Examples sheets

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A summary of examples sheets, with interesting results and proofs.

Contents

1 Algebraic geometry

1.1 Sheet 1

Question 7

The decomposition of an algebraic variety into irreducible components is unique, that is, if

$$V = V_1 \cup \cdots \cup V_n = V_1' \cup \cdots \cup V_m'$$

then $m = n$, and up to reordering, $V_i = V_i'$ for all i. To see this, we note that $V_i \cap V_j'$ is a subvariety of V_i, so if it is nonempty, then we must have that $V_i \cap V_j' = V_i$, i.e. $V_i \subseteq V_j'$. But we must also have $V_j' \subseteq V_i$, so $V_i = V_j'$.

Question 10

The product of affine varieties is an affine variety, and the projection maps are morphisms. One way of seeing this is that if we have

$$V = \mathbb{V}(I) \subseteq \mathbb{A}^n$$

where $I \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$, then if we consider the embedding $\mathbb{C}[x_1, \dots, x_n] \leq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$, we get a corresponding ideal $I^c \trianglelefteq \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$. Then we have that

$$V^c = \mathbb{V}(I^c) = \mathbb{V}(I) \times \mathbb{A}^m \subseteq \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$$

Which is a cylinder for V. Taking the intersection with the other cylinder gives the product.

Question 12

If V is the union of three lines in $\mathbb{A}^2$ through the origin, and W is the union of three lines through the origin in $\mathbb{A}^3$, where there is no plane containing all three lines, then V and W are not isomorphic. To see this, we can just compute the dimension of the tangent space of V and W at the origin. V has dimension 2 and W has dimension 3. In particular, there cannot be any point on V where the tangent space is 3-dimensional, so there cannot be any isomorphism between V and W.

1.2 Sheet 2

Question 5

Consider the nodal cubic, which is the projective closure of $y^2 = x^3$. The curve is smooth at every point except $x = y = 0$. Therefore, by considering this curve in a different affine, we can see that the projective closure of a smooth affine variety does not have to be smooth.

Question 11

Any (irreducible) quadric hypersurface $Q = \mathbb{V}(f) \subseteq \mathbb{P}^{n+1}$ is birational to $\mathbb{P}^n$. To see this, note that by a $\mathrm{PGL}_{n+2}(\mathbb{C})$ change of coordinates (i.e. diagonalisation of quadratic forms), we may assume that

$$f(X_0, \ldots, X_{n+1}) = -X_0X_1 + X_2^2 + \cdots + X_{n+1}^2$$

.

Define $\pi : Q \to \mathbb{P}^n$ to be the projection from $(1 : 0 : \cdots : 0) \in Q$, i.e.

$$\pi(x_0 : x_1 : \cdots : x_{n+1}) = (x_1 : \cdots : x_n)$$

and define $\psi : \mathbb{P}^n \to Q$ by

$$\psi(y_1 : \cdots : y_{n+1}) = (y_2^2 + \cdots + y_n + 1^2 : y_1^2 : y_1y_2 : \cdots : y_1y_{n+1})$$

These two define birational maps between Q and $\mathbb{P}^n$.

1.3 Sheet 3

Question 3

If we consider a degree d curve V, $P \in \mathbb{P}^n$ a point, and we consider the projection map π from P onto a hyperplane H.

If $P \notin V$, then $\deg(\pi) = d$, since for any $Q \in V$, the line L containing Q and $\pi(Q)$ intersects V at d points, so $\pi(Q)$ has d preimages for generic Q.

However, if $P \in V$, then $\deg(\pi) = d - 1$, since we can't have the preimage P. Therefore in this case, $P = (1 : 0 : \cdots : 0) \in Q$ and $d = 2$ means that $\deg(\pi) = 1$.

Questions 6-8

Recall the Segre embedding is the map

$$\sigma_{mn} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$$

Given by the $(m+1)(n+1)$ X_iY_j. Then for any varieties $V \subseteq \mathbb{P}^m$, $W \subseteq \mathbb{P}^n$, we can define

$$V \times W = \sigma_{mn}(V, W)$$

These questions are in the special case $m = n = 1$, where we show that a product variety is the same as a closed subset of $\Sigma = \sigma_{11}(\mathbb{P}^1 \times \mathbb{P}^1)$. Moreover, if $V \subseteq \Sigma$ closed, then $\sigma_{11}^{-1}(V) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is the vanishing locus of bihomogeneous polynomials in $\mathbb{C}[X_0, X_1, Y_0, Y_1]$.

1.4 Sheet 4

Question 2

This question follows immediately from Riemann Roch. Essentially, what we want is $n \geq 0$ such that

$$\ell(nP) \geq 2$$

since we know $\ell(nP) \geq 1$, where we have the constant functions. So if we have something which is linearly independent from the constant functions, it must be non-constant.

