Differential geometry

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Contents

1 Smooth manifolds and smooth maps

1.1 Definitions

Definition 1.1 (smooth map)
Let $X \subseteq \mathbb{R}^N$. Then $f : X \to \mathbb{R}^m$ is smooth if for all $x \in X$, there exists an open neighbourhood $U \subseteq \mathbb{R}^n$ of x, and $F : U \to \mathbb{R}^m$ smooth, such that $F|_{U\cap X} = f|_{U\cap X}$.

Definition 1.2 ((embedded) manifold, parametrisation, charts, coordinate functions)
$X \subseteq \mathbb{R}^N$ is a k-dimensional manifold if each $x \in X$ has a neighbourhood V which is diffeomorphic to an open set in $\mathbb{R}^k$.

If $\phi : U \to V$ is the diffeomorphism, we say that ϕ is a parametrisation of V. The inverse map $\phi^{-1} : V \to U$ is called a chart on V. If $\phi^{-1} = (x_1, \ldots, x_k)$, the $x_i : V \to \mathbb{R}$ are called coordinate functions.

Definition 1.3 (submanifold)

If X, Z are manifolds in $\mathbb{R}^N$, with $Z \subseteq X$, then Z is a submanifold of X. The codimension of Z in X is $\operatorname{codim}_X(Z) = \dim(X) - \dim(Z)$.

Definition 1.4 (tangent space)

Let $X \subseteq \mathbb{R}^N$ be a manifold, $\phi : U \to X$ a parametrisation around $x \in X$, with $\phi(0) = x$. Then define

$$T_xX = \mathrm{d}\phi_0(\mathbb{R}^k)$$

where $\mathrm{d}\phi_0 : \mathbb{R}^k \to \mathbb{R}^N$ is the derivative of ϕ at 0.

Lemma 1.5. The tangent space is well defined, that is, T_xX is independent of the choice of ϕ. Furthermore, $\dim(T_xX) = \dim(X)$.

Proof. Suppose we had another parametrisation $\psi : V \to X$, with $\psi(0) = x$. By shrinking U, V, wlog $\phi(U) = \psi(V)$. Then $h = \psi^{-1} \circ \phi : U \to V$ is a diffeomorphism. Then, by the chain rule,

$$\mathrm{d}\phi_0 = \mathrm{d}\psi_0 \circ \mathrm{d}h_0$$

and $\mathrm{d}h_0$ is an invertible linear map, so $\operatorname{im}(\mathrm{d}\phi_0) = \operatorname{im}(\mathrm{d}\psi_0)$. Now since $\phi^{-1} : \phi(U) \to U$ is smooth, we can choose $W \subseteq \mathbb{R}^N$ open neighbourhood of x, and a smooth map $\Phi : W \to \mathbb{R}^k$ with $\Phi|_{\phi(U)} = \phi^{-1}$. Then $\Phi \circ \phi = \mathrm{id}_U$, so by the chain rule, we have

$$\mathbb{R}^k \xrightarrow{\mathrm{d}\phi_0} T_xX \xrightarrow{\mathrm{d}\Phi_x} \mathbb{R}^k$$

is the identity map on $\mathbb{R}^k$, so $\mathrm{d}\phi_0 : \mathbb{R}^k \to T_xX$ is an isomorphism. □

Definition 1.6 (derivative)

Let $f : X \to Y$ be smooth. Then the derivative of f at x is $\mathrm{d}f_x : T_xX \to T_{f(x)}Y$, given by

- Choose parametrisations ϕ near x, ψ near $f(x)$, with

$$\begin{array}{ccc} (X, x) & \xrightarrow{f} & (Y, f(x)) \\ \phi \uparrow & & \uparrow \psi \\ (U, 0) & \xrightarrow[h := \psi^{-1} \circ f \circ \phi]{} & (V, 0) \end{array}$$

- Then define $\mathrm{d}f_x : T_xX \to T_{f(x)}Y$ by

$$\begin{array}{ccc} T_xX & \xrightarrow{\mathrm{d}f_x} & T_{f(x)}Y \\ \mathrm{d}\phi_0 \uparrow & & \uparrow \mathrm{d}\psi_0 \\ \mathbb{R}^k & \xrightarrow[\mathrm{d}h_0]{} & \mathbb{R}^\ell \end{array}$$

i.e. $\mathrm{d}f_x := \mathrm{d}\psi_0 \circ \mathrm{d}h_0 \circ \mathrm{d}\phi_0^{-1}$.

Lemma 1.7. The definition of the derivative is independent of the choice of ϕ and ψ.

Lemma 1.8 (chain rule)**.** If we have smooth maps

$$(X, x) \xrightarrow{\quad f \quad} (Y, y) \xrightarrow{\quad g \quad} (Z, z)$$

then

$$\mathrm{d}(g \circ f)_x = \mathrm{d}g_{f(x)} \circ \mathrm{d}f_x$$

Proof. By the chain rule for functions between Euclidean spaces. □

Theorem 1.9 (inverse function)**.** Suppose $f : X \to Y$ is a smooth map, with $\mathrm{d}f_x$ being an isomorphism. Then f is a diffeomorphism in a neighbourhood of x.

Proof. Follows from the inverse function theorem for smooth functions between Euclidean spaces. □

1.2 Regular values, Sard's theorem

Definition 1.10 (critical point, critical value, regular value)
Let $f : X \to Y$ be smooth. Then

(i) $x \in X$ is a critical point if $\mathrm{d}f_x : T_xX \to T_{f(x)}Y$ is not surjective. We write $\mathrm{Crit}(f)$ for the set of all critical points.

(ii) $y \in f(\mathrm{Crit}(f))$ is called a critical value,

(iii) $y \in Y \setminus f(\mathrm{Crit}(f))$ is called a regular value of f.

Proposition 1.11. If $\dim(X) < \dim(Y)$, then $\mathrm{Crit}(f) = X$, and the preimage of a regular value is the empty set.

Proof. Since $\mathrm{rank}(\mathrm{d}f_x) \leq \dim(X) < \dim(Y) = \dim(T_{f(x)}Y)$, the derivative is not surjective at any point. □

Theorem 1.12 (preimage)**.** Let y be a regular value of $f : X \to Y$, with $\dim(X) \geq \dim(Y)$. Then $f^{-1}(y)$ is a submanifold of y, with

$$\dim(f^{-1}(y)) = \dim(X) - \dim(Y)$$

Proof. Let $x \in f^{-1}(y)$. Since y is a regular value, $\mathrm{d}f_x : T_xX \to T_yY$ is surjective. Let $K = \ker(\mathrm{d}f_x) \leq T_xX$, with $d = \dim(K) = \dim(X) - \dim(Y)$.

Consider an embedding $X \subseteq \mathbb{R}^N$, and let $T : \mathbb{R}^N \to \mathbb{R}^d$ be any linear map such that $\ker(T) \cap K = 0$[1]. Then consider the map $F : X \to Y \times \mathbb{R}^d$, given by $F(z) = (f(z), T(z))$. Then we have that

$$\mathrm{d}F_x(v) = (\mathrm{d}f_x(v), T(v))$$

which is an injective linear map between two vector spaces of the same dimension, so it must be an isomorphism. Thus, by the inverse function theorem, F is a local diffeomorphism at x. That is, there exists neighbourhoods U of x, V of $(y, T(x))$ such that $F : U \to V$ is a diffeomorphism. So we have that

[1] Which exists, by basic linear algebra arguments if N is sufficiently large. For example, projection onto the last d coordinates, where we embed X into $\mathbb{R}^{N-d} \times 0$.

$$F : f^{-1}(y) \cap U \to (\{y\} \times \mathbb{R}^d) \cap V$$

is a diffeomorphism. Hence $f^{-1}(y)$ is a manifold, with $\dim(f^{-1}(y)) = d$. □

Corollary 1.13. If $f : X \to Y$ is a smooth map, $\dim(X) = \dim(Y)$, X compact and y is a regular value of f, then $f^{-1}(y)$ is a finite set of points.

Proof. $f^{-1}(y)$ is a 0-dimensional manifold, that is, a discrete set of points. But f is continuous and Y Hausdorff, so $f^{-1}(y)$ is closed. But a closed discrete subset of a compact space is finite. □

In fact, near regular values, smooth maps are covering maps.

Theorem 1.14 (stack of records)**.** Let $f : X \to Y$ be a smooth map, $\dim(X) = \dim(Y)$ and X compact. Let y be a regular value of f. Say $f^{-1}(y) = \{x_1, \ldots, x_k\}$. Then there exists an open neighbourhood U of u, and open neigbourhoods V_i of x_i, such that

$$f^{-1}(U) = \bigsqcup_{i=1}^{k} V_i \quad \text{and} \quad f|_{V_i} : V_i \to U \text{ is a diffeomorphism}$$

Proof. By the inverse function theorem, we can choose disjoint neighbourhoods W_i of x_i such that f maps W_i diffeomorphically to a neighbourhood of y. Now notice that $f\left(X \setminus \bigcup_i W_i\right)$ is a compact set which does not contain y, so we can take

$$U = \bigcup_i f(W_i) \setminus f\left(X \setminus \bigcup_i W_i\right)$$

□

With this, we have a result which is akin to the valency theorem from Riemann surfaces, or the degree of a (branched) covering.

Corollary 1.15. The function $y \mapsto \left|f^{-1}(y)\right|$ is locally constant as y ranges over the regular values of f.

Theorem 1.16 (Sard)**.** Let $f : X \to Y$ be a smooth map. Then $\mathrm{Crit}(f)$ has measure zero[a].

[a] Formally, we haven't defined a measure, or even σ-algebra on manifolds. However, in this case, we say that a set $A \subseteq X$ has measure zero if for all parametrisations ϕ, $\phi^{-1}(A)$ has measure zero in $\mathbb{R}^k$. Since a measure would be countably subadditive, and manifolds are second countable, this notion of "measure zero" makes sense.

Corollary 1.17. The set of regular values of $f : X \to Y$ is dense.

