then consider the Dirichlet problem

$$\begin{cases} -\Delta u + Vu & = f \text{ in } U \\ u & = 0 \text{ on } \partial U \end{cases} \tag{*}$$

Definition 6.25 (weak solution)
$u \in H_0^1(U)$ is a weak solution to (*) if

$$\int_U \left(\overline{\nabla u} \cdot \nabla v + V\overline{u}v\right) \mathrm{d}x = \int_U \overline{f}v \mathrm{d}x \tag{†}$$

for all $v \in H_0^1(U)$.

Proposition 6.26. Either

(i) there exists $\omega \in H_0^1 \cap C^\infty(U)$ nonzero such that

$$-\Delta\omega + V\omega = 0$$

(ii) or for all $f \in L^2(U)$, there exists a unique $u \in H_0^1(U)$ such that (*) holds.

Proof. Notice that (†) is equivalent to

$$\int_U (\overline{\nabla u} \cdot \nabla v + \overline{u} v)\, \mathrm{d}x = \int_U \overline{(f + (1 - V))\, u} v \mathrm{d}x$$

Let $S : L^2(U) \to H_0^1(U)$ be the solution operator for $V = 1$ from above. Then

$$(\dagger) \iff u = S(f + (1 - v)u) \iff (I - K)u = Sf$$

where $Ku = S((I - V)u)$. Since $K : L^2(U) \to H_0^1(U)$ is bounded, $K : L^2(U) \to L^2(U)$ is compact. Then either

(a) $\ker(I - K) \neq 0$, so there exists $\omega \in L^2(U)$ nonzero such that $(I - K)\omega = 0$,

(b) $\operatorname{im}(I - K) = L^2(U)$, so there exists a unique u such that $(I - K)u = Sf$.

Moreover, in (a), $\ker(I - K)$ is finite dimensional, and $\omega = S((1 - v)\omega)$, so $\omega \in H_0^1(U)$. By repeating the above argument, $\omega \in H_0^1(U) \cap C^\infty(U)$ by the Sobolev embedding theorem.

In (b), $u = S(f + (1 - v)u) \in H_0^1(U)$, so u is a weak solution to (*). □

Theorem 6.27. There exists an orthonormal basis $\{\psi_k\}$ of $L^2(U)$ such that

(i) $\psi_k \in H_0^1(U) \cap C^\infty(U)$,

(ii) $-\Delta \psi_k = \lambda_k \psi_k$ in U,

where $0 \le \lambda_1 \le \lambda_2 \le \ldots$, with $\lambda_k \to \infty$.

Proof. By the spectral theorem for compact self adjoint operators, we have that

$$\sigma(S) = \{0, \mu_1, \mu_2, \ldots\}$$

with $\mu_k \in \mathbb{R}$, and the only accumulation point is at 0. We also have a corresponding orthonormal basis of eigenvectors $\{\psi_k\}$ for S.

Now as $S\psi_k = \mu_k \psi_k$, $\psi_k \in H_0^1(U)$ as $\psi_k \in \operatorname{im}(S)$. Moreover,

$$\langle \psi_k, v \rangle_{L^2} = \langle S\psi_k, v \rangle_{H^1} = \mu_k \langle \psi_k, v \rangle_{H^1}$$

for all $v \in H_0^1(U)$. Setting $v = \psi_k$, we get that $1 = \mu_k \|\psi_k\|_{H_1}^2$. So $\mu_k > 0$. Moreover, as $1 = \|\psi_k\|_{L^2} \le \|\psi_k\|_{H^1}$, we have that $\mu_k \le 1$. Then notice that ψ_k is a weak solution to

$$\begin{cases} -\Delta \psi_k = \lambda_k \psi_k \text{ in } U \\ \psi_k = 0 \text{ on } \partial U \end{cases}$$

where $\lambda_k = 1/\mu_k - 1 \ge 0$. Since μ_k has only 0 as an accumulation point, $\lambda_k \to \infty$. Elliptic regularity implies that $\psi \in C^\infty(U)$. □