In this case, if $n > 2g - 2$, then

$$\ell(nP) = 1 - g + \deg(D) = 1 - g + n$$

Hence a sufficient condition is $n \geq \max\{1 + g, 2g - 2\}$.

Question 3

Recall that given a divisor D, we can define an associated morphism $\varphi_D : V \to \mathbb{P}^n$, where $n = \ell(D) - 1$. We do this by choosing a basis $\{f_0, \dots, f_n\}$ for $L(D)$, and defining

$$\varphi_D = (f_0 : \cdots : f_n)$$

In particular, if $\deg(D) > 2g$, then φ_D is an embedding. Therefore, consider the divisor $D = nP$, where $n > 2g$. Then we have a morphism

$$\varphi_D : V \to \mathbb{P}^m$$

Suppose wlog that $f_0 = 1$. Any rational function which is regular on all of V is constant, so we must have that $f_1, \dots, f_m$ have a pole at P. Hence we have that

$$\varphi_D(P) = (0 : a_1 : \cdots : a_m)$$

and for $Q \neq P$, we have that

$$\varphi_D(Q) = (1 : b_1 : \cdots : b_m)$$

Let $W = \varphi_D(V)$ and $P' = \varphi_D(P)$. Then we have that V and W are isomorphic, and $W \setminus \{P'\} \subseteq \{X_0 \neq 0\}$, so it is affine.

Question 4

If $\varphi : V \to \mathbb{P}^1$ has degree 2, then we get a corresponding effective divisor D on V of degree 2, by considering the zeroes of φ. Conversely, if we have an effective divisor D with degree 2, $\ell(D) \geq 2$, choosing a nonconstant element of $L(D)$ gives us a degree 2 morphism $V \to \mathbb{P}^1$.

Question 5

For a smooth plane quartic V, we have a basis for $L(D)$, given by

$$\left\{ \frac{x^r y^s}{\frac{\partial f}{\partial Y}} \;\middle|\; 0 \leq r + s \leq d - 3 = 1 \right\}$$

which means that up to clearing denominators, and permuting the coordinates, the morphism associated to the canonical divisor K is $\varphi_K = \mathrm{id}$. Hence it is an embedding, so we have that for any $P, Q \in V$,

$$\ell(K - P - Q) = \ell(K) - 2 = g - 2 = 0$$

Hence by Riemann Roch, for any degree 2 divisor, we have that

$$\ell(D) = 1 - g + \deg(D) = 1 - 2 + 2 = 1$$

So V is not hyperelliptic.

Question 8

This follows from the fact that for curves, being birational is the same as being isomorphic, and that two curves are birational if and only if their function fields are isomorphic.

2 Algebraic topology

2.1 Sheet 1

Question 4

A retract of a contractible space is contractible.

Question 10

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Question 11

From the covering map $\mathbb{R}^2 \to T^2$, we get that any homomorphism $\varphi : \pi_1(T^2, x_0) \to \pi_1(T^2, x_0)$ gives us a matrix $A \in M_2(\mathbb{Z})$, which then gives us a continuous map $f_A : T^2 \to T^2$, with $f_{A*} = \varphi$.

Question 13

We have a double cover of the Klein bottle by the torus, by cutting the torus in half. However, by the Galois correspondence, we can see that the torus does not have a covering space homeomorphic to the Klein bottle, since $\pi_1(K)$ is non-abelian.

2.2 Sheet 2

Question 6

We can construct a covering map $p : \hat{X} \to X$ of $X = S^1 \vee S^1$ which is the wedge of n-circles, which has $\pi_1(\hat{X}) = F_n$. So $p_*\pi_1(\hat{X}) \leq F_2$ is a subgroup of F_2 which is isomorphic to F_n.

To show there is no surjective homomorphism $F_m \to F_n$ for $m < n$, one way is by considering the abelianisation of F_n, i.e. we get a map $F_m \to \mathbb{Z}^n$, which can't be surjective by GRM.

Question 7

If X is a Hausdorff space, G a group acting on X by homeomorphisms, freely and properly discontinuously, then

(i) The quotient map $X \to G\backslash X$ is a covering map, $G\backslash X$ is Hausdorff

(ii) and if X is simply connected, then $\pi_1(G\backslash X) \simeq G$.

The proof is in the Riemann surfaces course.