Proof. A set of measure zero can't contain any nonempty open set. □

1.3 Transversality

Definition 1.18 (transversal)

A smooth map $f : X \to Y$ is transversal to a submanifold $Z \leq Y$ if for every $x \in f^{-1}(Z)$, we have that

$$\mathrm{im}(\mathrm{d}f_x) + T_{f(x)}Z = T_{f(x)}Y$$

In this case, we write $f \pitchfork Z$.

Proposition 1.19. $f \pitchfork \{y\}$ if and only if y is a regular value for f.

Theorem 1.20. Suppose $f : X \to Y$ is transversal to a submanifold $Z \leq Y$. Then $f^{-1}(Z)$ is a submanifold of X. Moreover,

$$\operatorname{codim}_X(f^{-1}(Z)) = \operatorname{codim}_Y(Z)$$

Proof. Nonexaminable, so omitted. □

Definition 1.21 (transversality of submanifolds)
Suppose X, Z are submanifolds of Y. Then we say that X, Z are transversal, written $X \pitchfork Z$ if

$$T_xX + T_xZ = T_xY$$

for all $x \in X \cap Z$. Equivalently, the inclusion map $\iota : X \hookrightarrow Y$ is transversal to Z.

Proposition 1.22. Suppose X, Z are transversal submanifolds of Y. Then $X \cap Z$ is a submanifold of Y, with

$$\operatorname{codim}_Y(X \cap Z) = \operatorname{codim}_Y(X) + \operatorname{codim}_Y(Z)$$

1.4 Manifolds with boundary

Definition 1.23 (closed half space)
The closed half space $\mathbb{H}^k$ is defined by

$$\mathbb{H}^k = \left\{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_k \geq 0\right\}$$

with boundary

$$\partial\mathbb{H}^k = \{x_k = 0\}$$

Definition 1.24 ((embedded) manifold with boundary)
A subset $X \subseteq \mathbb{R}^N$ is called a k-manifold with boundary if each $x \in X$ has a neighbourhood diffeomorphic to an open set in $\mathbb{H}^k$. The boundary of X, denoted ∂X, is the set of points in the image of $\partial\mathbb{H}^k$ under some parametrisation. We write $\operatorname{Int}(X) = X \setminus \partial X$ for the interior.

Proposition 1.25. The tangent space T_xX, as defined for manifolds, is well defined for manifolds with boundary.

Proposition 1.26. $\operatorname{Int}(X)$ is a k-manifold without boundary, and ∂X is a $(k-1)$-manifold without boundary.

Lemma 1.27. Let X be a manifold, $f : X \to \mathbb{R}$ a smooth function with 0 as a regular value. Then $\{x \mid f(x) \geq 0\}$ is a smooth manifold with boundary $f^{-1}(0)$.

Proof. The set $\{x \mid f(x) > 0\}$ is open, so it is a submanifold of the same dimension as X. For a point $x \in f^{-1}(0)$, the same proof as in the preimage theorem shows that x has a neighbourhood diffeomorphic to a neighbourhood of a point in $\partial\mathbb{H}^k$. □

Theorem 1.28 (preimage)**.** Let $f : X \to Y$ be a smooth map from an m-manifold with boundary to an n-manifold, with $m > n$. Suppose $y \in Y$ is a regular value for f, and $f|_{\partial X}$. Then $f^{-1}(y)$ is a smooth $(m-n)$-manifold with boundary $f^{-1}(y) \cap \partial X$.

Proof. Since being a submanifold is a local property, we may assume wlog that $X = \mathbb{H}^m$, $Y = \mathbb{R}^n$. Now consider $z \in f^{-1}(y)$. If $z \in \mathrm{Int}(\mathbb{H}^m)$, then the preimage theorem shows that $f^{-1}(y)$ is a smooth $(m-n)$-manifold near z.

Now suppose $z \in \partial\mathbb{H}^m$. Since f is smooth, we have a neighbourhood U of $\mathbb{R}^m$, $F : U \to \mathbb{R}^n$ smooth such that $F|_{U\cap\mathbb{H}^m} = f$. Since y is a regular value for f, and

$$\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial x_i} \text{ for } i = 1, \dots, k-1$$

Thus, y is a regular value for F. Hence $F^{-1}(y)$ is a $(m-n)$-manifold. Now let $\pi : F^{-1}(y) \to \mathbb{R}$ be the projection $\pi(x_1, \dots, x_m) = x_m$. But then we have that for $x \in \pi^{-1}(0)$,

$$T_x F^{-1}(y) = \ker(\mathrm{d}F_x) = \ker(\mathrm{d}f_x)$$

So 0 is a regular value for π, as y is a regular value for $f|_{\partial\mathbb{H}^m}$. Finally, notice that

$$F^{-1}(y) \cap \mathbb{H}^m = f^{-1}(y) \cap U = \left\{x \in F^{-1}(y) \mid \pi(x) \geq 0\right\}$$

so it is a smooth manifold with boundary $\pi^{-1}(0)$. □

Theorem 1.29. Suppose X is a manifold with boundary, Y a manifold and Z a submanifold of Y. Given $f : Y \to X$ and $f|_{\partial X} : \partial X \to Y$ are both transversal to Z, then $f^{-1}(Z)$ is a manifold with boundary $f^{-1}(Z) \cap \partial X$ and $\mathrm{codim}_X(f^{-1}(Z)) = \mathrm{codim}_Y(Z)$.

1.5 Degree modulo 2

Definition 1.30 (smooth homotopy)

Given smooth maps $f, g : X \to Y$, a smooth homotopy between f and g is a smooth map $H : X \times I \to Y$, with $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. If such a map exists, we say f and g are smoothly homotopic.

Proposition 1.31. Smooth homotopy is an equivalence relation.

Notation 1.32. We write $f_t = H(\cdot, t)$ for the one-parameter family of maps given by a smooth homotopy.

Definition 1.33 (smooth isotopy)

A smooth isotopy between diffeomorphisms $f, g : X \to Y$ is a homotopy $H : X \times [0, 1] \to Y$ between f and g, such that $f_t = H(\cdot, t)$ is a diffeomorphism for all $t \in [0, 1]$. If such a map exists, we say f and g are smoothly isotopic.

Theorem 1.34 (classification of 1-manifolds)**.** Every compact connected 1-manifold with boundary is diffeomorphic to $[0, 1]$ or S^1.

Corollary 1.35. The boundary of any compact 1-manifold with boundary consists of an even number of points.

Proof. Every compact manifold is the disjoint union of finitely many compact connected manifolds. □

Lemma 1.36 (homotopy lemma)**.** Suppose $f, g : X \to Y$ are smoothly homotopic, X compact without boundary, $\dim(X) = \dim(Y)$. If y is a regular value for f and g, then

$$\left|f^{-1}(y)\right| \equiv \left|g^{-1}(y)\right| \pmod 2$$

Proof. Let $F : X \times I \to Y$ be a smooth homotopy between f and g. First suppose y is a regular value for F. Then $F^{-1}(y)$ is a compact 1-manifold with boundary

$$F^{-1}(y) \cap (X \times \{0\} \cup X \times \{1\}) = f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}$$

Therefore, we have that $\left|\partial F^{-1}(y)\right| = \left|f^{-1}(y)\right| + \left|g^{-1}(y)\right|$. But $F^{-1}(y)$ is a compact 1-manifold with boundary, so the number of points in the boundary is even, which gives us the required result.

On the other hand, if y is not a regular value for F, by the stacks of records theorem, we know that $\left|f^{-1}(w)\right|, \left|g^{-1}(w)\right|$ are locally constant as w ranges over regular values. Therefore, there are neighbourhoods V, W of y, consisting of regular values of f, g respectively, then

$$\left|f^{-1}(w)\right| = \left|f^{-1}(y)\right| \text{ for all } w \in V$$

and

$$\left|g^{-1}(w)\right| = \left|g^{-1}(y)\right| \text{ for all } w \in W$$

Now by Sard's theorem, we can choose a regular value $z \in V \cap W$ of F, then

$$\left|f^{-1}(y)\right| = \left|f^{-1}(z)\right| \equiv \left|g^{-1}(z)\right| = \left|g^{-1}(y)\right| \pmod 2$$

□

Lemma 1.37 (homogeneity)**.** Let X be a smooth connected manifold, possibly with boundary. Let $y, z \in \operatorname{Int}(X)$. Then there exists a diffeomorphism $h : X \to X$ smoothly isotopic to id_X such that $h(y) = z$.

Proof. Since X is connected, suffices to check that the result holds locally. Choose a small neighbourhood of y which is diffeomorphic to $\mathbb{R}^k$. So we only need to construct $F : \mathbb{R}^k \to \mathbb{R}^k$ smoothly isotopic to the identity, such that the isotopy restricts to the identity on $\mathbb{R}^n \setminus B_1(0)$.

Let $\varphi : \mathbb{R}^k \to \mathbb{R}$ be a smooth bump function, with

(i) $\varphi(x) > 0$ for $|x| < 1$,

(ii) $\varphi(x) = 0$ for $|x| \geq 1$.

Then given a unit vector $u \in \mathbb{R}^k$, consider the ODE in $\mathbb{R}^k$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\varphi(x)$$

By standard ODE theory, if $F_t : \mathbb{R}^k \to \mathbb{R}^k$ is the flow of this differential equation, that is, $t \mapsto F_t(x)$ is the solution to the ODE with $F_t(0) = x$. Then we have that

(i) F_t is defined for all $x \in \mathbb{R}^k$, for all $t \geq 0$ and smooth.

(ii) $F_0 = \mathrm{id}$,

(iii) $F_{t+s} = F_t \circ F_s$.

Furthermore, F_t leaves all points outside $B_1(0)$ fixed. Finally, for appropriate choices of u, t, F_t will map the origin to any point in the open unit ball. □

Theorem 1.38 (degree mod 2). Suppose X compact manifold without boundary, Y connected, $\dim(X) = \dim(Y)$, $f : X \to Y$ smooth. Then if y, z regular values of f, we have that

$$\left|f^{-1}(y)\right| \equiv \left|f^{-1}(z)\right| \pmod 2$$

Furthermore, this value only depends on the homotopy class of f.

Proof. Given y, z, by the homogeneity lemma, we have a diffeomorphism h smoothly isotopic to the identity such that $h(y) = z$. Now notice that z is also a regular value for $h \circ f$. Since $h \circ f$ is homotopic to f, the homotopy lemma tells us that

$$\left|f^{-1}(y)\right| = \left|(h \circ f)^{-1}(z)\right| \equiv \left|f^{-1}(z)\right| \pmod 2$$

Now suppose g is smoothly isotopic to f. Then by Sard's theorem, there exists a point $y \in Y$ which is a regular value for f and g, since the (finite) union of measure zero sets has measure zero. Thus, by the homotopy lemma, we have that

$$\left|f^{-1}(y)\right| \equiv \left|g^{-1}(y)\right| \pmod 2$$

□

Definition 1.39 (degree mod 2)
The degree modulo 2 for a smooth map $f : X \to Y$ is defined by

$$\deg_2(f) = \left|f^{-1}(y)\right| \mod 2$$

for any regular value y of f.

Theorem 1.40 (Brouwer). Any smooth map $f : D^k \to D^k$ has a fixed point.

Proof. Suppose not. Then (as in the proof for $k = 2$ from algebraic topology), we have a retraction $g : D^k \to S^{k-1}$. Then $H(x, t) = f(tx)$ is a homotopy between a constant map $S^{k-1} \to S^{k-1}$ and $\mathrm{id}_{S^{k-1}}$. But the first has $\deg_2 = 0$ and the second has $\deg_2 = 1$. □

Remark 1.41. Morally this is the same as the homology proof of Brouwer. The degree modulo 2 is just induced map on the top homology group with coefficients in $\mathbb{F}_2$.

1.5.1 Intersection numbers modulo 2

Now suppose

(i) X is a compact manifold without boundary,

(ii) Y is a connected manifold,

(iii) $Z \subseteq Y$ is a closed[2] submanifold without boundary,

[2] As in a closed subset, not a closed manifold.

(iv) $f : X \to Y$ smooth, with $f \pitchfork Z$,

(v) $\dim(X) + \dim(Z) = \dim(Y)$.

In this case, $f^{-1}(Z)$ is a closed 0-dimensional submanifold of a compact manifold X, so it is a finite set.

Definition 1.42 (mod 2 intersection number)
The mod 2 intersection number of f with Z is

$$I_2(f, Z) = \left|f^{-1}(Z)\right| \mod 2$$

Proposition 1.43. If f_0, f_1 are transversal to Z and homotopic, then $I_2(f_0, Z) = I_2(f_1, Z)$.

Proposition 1.44. For any map $f : X \to Y$, we can find $g : X \to Y$ homotopic to f, such that g is transversal to Z. Therefore we can define $I_2(f, Z) = I_2(g, Z)$.

Definition 1.45 (mod 2 intersection number of submnifolds)
If f is the inclusion map $X \hookrightarrow Z$, define $I_2(X, Z) = I_2(f, Z)$.

Proposition 1.46. If $X \pitchfork Z$, then $I_2(X, Z) = |X \cap Z| \mod 2$.

2 Geometry of curves and surfaces

2.1 Curves

Definition 2.1 ((regular) curve)
Let $I \subseteq \mathbb{R}$ be an interval, X be manifold. A curve in X is a smooth map $\alpha : I \to X$. We say that α is regular if $\dot{\alpha} \in T_{\alpha(t)}X$ is never zero.

Definition 2.2 (arc length)
Given $t \in I$, the arc length of $\alpha : I \to \mathbb{R}^3$ from $t_0 \in I$ is given by

$$s(t) = \int_{t_0}^{t} |\dot{\alpha}(\tau)| \mathrm{d}\tau$$

Proposition 2.3. Suppose α is a regular curve. Then s is a strictly increasing function, and so has a smooth inverse. Then the curve $\beta(s) = \alpha(t(s))$ is parametrised by arc length, that is, $\left|\dot{\beta}\right| = 1$.

From now on, all curves will be parametrised by arc length unless otherwise specified.

Definition 2.4 (tangent)

Let $\alpha : I \to \mathbb{R}^3$ be a curve, the tangent vector of α at s is $t(s) = \dot{\alpha}(s)$.

Definition 2.5 (curvature, normal, osculating plane)
Let $\alpha : I \to \mathbb{R}^3$ be a curve, the curvature of α at $s \in I$ is defined by

$$\kappa(s) = |\ddot{\alpha}(s)|$$

If $\kappa(s) \neq 0$, then the unit normal vector to α is $n(s)$ given by

$$\ddot{\alpha}(s) = \kappa(s)n(s)$$

The plane spanned by $t(s)$ and $n(s)$ is called the osculating plane at s.

Definition 2.6 (binormal, torsion)
The binormal vector of α is

$$b(s) = t(s) \wedge n(s)$$

then we have that

$$\dot{b}(s) = \tau(s)n(s)$$

where $\tau(s)$ is the torsion of α at s.

Theorem 2.7 (Frenet formulae).

$$\begin{aligned}\dot{t} &= \kappa n\\ \dot{n} &= -\kappa t - \tau b\\ \dot{b} &= \tau n\end{aligned}$$

Proof. Easy differentiation. ☐

Theorem 2.8 (fundamental theorem of curves). Given smooth functions $\kappa(s) > 0$ and $\tau(s)$, there exists a regular curve α such that s is the arc length, $\kappa(s)$ is the curvature, $\tau(s)$ is the torsion of α. Moreover, α is unique up to a rotation and/or a translation.

Proof. The result follows from the existence and uniqueness of solutions to ODEs, and the Frenet formulae. Then we can see that the solution is unique given initial conditions. ☐

2.2 Isoperimetric inequality in $\mathbb{R}^2$

Lemma 2.9 (Wirtinger's inequality). Let $f : \mathbb{R} \to \mathbb{R}$ be C^1, f periodic with period L. Suppose

$$\int_0^L f(t)\mathrm{d}t = 0$$

then

$$\int_0^L |f'(t)|^2 \mathrm{d}t \geq \frac{4\pi^2}{L^2} \int_0^L |f(t)|^2 \mathrm{d}t$$

with equality if and only if there exists constants a_{-1}, a_1 such that

$$f(t) = a_{-1}e^{-2\pi it/L} + a_1 e^{2\pi it/L}$$

Proof. Consider the Fourier expansions of f and f', that is

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi ikt/L} \quad \text{and} \quad f'(t) = \sum_{k=-\infty}^{\infty} b_k e^{2\pi ikt/L}$$

The Fourier coefficients are given by

$$a_k = \frac{1}{L}\int_0^L f(t)e^{-2\pi ikt/L}\mathrm{d}t \quad \text{and} \quad b_k = \frac{1}{L}\int_0^L f'(t)e^{-2\pi ikt/L}\mathrm{d}t$$

The hypotheses imply $a_0 = b_0 = 0$, and by integration by parts, we find that

$$b_k = \frac{2\pi ik}{L}a_k$$

Then, by Parseval's identity, we have that

$$\begin{aligned}\int_0^L |f'|^2 \mathrm{d}t &= L\sum_k |b_k|^2 \\ &= \frac{4\pi^2}{L}\sum_k k^2|a_k|^2 \\ &\geq \frac{4\pi^2}{L}\sum_k |a_K|^2 \\ &= \frac{4\pi^2}{L^2}\int_0^L |f|^2\mathrm{d}t\end{aligned}$$

and equality holds if and only if $a_k = 0$ for all $|k| > 1$. □

Theorem 2.10 (isoperimetric inequality). Let $\Omega \subseteq \mathbb{R}^2$ be a connected, bounded open set, with $\partial\Omega$ a connected 1-manifold of class C^1. Then

$$\ell(\partial\Omega)^2 \geq 4\pi|\Omega|$$

with equality if and only if Ω is a disk.

Proof. Define the vector field $X(x, y) = (x, y)$, and let n be the outward pointing normal vector field along $\partial\Omega$. The divergence theorem gives us that

$$\int_\Omega \operatorname{div}(X)\mathrm{d}A = \int_{\partial\Omega}\langle X, n\rangle \,\mathrm{d}s$$

But $\operatorname{div}(X) = 2$. Combining this with the Cauchy-Schwarz inequality, we have that

$$2|\Omega| = \int_{\partial\Omega}\langle X, n\rangle \,\mathrm{d}s \leq \int_{\partial\Omega}|X|\mathrm{d}s \tag{*}$$

Now by the Cauchy-Schwarz inequality again, we have that

$$2|\Omega| \leq \left(\int_{\partial\Omega}|X|^2\mathrm{d}s\right)^{1/2}\left(\int_{\partial\Omega}\mathrm{d}s\right)^{1/2} = \ell(\partial\Omega)^{1/2}\left(\int_{\partial\Omega}|X|^2\mathrm{d}s\right)^{1/2} \tag{**}$$

Since we parametrise $\partial\Omega$ by arc length, $X(s) = (x(s), y(s))$ along $\partial\Omega$ are C^1, and periodic with period $L = \ell(\partial\Omega)$. Hence by Wirtinger's inequality, we have that

$$\left(\int_{\partial\Omega}|X|^2\mathrm{d}s\right)^{1/2} \leq \left(\frac{\ell(\partial\Omega)^2}{4\pi^2}\int_{\partial\Omega}|X'|^2\mathrm{d}s\right)^{1/2} = \frac{\ell(\partial\Omega)^{3/2}}{2\pi} \tag{***}$$

Combining (**) and (***) gives the required result. Equality holds if and only if we have equality in (*), (**) and (***). But equality in (**) implies that $s \mapsto |X(s)|$ is constant, so Ω is a disc. □

2.3 First fundamental form and area

Definition 2.11 (first fundamental form)
Let $S \subseteq \mathbb{R}^3$ eba surface. The first fundamental form of S at p is thq quadratic form $I_p : T_pS \to \mathbb{R}$ defined by

$$I_p(w) = \langle w, w \rangle = |w|^2$$

Definition 2.12 (isometric)
Surfaces S_1, S_2 are isometric if there exists a diffeomorphism $f : S_1 \to S_2$ such that for all $p \in S_1$, $df_p : T_pS_1 \to T_{f(p)}S_2$ is an isometry.