2.3 Sheet 3

Question 3 (ii)

By the simplicial approximation theorem, we get that there is an isomorphism of fundamental groups between

$$\pi_1(|K|) \simeq \pi_1(|K_2|)$$

where K_2 is the 2-skeleton of K. Intuitively, the higher simplices do not contribute to the fundamental group, since any homotopy can be approximated by a simplicial map in the 2-skeleton. Furthermore, we can think of this as (in terms of cell complexes, not simplicial complexes...)

1. adding 1-cells adds a generator,
2. adding 2-cells adds a relation.

Question 7

If K is a triangulation of a compact n-manifold, i.e.

1. every $n-1$ simplex is a face of exactly two n-simplices,
2. every pair of n simplices can be joined by a sequence of n-simplices, with adjacent terms sharing a face.

Then we have that

$$H_n(K) = \begin{cases} \mathbb{Z} & \text{if } K \text{ is orientable} \\ 0 & \text{if } K \text{ is non-orientable} \end{cases}$$

Furthermore, from examples such as $\mathbb{RP}^2$, we can see that the $n-1$ homology has torsion in the non-orientable case.

2.4 Sheet 4

Question 1

One useful thing to know is that if we have a SES

$$0 \longrightarrow A \longrightarrow B \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

then $B \simeq A \oplus \mathbb{Z}^n$.

Question 2

This question is the five lemma, that is, if we have

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "h_1"] & A_2 \arrow[r] \arrow[d, "h_2"] & A_3 \arrow[r] \arrow[d, "h_3"] & A_4 \arrow[r] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

where each row is exact, h_1, h_2, h_4, h_5 are isomorphisms, then h_3 is an isomorphism.

Question 4

A covering space of a triangulable space is triangulable. To see this, by the Lebesgue covering lemma and barycentric subdivision, we can assume that each simplex is contained in an evenly covered neighbourhood. Taking preimages gives us a triangulation of the covering space.

Question 5

To compute the Lefschetz number of the antipodal map, we can just note that if $L(a) \neq 0$, then by the Lefschetz fixed point theorem a has a fixed point, which it can't. Hence we must have $L(a) = 0$.

Alternatively, notice that since most of the homology groups are zero, we have that $L(a) = 1 + (-1)^n d$, where $a_* : H_n(S^n) \to H_n(S^n)$ is given by $x \mapsto dx$. In particular, $d = \pm 1$. By considering a as $n+1$ reflections and using functoriality, we must have that $d = (-1)^{n+1}$. So $L(a) = 0$.

Question 8

We have that $H_1(X)$ is the abelianisation of $\pi_1(X)$.

3 Analysis of functions

3.1 Sheet 1

Question 1

Here we have a generalisation of Hölder's inequality, which is

$$\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

where $p^{-1} + q^{-1} = r^{-1}$. This follows from the $r = 1$ case.

Question 7

Suppose $f \in L^r$ for some $r < \infty$. Then

$$\|f\|_{L^\infty} = \lim_{p \to \infty} \|f\|_{L^p}$$

3.2 Sheet 2

Question 2

Any non-constant element of X' is an open map, even if X is not complete.

Question 4

Riesz's lemma says that if X is a normed vector space, $V \leq X$ a closed proper subspace, $0 < \alpha < 1$, then there exists $x \in X$, with $\|x\| = 1$ and $\|x - y\| \geq \alpha$ for all $y \in V$.

To see this, as V is a proper subspace, choose $x \in X \setminus V$. As $X \setminus V$ open, we have that

$$d = \inf_{y \in V} \|x - y\| \geq \varepsilon > 0$$

Fix $\varepsilon > 0$, then choose $y \in V$ such that $d \leq \|x - v\| \leq d + \varepsilon$. Let $z = \frac{x-y}{\|x-y\|}$, then $\|z\| = 1$, and

$$\inf_{v \in V} \|v - z\| = \frac{1}{\|x - y\|} \inf_{v \in V} \left\| \left(\|x - y\| v + y \right) - x \right\| = \frac{\inf_{v \in V} \|v - x\|}{\|x - y\|} \geq \frac{d}{d + \varepsilon}$$

Taking ε sufficiently small gives the result.

Question 5

If $\mathscr{P}$ is a separating family of seminorms on X, then $x_n \to 0$ if and only if $p(x_n) \to 0$ for all $p \in \mathscr{P}$.

Question 6

If X is a Banach space, $(\Lambda_k) \subseteq X'$ a sequence, then

$$\Lambda_k \to \Lambda \implies \Lambda_k \rightharpoonup \Lambda \implies \Lambda_k \xrightharpoonup{*} \Lambda$$

Where for the second implication, we use the canonical embedding $\Phi : X \to X''$. In fact, we have that

$$\tau_{w*} \subseteq \tau_w \subseteq \tau_s$$

Question 8 (i)

Suppose H is a Hilbert space, then $x_i \rightharpoonup x$ if and only if for all $y \in H$, $\langle y, x_i \rangle \to \langle y, x \rangle$. This follows from Riesz's representation theorem.