Let $\phi : U \to S$ be a parametrisation of a neighbourhood of $p \in S$. Let (u, v) be coordinates in u, and define

$$\phi_u(u, v) = \frac{\partial \phi}{\partial u} \in T_{\phi(u,v)}S \quad \text{and} \quad \phi_v(u, v) = \frac{\partial \phi}{\partial v} \in T_{\phi(u,v)}S$$

Define

$$E(u, v) = \langle \phi_u(u, v), \phi_u(u, v) \rangle , \qquad F(u, v) = \langle \phi_u(u, v), \phi_v(u, v) \rangle \quad \text{and} \quad G(u, v) = \langle \phi_v(u, v), \phi_v(u, v) \rangle$$

Proposition 2.13. Suppose $\alpha(t) = \phi(u(t), v(t))$. Then we have that

$$I_p(\dot{\alpha}(0)) = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

Proof. Chain rule. □

Proposition 2.14. The length of a curve $\alpha(t) = \phi(u(t), v(t))$ is given by

$$\ell(\alpha) = \int \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} \mathrm{d}t$$

Proposition 2.15. $|\phi_u \wedge \phi_v| = \sqrt{EG - F^2}$.

Lemma 2.16. Suppose $\Omega \subseteq S$ is open, connected, bounded[a]. Furthermore, suppose Ω is contained in the image of a parametrisation $\phi : U \to S$. Then

$$\int_{\phi^{-1}(\Omega)} |\phi_u \wedge \phi_v| \mathrm{d}u\mathrm{d}v$$

is independent of the choice of ϕ.

[a] $\Omega \subseteq \mathbb{R}^3$ is bounded.

Proof. Suppose $\psi : W \to S$ is another parametrisation, with $\omega \subseteq \psi(W)$, then let $J(x, y)$ be the Jacobian of $h = \phi^{-1} \circ \psi$. Then we have that

$$|\psi_x \wedge \psi y| = |\det(J)||\phi_u \wedge \phi_v| \circ h$$

the result follows from the change of variables formula for multiple integrals. □

Definition 2.17 (area)

$$A(\Omega) = \int_{\phi^{-1}(\Omega)} |\phi_u \wedge \phi_v| \mathrm{d}u\mathrm{d}v$$

is called the area of Ω.

Definition 2.18 (Riemannian measure)
For $f : S \to \mathbb{R}$ continuous, $\phi : U \to S$ a parametrisation which covers S[a], we can define

$$\int_{\mathbb{R}^n} f \mathrm{d}A = \int_U f(u,v)\sqrt{EG - F^2} \mathrm{d}u\mathrm{d}v$$

[a]Up to some null sets.

2.4 Gauss map

Definition 2.19 (Gauss map)
Let $S \subseteq \mathbb{R}^3$ be a surface. Then a smooth map $N : S \to S^2$, with $N(p) \perp T_pS$ for all $p \in S$ is called a Gauss map of S.

Proposition 2.20. Suppose $\phi : U \to S$ is a parametrisation, then $N : \phi(U) \to S^2$ defined by

$$N(p) = \frac{\phi_u \wedge \phi_v}{|\phi_u \wedge \phi_v|}$$

is a Gauss map.

The derivative of the Gauss map N is given by $\mathrm{d}N_p : T_pS \to T_{N(p)}S^2$. But by definition, $N(p) \perp T_{N(p)}(S^2)$, so in fact (as subspaces of the ambient $\mathbb{R}^3$), $\mathrm{T}_{N(p)}S^2 = T_pS$. So we will write $\mathrm{d}N_p : T_pS \to T_pS$.

Furthermore, when working with a parametrisation $\phi : U \to S$, we will abuse notation and write $N : U \to S^2$ where $N(u,v) = N(\phi(u,v))$, and accordingly,

$$N_u(u,v) = \frac{\partial(N \circ \phi)}{\partial u} \quad \text{and} \quad N_v(u,v) = \frac{\partial(N \circ \phi)}{\partial v}$$

Finally, notice that by chain rule, $N_u = \mathrm{d}N(\phi_u)$ and $N_v = \mathrm{d}N(\phi_v)$ are in T_pS.

Proposition 2.21. The linear map $\mathrm{d}N_p : T_pS \to T_pS$ is self adjoint.

Proof. Let $\phi : U \to S$ be a parametrisation around p. If $\alpha(t) = \phi(u(t), v(t))$, with $\alpha(0) = p$, then we have that

$$\mathrm{d}N_p(\dot\alpha(0)) = \mathrm{d}N_p(\dot u(0)\phi_u + \dot v(0)\phi_v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} N(u(t), v(t)) = \dot u(0)N_u + \dot v(0)N_v$$

In particular, this means that $\mathrm{d}N_p(\phi_u) = N_u$ and $\mathrm{d}N_p(\phi_v) = N_v$. Since ϕ_u, ϕ_v is a basis for T_pS, we only need to show that

$$\langle N_u, \phi_v \rangle = \langle N_v, \phi_u \rangle$$

But notice that $\langle N, \phi_u \rangle = \langle N, \phi_v \rangle = 0$. Taking derivatives with respect to v and u respectively, we get

$$\langle N_v, \phi_u \rangle + \langle N, \phi_{uv} \rangle = 0 \quad \text{and} \quad \langle N_u, \phi_v \rangle + \langle N, \phi_{vu} \rangle = 0$$

which gives the result by the symmetry of mixed partial derivatives. ☐

Definition 2.22 (second fundamental form)
The quadratic form $II_p : T_pS \to \mathbb{R}$ defined by

$$II_p(w) = -\left\langle dN_p(w), w \right\rangle$$

is called the second fundamental form of S at p.

Definition 2.23 (normal curvature)
Let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a curve, $\alpha(0) = p$. Then the normal curvature of α at p is defined by

$$\kappa_n(p) = \langle N, \kappa n \rangle$$

where N is the Gauss map, κ the curvature of α and n the unit normal to α at p (i.e. $\kappa n = \ddot{\alpha}$).

Proposition 2.24. $\kappa_n(p) = II_p(\dot{\alpha}(0))$. In particular, it only depends on $\dot{\alpha}(0)$.

Proof. Write $N(s) = N(\alpha(s))$. Then we have that $\langle N(s), \dot{\alpha}(s) \rangle = 0$ for all s. Differentiating this, we get

$$\langle N(s), \ddot{\alpha}(s) \rangle = -\left\langle \dot{N}(s), \dot{\alpha}(s) \right\rangle$$

But we have that by chain rule, $II_p(\dot{\alpha}(0)) = -\left\langle \dot{N}(0), \dot{\alpha}(0) \right\rangle$, which means that

$$II_p(\dot{\alpha}(0)) = \langle N(0), \ddot{\alpha}(0) \rangle = \langle N, \kappa n \rangle$$

□

Definition 2.25 (principal curvatures, principal directions)
As $dN_p : T_pS \to T_pS$ is self adjoint, it can be diagonalised. Let $e_1, e_2 \in T_pS$ be such that, with respect to this basis, we have

$$dN_p = \begin{pmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{pmatrix}$$

where $\kappa_1 \geq \kappa_2$. We call κ_1, κ_2 the princial curvatures, and e_1, e_2 the principal directions.

Proposition 2.26. κ_1 (resp. κ_2) is the maximum (resp. minimum) value of II_p on the set of unit vectors in T_pS. That is, they are the extreme values of the normal curvature at p.

Proof. Linear algebra. □

Definition 2.27 (Gaussian curvature)
The Gaussian curvature of S at p is

$$K(p) = \det\left(dN_p\right) = \kappa_1 \kappa_2$$

Definition 2.28 (mean curvature)
The mean curvature of S at p is

$$H(p) = -\frac{1}{2}\operatorname{tr}\left(\mathrm{d}N_p\right) = \frac{\kappa_1+\kappa_2}{2}$$

Definition 2.29 (elliptic, hyperbolic, parabolic, planar)
A point $p \in S$ is

(i) elliptic if $K(p) > 0$,

(ii) hyperbolic if $K(p) < 0$,

(iii) parabolic if $K(p) = 0$ and $\mathrm{d}N_p \neq 0$,

(iv) planar if $\mathrm{d}N_p = 0$,

Definition 2.30 (umbilical)
A point $p \in S$ is umbilical if $\kappa_1 = \kappa_2$.

Proposition 2.31. If all points on a connected surface S are umbilical, then S is contained in a sphere or a plane.

2.5 Local coordinate computations

Let $\phi : U \to S$ be a parametrisation about a point $p \in S$. Define

$$e = \langle N, \phi_{uu}\rangle\,, \quad f = \langle N, \phi_{uv}\rangle = \langle N, \phi_{vu}\rangle \quad \text{and} \quad g = \langle N, \phi_{vv}\rangle$$

Proposition 2.32.

$$e = -\langle N_u, \phi_u\rangle\,, \quad f = -\langle N_v, \phi_u\rangle = -\langle N_u, \phi_v\rangle \quad \text{and} \quad g = -\langle N_v, \phi_u\rangle$$

Proof. Differentiate $\langle N, \phi_u\rangle = \langle N, \phi_v\rangle = 0$. □

Proposition 2.33. If $\alpha(t) = \phi(u(t), v(t))$ is a curve, with $\alpha(0) = 0$, then

$$II_p(\dot\alpha(0)) = e\dot u^2 + 2f\dot u\dot v + g\dot v^2$$

With respect to the basis ϕ_u, ϕ_v, we can express $\mathrm{d}N_p$ as a matrix, namely[3]

$$\begin{aligned} \mathrm{d}N_p(\phi_u) &= N_u = a_{11}\phi_u + a_{21}\phi_v \\ \mathrm{d}N_p(\phi_v) &= N_v = a_{12}\phi_u + a_{22}\phi_v \end{aligned}$$

Taking inner products of the above equations with ϕ_u, ϕ_v we get

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

But with respect to the basis ϕ_u, ϕ_v, $\mathrm{d}N_p$ has matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Therefore, we have

[3]This numbering of the a_{ij} corresponds to matrix notation.

Corollary 2.34.

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{eG - 2fF + gE}{2(EG - F^2)}$$

2.6 Theorema Egregium

Theorem 2.35 (Theorema Egregium)**.** The Gaussian curvature K is invariant under isometries. Equivalently, it can be expressed in local coordinates in terms of E, F, G and their derivatives.

Proof. Let $\phi : U \to S$ be a parametrisation. Then at each $p \in \phi(U)$, we have a basis ϕ_u, ϕ_v, N of $\mathbb{R}^3$. Hence we can express the derivatives of ϕ_u, ϕ_v in this basis, in terms of the Christoffel symbols.

$$\phi_{uu} = \Gamma^1_{11}\phi_u + \Gamma^2_{11}\phi_v + eN \tag{I}$$
$$\phi_{uv} = \Gamma^1_{12}\phi_u + \Gamma^2_{12}\phi_v + fN \tag{II}$$
$$\phi_{vv} = \Gamma^1_{22}\phi_u + \Gamma^2_{22}\phi_v + fN \tag{III}$$
$$\phi_{vu} = \Gamma^1_{21}\phi_u + \Gamma^2_{21}\phi_v + gN \tag{IV}$$

By symmetry of mixed partial derivatives, $\Gamma^1_{12} = \Gamma^1_{21}$ and $\Gamma^2_{12} = \Gamma^2_{21}$. Take inner products of (I) with ϕ_u and ϕ_v respectively, we get that

$$\Gamma^1_{11}E + \Gamma^2_{11}F = \langle \phi_{uu}, \phi_u \rangle = \frac{1}{2}E_u$$
$$\Gamma^1_{11}F + \Gamma^2_{11}G = \langle \phi_{uu}, \phi_v \rangle = F_u - \frac{1}{2}E_v$$

Since $EG - F^2 \neq 0$, we can solve for Γ^1_{11} and Γ^2_{11} in terms of E, F, G, E_u, E_v, F_u. Similarly, we can express all of the Christoffel symbols in terms of E, F, G and their first derivatives.

Now if we differentiate (I) with respect to v, and (II) with respect to u, we get that

$$\begin{aligned}\Gamma^1_{11}\phi_{uv} + \Gamma^2_{11}\phi_{vv} + eN_v + (\Gamma^1_{11})_v\phi_u + (\Gamma^2_{11})_v\phi_v + e_vN &= \phi_{uuv} \\ &= \phi_{uvu} \\ &= \Gamma^1_{12}\phi_{uu} + \Gamma^2_{12}\phi_{uv} + fN_u + (\Gamma^1_{12})_u\phi_u + (\Gamma^2_{12})_u\phi_v + f_uN\end{aligned}$$

Using (I), (II) and (IV), and the (a_{ij}) from the previous section, and equating coefficients, we get

$$\Gamma^1_{11}\Gamma^1_{12} + \Gamma^2_{11}\Gamma^1_{22} + ea_{12} + (\Gamma^1_{11})_v = \Gamma^1_{11}\Gamma^1_{12} + \Gamma^1_{12}\Gamma^2_{12} + fa_{11} + (\Gamma^1_{12})u \tag{1}$$
$$\Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} + ea_{22} + (\Gamma^2_{11})v = \Gamma^2_{11}\Gamma^1_{12} + \Gamma^2_{12}\Gamma^2_{12} + fa_{21} + (\Gamma^2_{12})_u \tag{2}$$
$$\Gamma^1_{11}f + \Gamma^2_{11}g + e_v = \Gamma^1_{12}e + \Gamma^2_{12}f + f_u \tag{3}$$

Fortunately, we only need (2), since if $A = (a_{ij})$, then

$$K\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} A\,\mathrm{adj}(A) = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\mathrm{adj}(A) = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

So $EK = -ea_{22} + fa_{21}$, which we can express in terms of Christoffel symbols. □

Definition 2.36 (isothermal)

A parametrisation is isothermal if $E = G = \lambda(u,v)^2$, $F = 0$.

Proposition 2.37. In isothermal coordinates,

$$K = -\frac{1}{\lambda^2}\Delta(\log(\lambda(u, v)))$$

3 Riemannian geometry

3.1 Geodesics

Let $S \subseteq \mathbb{R}^3$ be a surface, $p, q \in S$. Let $\Omega(p, q)$ be the set of all curves $\alpha : [0, 1] \to S$, *which do not have to be parametrised by arc length*, with $\alpha(0) = p$ and $\alpha(1) = q$.

Definition 3.1 (length functional)
The length functional is

$$\ell(\alpha) = \int_0^1 |\dot{\alpha}| \mathrm{d}t$$

Definition 3.2 (energy functional)
The energy of a curve is

$$E(\alpha) = \frac{1}{2}\int_0^1 |\dot{\alpha}|^2 \mathrm{d}t$$

Proposition 3.3. $\ell(\alpha) \leq \sqrt{2E(\alpha)}$, with equality if and only if α is parametrised proportional to arc length.

Proof. Cauchy-Schwarz. □

Let $\alpha_s \in \Omega(p, q)$ be a smooth one parameter family of curves, with $s \in (-\varepsilon, \varepsilon)$. Let $E(s) = E(\alpha_s)$. Then we have that

$$\frac{\mathrm{d}E}{\mathrm{d}s} = \int_0^1 \left\langle \frac{\partial}{\partial s}\frac{\partial \alpha_s}{\partial t}, \frac{\partial \alpha_s}{\partial t} \right\rangle \mathrm{d}t$$

Integrating by parts[4] we get

$$\frac{\mathrm{d}E}{\mathrm{d}s}|_{s=0} = \langle J(1), \dot{\alpha}(1)\rangle - \langle J(0), \dot{\alpha}(0)\rangle - \int_0^1 \langle J(t), \ddot{\alpha}(t)\rangle \,\mathrm{d}t$$

where[5]

$$J(t) = \frac{\partial \alpha_s(t)}{\partial s}|_{s=0}$$

Since $\alpha_s \in \Omega(p, q)$, $J(0) = J(1) = 0$. So we get that

$$\frac{\mathrm{d}E}{\mathrm{d}s}|_{s=0} = -\int_0^1 \langle J(t), \ddot{\alpha}(t)\rangle \,\mathrm{d}t$$

Now notice that for each $t \in [0, 1]$, $J(t) \in T_{\alpha(t)}S$, since $s \mapsto \alpha_s(t)$ is a curve in s. So if α is such that $\ddot{\alpha} \perp T_{\alpha(t)}S$ for all t, then α extremises E.

[4] That is,

$$\int_a^b \left\langle \frac{\partial f}{\partial t}, g \right\rangle \mathrm{d}t = [\langle f, g\rangle]_a^b - \int_a^b \left\langle f, \frac{\partial g}{\partial t} \right\rangle \mathrm{d}t$$

which follows from $\frac{\partial}{\partial t}\langle f, g\rangle = \left\langle \frac{\partial f}{\partial t}, g \right\rangle + \left\langle f, \frac{\partial g}{\partial t} \right\rangle$.

[5] In Paternain's notes it's W, I renamed it to J since it is a Jacobi field.

Definition 3.4 (geodesic)
A curve $\alpha : I \to S$ is a geodesic if for all $t \in I$, $\ddot{\alpha}(t)$ is orthogonal to $T_{\alpha(t)}S$.

3.2 Covariant derivative, parallel transport

Definition 3.5 (vector field)
Let $\alpha : I \to S$ be a curve. A vector field along α is a smooth map $V : I \to \mathbb{R}^3$ such that for all t, $V(t) \in T_{\alpha(t)}S$.

Definition 3.6 (covariant derivative)
The covariant derivative of a vector field V along α is

$$\frac{\mathrm{D}V}{\mathrm{d}t}(t) = \mathrm{proj}_{T_{\alpha(t)}S}\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)$$

where $\mathrm{proj}_{T_{\alpha(t)}S}$ is the orthogonal projection onto $T_{\alpha(t)}S$.

Proposition 3.7. A curve α is a geodesic if and only if $\frac{\mathrm{D}\dot{\alpha}}{\mathrm{d}t} = 0$ for all t.

Definition 3.8 (parallel)
A vector field V along α is parallel if $\frac{\mathrm{D}V}{\mathrm{d}t} = 0$.

Proposition 3.9. Let V, W be parallel vector fields along α. Then $\langle V(t), W(t)\rangle$ is constant.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle V(t), W(t)\rangle = \left\langle \frac{\mathrm{d}V}{\mathrm{d}t}(t), W(t)\right\rangle + \left\langle V(t), \frac{\mathrm{d}W}{\mathrm{d}t}(t)\right\rangle$$

But as V, W are parallel, $\frac{\mathrm{d}V}{\mathrm{d}t}, \frac{\mathrm{d}W}{\mathrm{d}t}$ are orthogonal to $T_{\alpha(t)}S$, so $\frac{\mathrm{d}}{\mathrm{d}t}\langle V(t), W(t)\rangle = 0$. □

Corollary 3.10. If α is a geodesic, then $|\dot{\alpha}|$ is constant. So geodesics are parametrised proportional to arc length.

3.2.1 Local coordinate expressions

Let $\phi : U \to S$ be a parametrisation, $\alpha : I \to S$ a curve, with $\alpha(I) \subseteq \phi(U)$. Write $\alpha(t) = \phi(u(t), v(t))$. Let V be a vector field along α. Then there are functions $a(t), b(t)$ such that

$$V(t) = a(t)\phi_u + b(t)\phi_v$$

Differentiating this, we get that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = a(\phi_{uu}\dot{u} + \phi_{uv}\dot{v}) + b(\phi_{vu}\dot{u} + \phi_{vv}\dot{v}) + \dot{a}\phi_u + \dot{b}\phi_v$$

The covariant derivative is just the ϕ_u and ϕ_v components of this, since N is orthogonal to $T_{\alpha(t)}S$. Therefore, in terms of Christoffel symbols, we have that

$$\frac{\mathrm{D}V}{\mathrm{d}t} = (\dot a + a\dot u\Gamma^1_{11} + a\dot v\Gamma^1_{12} + b\dot u\Gamma^1_{12} + b\dot v\Gamma^1_{22})\phi_u + (\dot b + a\dot u\Gamma^2_{11} + a\dot v\Gamma^2_{12} + b\dot u\Gamma^2_{12} + b\dot v\Gamma^2_{22})\phi_v \qquad (*)$$

From this expression, we see that the covariant derivative only depends on the first fundamental form[6].

Proposition 3.11 (geodesic equations)**.** $\alpha(t) = \phi(u(t), v(t))$ is a geodesic if and only if

$$\ddot u + \Gamma^1_{11}\dot u^2 + 2\Gamma^1_{12}\dot u\dot v + \Gamma^1_{22}\dot v^2 = 0$$
$$\ddot v + \Gamma^2_{11}\dot u^2 + 2\Gamma^2_{12}\dot u\dot v + \Gamma^2_{22}\dot v^2 = 0$$

Proof. Set $a = \dot u, b = \dot v$ in (*). ☐

Proposition 3.12 (parallel transport)**.** Given $v_0 \in T_{\alpha(t_0)}S$, there exists a unique parallel vector field $V(t)$ along $\alpha(t)$, with $V(t_0) = v_0$. We call $V(t_1)$ the parallel transport of v_0 along α at t_1.

Proof. (*) is a linear ODE in terms of (a, b), therefore we can apply standard ODE theory in terms of existence and uniqueness. ☐

Corollary 3.13. Given $p \in S$, $v \in T_pS$, there exists $\varepsilon > 0$, and a unique geodesic $\gamma : (-\varepsilon, \varepsilon) \to S$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$.

Proof. Standard ODE theory. ☐

Definition 3.14 (parallel transport)

Let $\alpha \in \Omega(p, q)$. Define $P : T_pS \to T_qS$ the map sending $v \in T_pS$ to the parallel transport of v along α at q.

Proposition 3.15. $P : T_pS \to T_qS$ is a linear isometry.

Proof. The fact that P is linear follows from (*) being a linear ODE for (a, b), and uniqueness of solutions. P being an isometry follows from the fact that if $V(t)$ is the parallel vector field, then $\langle V(t), V(t)\rangle$ is constant, so $\left\|P(v)\right\|^2 = \langle V(t_1), V(t_1)\rangle = \langle V(t_0), V(t_0)\rangle = ||v||^2$. ☐

3.3 Exponential map and geodesic polar coordinates

Proposition 3.16. Given $p \in S, v \in T_pS$, let $\gamma_v : (-\varepsilon, \varepsilon) \to S$ by the unique geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then $\gamma_{\lambda v}$ is defined on $(-\varepsilon/\lambda, \varepsilon/\lambda)$. Furthermore, $\gamma_{\lambda v}(t) = \gamma_v(\lambda t)$.

Proof. By uniqueness of such a geodesic, and the chain rule. ☐

Definition 3.17 (exponential map)

Suppose $v \in T_pS$ nonzero is such that $\gamma_v(1)$ is defined, we define

[6] Or in the language of Part III Differential Geometry, this definition of the covariant derivative agrees with the one coming from the Levi-Civita connection for a Riemannian manifold.

$$\exp_p(v) = \gamma_v(1)$$

Proposition 3.18. There exists $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \to S$ is well defined and smooth.

Proof. By existence of solutions to ODEs, and smooth dependence on the initial conditions. □

Proposition 3.19. If S is closed, then $\exp_p$ is defined on all of T_pS.

Proposition 3.20. $\exp_p : B_\varepsilon(0) \to S$ is a diffeomorphism onto its image in a neighbourhood $U \subseteq B_\varepsilon(0)$ of $0 \in T_pS$.

Proof. By the inverse function theorem, suffices to show $\mathrm{d}\left(\exp_p\right)_0$ is nonsingular. Let $\alpha(t) = tv$ for some fixed $v \in T_pS$. Then $\exp_p(tv) = \gamma_v(t)$ at $t = 0$ has tangent vector v. So $\mathrm{d}\left(exp_p\right)_0(v) = v$. □

Definition 3.21 (normal neighbourhood)
Let U be as in the previous proposition. Then $V = \exp_p(U)$ is called a normal neighbourhood of p.

Corollary 3.22. $\exp_p : U \to V$ is a parametrisation.

Proposition 3.23. If we choose cartesian coordinates on T_pS, then with the $\exp_p$ parametrisation, we have the first fundamental form

$$E(p) = G(p) = 1 \quad \text{and} \quad F(p) = 0$$

Definition 3.24 (geodesic polar coordinates)
If we choose polar coordinates (r, θ) for T_pS, then we have the geodesic polar coordinates. That is,

$$\phi(r, \theta) = \exp_p(r(\cos(\theta)e_1 + \sin(\theta)e_2)) = \exp_p(rv(\theta)) = \gamma_{v(\theta)}(t)$$

where $v(\theta) = \cos(\theta)e_1 + \sin(\theta)e_2$.

Remark 3.25. Recall that to define polar coordinates, we need to take a branch cut. But the above map makes sense, even though it is not a parametrisation without taking a branch cut.

Definition 3.26 (geodesic circles, radial geodesics)
The images of circles centred in the origin under the map ϕ are called geodesic circles (i.e. r = const). Similarly, the images of lines through the origin (i.e. θ = const) are called radial geodesics.

Proposition 3.27. The coefficients $E(r,\theta), F(r,\theta), G(r,\theta)$ satisfy

$$E = 1, \quad F = 0, \quad G(0,\theta) = 0 \quad \text{and} \quad (\sqrt{G})_r(0,\theta) = 1$$

Moreover, the Gaussian curvature can be written as

$$K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$$

Proof. By definition of ϕ, we have that $\phi_r = \dot{\gamma}_{v(\theta)}(r)$, so $E = 1$ as $v(\theta)$ is a unit vector and geodesics travel at constant speed. Now let $w = \frac{\mathrm{d}v}{\mathrm{d}\theta}$. Then by chain rule, we have that

$$\phi_\theta = \mathrm{d}\left(\exp_p\right)_{rv}(rw) = r\,\mathrm{d}\left(\exp_p\right)_{rv}(w)$$

So we have that

$$F = r\left\langle \dot{\gamma}_v(r), \mathrm{d}\left(\exp_p\right)_{rv}(w)\right\rangle$$
$$G = r^2\left|\mathrm{d}\left(\exp_p\right)_{rv}(w)\right|^2$$

Clearly $F(0,\theta) = 0$, and from the last equality, we find that

$$(\sqrt{G})_r(0,\theta) = \left|\mathrm{d}\left(\exp_p\right)_0(w)\right| = |w| = 1$$

Finally, we can compute

$$\begin{aligned} F_r &= \langle \phi_{rr}, \phi_\theta\rangle + \langle \phi_r, \phi_{\theta r}\rangle \\ &= \langle \phi_r, \phi_{\theta r}\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial\theta}\langle \phi_r, \phi_r\rangle \\ &= \frac{1}{2}E_\theta \\ &= 0 \end{aligned}$$

where we used the fact that $\phi(\cdot,\theta) = \gamma_v$ is a geodesic, so $\phi_{rr} = \ddot{\gamma}_v$ is normal to T_pS. So $F = 0$ identically. We omit the computation for K, and note that it can be computed using Christoffel symbols. ☐

3.4 Geodesic curvature

Definition 3.28 (algebraic value of the covariant derivative)
Let W be a differentiable field fo unit vectors along a curve $\alpha : I \to S$ along an oriented surface S. Then

$$\left[\frac{\mathrm{D}W}{\mathrm{d}t}\right] = \left\langle \frac{\mathrm{d}W}{\mathrm{d}t}, N \wedge W\right\rangle$$

Note that this definition depends on the orientation of S, but only up to a sign.

Proposition 3.29. Let W be a field of unit vectors along α. Then $\frac{\mathrm{D}W}{\mathrm{d}t}$ is parallel to $N \wedge W$, and we have that

$$\frac{\mathrm{D}W}{\mathrm{d}t} = \left[\frac{\mathrm{D}W}{\mathrm{d}t}\right](N \wedge W)$$

Proof. $\langle W, W\rangle = 1$, so $\left\langle \frac{\mathrm{d}W}{\mathrm{d}t}, W\right\rangle = 0$. By definition, $\frac{\mathrm{D}W}{\mathrm{d}t}$ is orthogonal to N, hence by the above, it must be parallel to $N \wedge W$. ☐

Definition 3.30 (geodesic curvature)
Let $\alpha : I \to S$ be a regular curve parametrised by arc length. The algebraic value of the covariant derivative

$$\kappa_g(s) = \left[\frac{\mathrm{D}\dot{\alpha}}{\mathrm{d}t}\right] = \langle \ddot{\alpha}, N \wedge \dot{\alpha} \rangle$$

is called the geodesic curvature of α at $\alpha(s)$.

Proposition 3.31. α is a geodesic if and only if its geodesic curvature is identically zero.

Proposition 3.32. Let κ and n be the curvature and unit normal for α. Then we have that

$$\ddot{\alpha} = \kappa_n N + \kappa_g (N \wedge \dot{\alpha})$$

where κ_n, κ_g are the normal and geodesic curvatures respectively.

Proof. Since W has norm 1, we have that $\langle W, W \rangle = 0$, so $\left\langle \frac{\mathrm{d}W}{\mathrm{d}t}, W \right\rangle = 0$. Hence $\frac{\mathrm{d}W}{\mathrm{d}t}$ is perpendicular to W. Thus, $\frac{\mathrm{D}W}{\mathrm{d}t}$ must be perpendicular to both W and N, so it is parallel to $N \wedge W$. ☐

Definition 3.33 (perpendicular vector field)
Let V be a unit vector field along $\alpha : I \to S$. Let $iV(t)$ be the unique vector field along α such that for every $t \in I$, $V(t), iV(t), N(t)$ forms a positively oriented orthonormal basis of $\mathbb{R}^3$. That is,

$$V(t) \wedge iV(t) = N(t)$$

Proposition 3.34. Let V, W be unit vector fields along $\alpha : I \to S$. Then there exists smooth functions a, b, such that

$$W(t) = a(t)V(t) + b(t)iV(t)$$

with $a^2 + b^2 = 1$. Furthermore, if we fix $t_0 \in I$, and let φ_0 be such that

$$a(t_0) = \cos(\varphi_0) \quad \text{and} \quad b(t_0) = \sin(\varphi_0)$$

then there exists a smooth function $\varphi : I \to S$ such that

$$a(t) = \cos(\varphi(t)), \quad b(t) = \sin(\varphi(t)) \quad \text{and} \quad \varphi(t_0) = \varphi_0$$

Proof. $V(t), iV(t)$ is an orthonormal basis of $T_{\alpha(t)}S$. The construction of φ is as in the construction of the winding number in Complex Analysis. ☐

Definition 3.35 (smooth determination of angle)
φ from the previous proposition is called a smooth determination of the angle from V to W.

Proposition 3.36. Let V, W be unit vector fields along $\alpha : I \to S$ and φ by a smooth determination of angle from V to W. Then

$$\left[\frac{DW}{dt}\right] - \left[\frac{DV}{dt}\right] = \frac{d\varphi}{dt}$$

Proof. By definitions, we have that

$$\left[\frac{DW}{dt}\right] = \left\langle W', N \wedge W \right\rangle$$
$$\left[\frac{DV}{dt}\right] = \left\langle V', N \wedge V \right\rangle = \left\langle V', iV \right\rangle$$

Write $W = \cos(\varphi)V + \sin(\varphi)iV$, and differentiate, to get

$$W' = \varphi'(-\sin(\varphi)V + \cos(\varphi)iV) + \cos(\varphi)V' + \sin(\varphi)(iV)'$$

But $N \wedge W = \cos(\varphi)iV - \sin(\varphi)V$, so we get that

$$\begin{aligned}\left[\frac{DW}{dt}\right] &= \varphi' + \left\langle -\sin(\varphi)V + \cos(\varphi)iV, \cos(\varphi)V' + \sin(\varphi)(iV)' \right\rangle \\ &= \varphi' + \cos(\varphi)^2 \left\langle iV, V' \right\rangle - \sin(\varphi)^2 \left\langle V, (iV)' \right\rangle \\ &= \varphi' + \left[\frac{DV}{dt}\right]\end{aligned}$$

where we used the fact that $\langle V, iV \rangle = 0$, so $\langle V', iV \rangle + \langle V, (iV)' \rangle = 0$, and $\langle V, V \rangle = \langle iV, iV \rangle = 1$, so $\langle V', V \rangle = \langle iV, (iV)' \rangle = 0$. □

Proposition 3.37. Let $\alpha : I \to S$ be a curve parametrised by arc length, $V(s)$ a parallel unit vector field along α, φ a smooth determination of angle from V to $\dot{\alpha}$. Then

$$\kappa_g(s) = \frac{d\varphi}{ds}$$

Proof. $\left[\frac{DV}{dt}\right] = 0$ as V is parallel. □

Proposition 3.38. Let $\phi(u, v)$ be an orthogonal parametrisation (i.e. $F = 0$) of an oriented surface S, which is compatible with the orientation. Let W be a smooth vector field along the curve $\phi(u(t), v(t))$. Then

$$\left[\frac{DW}{dt}\right] = \frac{1}{2\sqrt{EG}}(G_u\dot{v} - E_v\dot{u}) + \frac{d\varphi}{dt}$$

where φ is the angle from ϕ_u to W in the given orientation.

Proof. Let $e_1 = \phi_u/\sqrt{E}$ and $e_2 = \phi_v/\sqrt{G}$. Then e_1, e_2 is a positively oriented orthonormal basis of the tangent plane. By the previous proposition,

$$\left[\frac{DW}{dt}\right] = \left[\frac{De_1}{dt}\right] + \dot{\varphi}$$

Computing this,

$$\left[\frac{De_1}{dt}\right] = \langle \dot{e}_1, N \wedge e_1 \rangle = \langle \dot{e}_1, e_2 \rangle = \langle (e_1)_u, e_2 \rangle \dot{u} + \langle (e_1)_v, e_2 \rangle \dot{v}$$

But then we have that

$$\langle (e_1)_u, e_2\rangle = \left\langle (\phi_u/\sqrt{E})_u, \phi_v/\sqrt{G}\right\rangle = \frac{1}{\sqrt{EG}}\langle \phi_{uu}, \phi_v\rangle + \left\langle \frac{d}{du}\left(\frac{1}{\sqrt{E}}\right)\phi_u, \phi_v\right\rangle = \frac{1}{\sqrt{EG}}\langle \phi_{uu}, \phi_v\rangle$$

as $F = \langle \phi_u, \phi_v\rangle = 0$. Differentiating $F = 0$, we get that $\langle \phi_{uu}, \phi_v\rangle = -\langle \phi_u, \phi_{uv}\rangle = -E_v/2$. Therefore, we have that

$$\langle (e_1)_u, e_2\rangle = -\frac{E_v}{2\sqrt{EG}}$$

similarly,

$$\langle (e_1)_v, e_2\rangle = \frac{G_u}{2\sqrt{EG}}$$

□

Corollary 3.39. If $\alpha :\to \phi(U)$ is a curve parametrised by arc length, then

$$\kappa_g(s) = \frac{1}{2\sqrt{EG}}(G_u\dot{v} - E_v\dot{u}) + \dot{\phi}$$

where φ is the angle from ϕ_u to $\dot{\alpha}$.

Proof. Set $W = \dot{\alpha}$ in the previous proposition. □

3.5 Gauss-Bonnet

Theorem 3.40 (Gauss's theorem for geodesic triangles)**.** Let T be a geodesic triangle on a surface S. Suppose T is small enough so that it is contained in a normal neighbourhood of one of its vertices, then

$$\int_T K\mathrm{d}A = \alpha_1 + \alpha_2 + \alpha_3 - \pi$$

where K is the Gaussian curvature of S, and $0 < \alpha_i < \pi$ are the internal angles of T.

Proof. We can assume without loss of generality that we have geodesic polar coordinates centred at one of the vertices of T, one of the edges corresponds to $\theta = 0$ and another corresponds to $\theta = \theta_0$. The remaining edge is a geodesic segment γ.

First notice that γ can be written in the form $r = h(\theta)$. Suppose not, then there exists s such that $\dot{\gamma}(s)$ is parallel to ϕ_r. But radial segments are geodesics, so this means that γ is radial. Contradiction. Hence we can write γ as $r = h(\theta)$. Then

$$\int_T K\mathrm{d}A = \int_T K\sqrt{G}\mathrm{d}r\mathrm{d}\theta = \int_0^{\theta_0}\left(\lim_{\varepsilon\to 0}\int_\varepsilon^{h(\theta)} K\sqrt{G}\mathrm{d}r\right)\mathrm{d}\theta$$

But in geodesic polar coordinates, we have $K\sqrt{G} = -(\sqrt{G})_{rr}$, and $\lim_{r\to 0}(\sqrt{G})_r = 1$, so

$$\lim_{\varepsilon\to 0}\int_\varepsilon^{h(\theta)} K\sqrt{G}\mathrm{d}r = 1 - (\sqrt{G})_r(h(\theta), \theta)$$

Now suppose $\gamma(s) = \phi(r(s), \theta(s))$ makes an angle $\varphi(s)$ with ϕ_r, that is, the curves $\theta =$ const. Then the previous corollary ($u = r, v = \theta$) gives that[7]

$$(\sqrt{G})_r\frac{\mathrm{d}\theta}{\mathrm{d}s} + \frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0$$

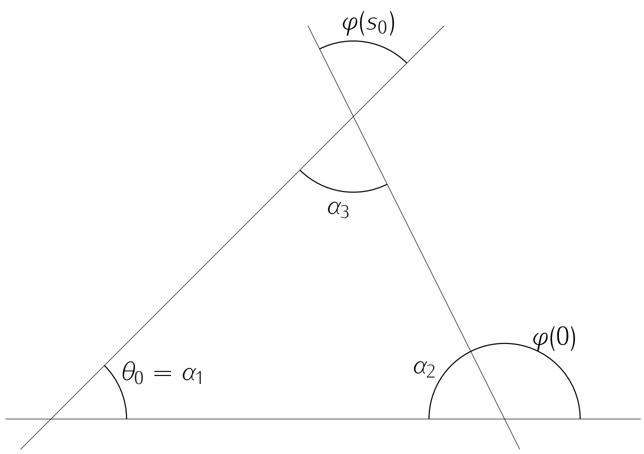
[7] Using

$$(\sqrt{G})_r = \frac{G_r}{2\sqrt{G}}$$

as γ is a geodesic. Therefore, we have that

$$\begin{aligned}
\int_T K \mathrm{d}A &= \int_0^{\theta_0} (1 - (\sqrt{G})_r(h(\theta), \theta))\mathrm{d}\theta \\
&= \int_0^{\theta_0} \mathrm{d}\theta - \int_0^{s_0} (\sqrt{G})_r \frac{\mathrm{d}\theta}{\mathrm{d}s}\mathrm{d}s \\
&= \theta_0 + \int_0^{s_0} \frac{\mathrm{d}\varphi}{\mathrm{d}s}\mathrm{d}s \\
&= \theta_0 + \int_{\varphi(0)}^{\varphi(s_0)} \mathrm{d}\varphi \\
&= \theta_0 + \varphi(s_0) - \varphi(0)
\end{aligned}$$

Finally, by the orientations, we have



Which gives the result. □

Definition 3.41 (triangulation)
Let S be a compact surface. A triangulation of S is a finite number of closed subsets $T_1, \ldots, T_n$ which cover S, each T_i is homeomorphic to a Euclidean triagngle in the plane. Moreover, any two distinct triangles are either disjoint, share a vertex, or share an edge.

Theorem 3.42. Triangulations always exist. Furthermore, we can choose it so that each T_i is diffeomorphic to a Euclidean triangle, and each edge is a geodesic segment.

Sketch Proof, requires Part II Algebraic topology. We omit the proof of existence. By barycentric subdivision, we can make the triangles in S arbitrarily small. Then by the Lebesgue covering lemma, if the mesh of the triangulation is sufficiently small, then each triangle is contained within a normal neighbourhood at one of its vertices. □

Definition 3.43 (Euler characteristic)
Given a triangulation of S, let F be the number of faces, E the number of edges, V the number of vertices. Then

$$\chi(S) = V - E + F$$

is the Euler characteristic of S.

Proposition 3.44. The Euler characteristic does not depend on the choice of triangulation.

Proof: Part II Algebraic Topology. This is just the homotopy invariance of homology. □

Proposition 3.45 (Classification of compact orientable surfaces)**.** All compact orientable surfaces are diffeomorphic to some Σ_g where g is a g-holed torus. g is called the genus of Σ_g. Furthermore,

$$\chi(\Sigma_g) = 2 - 2g$$

Proof. Part II Algebraic topology. □

Theorem 3.46 (Global Gauss-Bonnet)**.** Let S be a compact surface without boundary. Then

$$\int_S K \mathrm{d}A = 2\pi\chi(S)$$

Proof. Consider a triangulation by geodesic triangles $T_1, \ldots, T_F$. We can assume wlog that each T_i is contained in a normal neighbourhood of one of its vertices.

Let $\alpha_i, \beta_i, \gamma_i$ be the interior angles of T_i. Then by Gauss's theorem for triangles, we have that

$$\int_{T_i} K \mathrm{d}A = \alpha_i + \beta_i + \gamma_i - \pi$$

Summing over all i, we have that

$$\int_S K \mathrm{d}A = \sum_{i=1}^{F} (\alpha_i + \beta_i + \gamma_i) - \pi F$$

Now notice that the sum of the angles at every vertex is 2π, so

$$\sum_{i=1}^{F} (\alpha_i + \beta_i + \gamma_i) = 2\pi V$$

Finally, for a triangulation, every edge belongs to two triangles, so $2E = 3F$. Putting this all together we get that

$$\int_S K \mathrm{d}A = \pi(2V - F) = 2\pi\chi(S)$$

□

Theorem 3.47 (local Gauss-Bonnet)**.** Let $\phi : U \to S$ be an orthogonal parametrisation of an oriented surface S, U is a disc in $\mathbb{R}^2$, and ϕ is compatible with the orientation of S. Let $\alpha : I \to \phi(U)$ be a smooth simple closed curve enclosing a domain R. Suppose α is positively oriented and parametrised by arc length. Then

$$\int_I \kappa_g(s)\mathrm{d}s + \int_R K\mathrm{d}A = 2\pi$$

where κ_g is the geodesic curvature of α.

Proof. By our local coordinate computation, we have that

$$\kappa_g(s) = \frac{1}{2\sqrt{EG}}\left(G_u\frac{\mathrm{d}v}{\mathrm{d}s} - E_v\frac{\mathrm{d}u}{\mathrm{d}s}\right) + \frac{\mathrm{d}\varphi}{\mathrm{d}s}$$

where φ is the angle from ϕ_u to $\dot{\alpha}$. Without loss of generality, we may assume $I = [0, \ell]$. Integrating this, we get

$$\int_I \kappa_g(s)\mathrm{d}s = \int_I \frac{1}{2\sqrt{EG}}\left(G_u\frac{\mathrm{d}v}{\mathrm{d}s} - E_v\frac{\mathrm{d}u}{\mathrm{d}s}\right)\mathrm{d}s + \varphi(\ell) - \varphi(0)$$

By Green's theorem[8]

$$\int_I \kappa_g(s)\mathrm{d}s = \int_{\phi^{-1}(R)}\left(\left(\frac{G_u}{2\sqrt{EG}}\right)_u + \left(\frac{E_v}{2\sqrt{EG}}\right)_v\right)\mathrm{d}u\mathrm{d}v + \varphi(\ell) - \varphi(0)$$

But for an orthogonal parametrisation, we have that

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

So

$$\int_I \kappa_g(s)\mathrm{d}s = -\int_R K\mathrm{d}A + \varphi(\ell) - \varphi(0)$$

But $\varphi(\ell) - \varphi(0) = 2\pi$, which gives us the result. □

Theorem 3.48 (Gauss-Bonnet with boundary)**.** Let $R \subseteq S$ be a connected open relatively compact[a] subset. Suppose ∂R contains of n piecewise smooth simple closed curves $\alpha_i : I_i \to S$, where the images do not intersetc. Suppose the α_i are parametrised by arc length, and are positively oriented. Let θ_i be the external angles of the vertices of these curves. Then

$$\sum_{i=1}^{n}\int_{I_i}\kappa_g(s)\mathrm{d}s + \int_R K\mathrm{d}A + \sum_i \theta_i = 2\pi\chi(R)$$

[a] That is, the closure is compact

Proof. As for the global Gauss-Bonnet, but we need to treat the boundary vertices differently. □

Theorem 3.49. Suppose S is a compact orientable surface with $K > 0$. Then S is diffeomorphic to S^2. Moreover, if α, β are simple closed geodesics on S, then they must intersect.

Proof. Gauss-Bonnet gives us that $\chi(S) > 0$, so S is diffeomorphic to S^2. Now suppose α, β do not intersect. Then they bound a domain R with $\chi(R) = 0$. But then Gauss-Bonnet means that R must in fact have measure zero. Contradiction. □

[8] From IA Vector Calculus, which says that if $D \subseteq \mathbb{R}^2$ with ∂D sufficiently regular, then

$$\int_{\partial D}(L\mathrm{d}x + M\mathrm{d}y) = \int_D\left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

Theorem 3.50. Let S be a surface hoemomorphic to a cylinder, with negative Gaussian curvature everywhere. Then S has at most one simple closed geodesic.

4 Minimal surfaces

Note that the sections in the notes about the Weierstrass representation are non-examinable, and hence omitted.

Definition 4.1 (minimal surface)
A surface S is minimal if its mean curvature vanishes everywhere.

Definition 4.2 (normal variation)
Let $\phi : U \to S$ be a parametrisation, $D \subsetneq U$ bounded open connected, with $\overline{D} \subseteq U$. Let $h : \overline{D} \to \mathbb{R}$ be smooth. Then the normal variation of $\phi(\overline{D})$ determined by h is the map $\rho : \overline{D} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ given by

$$\rho(u, v, t) = \phi(u, v) + th(u, v)N(u, v)$$

For fixed $t \in (-\varepsilon, \varepsilon)$, consider the map $\rho^t : D \to \mathbb{R}^3$, given by $\rho^t(u, v) = \rho(u, v, t)$. Since $\rho^0 = \phi$, for ε small enough $\rho^t(D)$ is a smooth surface in $\mathbb{R}^3$, since ρ^t_u, ρ^t_v are linearly independent for ε small enough. Now by chain rule,

$$\begin{aligned}\rho^t_u &= \phi_u + thN_u + th_uN\\ \rho^t_v &= \phi_v + thN_v + th_vN\end{aligned}$$

and if we let E^t, F^t, G^t be the coefficients for the first fundamental form of $\rho^t(D)$, we get

$$\begin{aligned}E^t &= E + 2th\langle\phi_u, N_u\rangle + t^2h^2\langle N_u, N_u\rangle + t^2h_uh_u\\ F^t &= F + 2th\langle\phi_u, N_v\rangle + t^2h^2\langle N_u, N_v\rangle + t^2h_uh_v\\ G^t &= G + 2th\langle\phi_v, N_v\rangle + t^2h^2\langle N_v, N_v\rangle + t^2h_vh_v\end{aligned}$$

Computing the area functional to first order, we find

$$\begin{aligned}E^tG^t - (F^t)^2 &= EG - F^2 - 2th(Eg - 2Ff + Ge) + r(t)\\ &= (EG - F^2)(1 - 4thH) + r(t)\end{aligned}$$

where $r(t)$ is $\mathcal{O}(t^2)$, and we used the formula

$$H = \frac{eG - 2fF + Ge}{2(EG - F^2)}$$

If $A(t)$ is the area of $\rho^t(D)$, then we have

$$A(t) = \int_D \sqrt{E^tG^t - (F^t)^2}\mathrm{d}u\mathrm{d}v = \int_D \sqrt{1 - 4thH + \overline{r}}\sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v$$

where $\overline{r} = r/(EG - F^2)$. Clearly $t \mapsto A(t)$ is smooth and

$$A'(0) = -\int_D 2hH\sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v$$

Proposition 4.3. $\phi(U)$ is minimal if and only if $A'(0) = 0$ for all bounded domains $D \subseteq U$ and all normal variations of $\phi(D)$.

Proof. If $H = 0$, then clearly $A'(0) = 0$. Conversely, suppose $H(q) \neq 0$ for some $q \in D$. Let $h = H$. Then $A'(0) < 0$. Contradiction. □

Definition 4.4 (mean curvature vector)
The mean curvature vector is $\mathbf{H} = HN$.

Proposition 4.5. A normal variation in the direction of HN always has $A'(0) < 0$, provided H does not vanish.