Question 8 (iii)

Suppose X is a Banach space, and $x_i \rightharpoonup x$. Then the x_i are norm bounded. To see this, notice that for all $\Lambda \in X'$,

$$\Phi(x_n)(\Lambda) = \Lambda(x_n) \to \Lambda(x)$$

so $\{\Phi(x_n)\}$ is pointwise bounded. Hence by the uniform boundedness principle, it is norm bounded. But Φ is an isometry by Hahn-Banach, so we are done.

Next, let $f \in X'$ be a support functional for x, then

$$\|x\| = |f(x)| = \liminf_n |f(x_n)| \leq \liminf_n \|f\| \|x_n\| = \liminf_n \|x_n\|$$

as $\|f\| = 1$.

Finally, suppose that X is a Hilbert space, $x_n \rightharpoonup x$ and $\|x_i\| \to \|x\|$. Then

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2 \operatorname{Re} \langle x_n, x \rangle \to 0$$

as $n \to \infty$.

Question 13

Let X be a reflexive Banach space, Y a closed subspace. Then Y is reflexive.

Given $\phi \in Y''$, define $\tilde{\phi} \in X''$ by $\tilde{\phi}(\Lambda) = \phi(\Lambda|_Y)$. As X is reflexive, we have that $\tilde{\phi}(\Lambda) = \Lambda(x)$ for some $x_0 \in X$. Suppose $x \notin Y$. Then by Geometric Hahn-Banach, we have $\Lambda \in X'$ such that $\Lambda(x) = 1$ and $\Lambda|_Y = 0$. Contradiction. So we must have that $x_0 \in Y$.

Now for $\Lambda \in Y'$, let $\tilde{\Lambda} \in X'$ be a Hahn-Banach extension of Λ. Then

$$\phi(\Lambda) = \phi(\tilde{\Lambda}|_Y) = \tilde{\phi}(\tilde{\Lambda}) = \tilde{\Lambda}(x_0)$$

So we must have that $\phi = \Phi_Y(x_0)$, where $\Phi_Y : Y \to T''$ is the canonical embedding.

3.3 Sheet 3

Question 1

We have that

$$\text{convergence in } \mathscr{D} \implies \text{convergence in } \mathscr{S} \implies \text{convergence in } \mathscr{E}$$

and the reverse implications do not hold.

Questions 2 and 4

For each of $X = \mathscr{D}, \mathscr{S}, \mathscr{E}, \mathscr{D}', \mathscr{S}', \mathscr{E}'$, we have that

$$\tau_h \phi \to \phi \quad \text{in} \quad X \text{ as } h \to 0$$

and

$$\Delta_i^h \phi \to \nabla_i \phi \quad \text{in} \quad X \text{ as } h \to 0$$

Question 6

A linear map $u : \mathscr{S} \to \mathbb{C}$ is continuous if and only if there exists $N, k \in \mathbb{N}$ and $C > 0$ such that

$$\left|u(\phi)\right| \leq C \sup_{x \in \mathbb{R}^n, |\alpha| \leq k} \left|(1 + |x|)^N \nabla^\alpha \phi(x)\right|$$

for all $\phi \in \mathscr{S}$. One direction is clear since it shows that if $\phi_j \to 0$ in $\mathscr{S}$, then $u(\phi_j) \to 0$. Conversely, suppose that u is continuous. Then $u^{-1}(|z| < 1)$ is an open neighbourhood of 0. By considering the generating seminorms and the neighbourhood basis of 0, we get the required result.

Question 8

Suppose $f \in L^1(\mathbb{R}^n)$, with $\text{supp}(f) \subseteq B_R(0)$. Then $\hat{f} \in \mathbb{C}^\infty$, and

$$\sup_\xi \left|\nabla^\alpha \hat{f}(\xi)\right| \leq R^{|\alpha|} \left\|f\right\|_{L^1}$$

The proof follows easily from the fact that

$$\nabla_j \hat{f}(\xi) = -i\widehat{x_j f}(\xi)$$

In general, we have correspondences

$$\text{decay in } f \leftrightarrow \text{regularity of } \hat{f}$$

$$\text{decay in } \hat{f} \leftrightarrow \text{regularity of } f$$

In particular, this means often we want to consider small ξ (i.e. decay) and large ξ (i.e. regularity) separately.

Question 10

In this case, we have an uncertainty principle, which says that a function and its Fourier transform can't both be sharply located.

3.4 Sheet 4

Question 4 (a), (c), (d)

We have that $\mathscr{S} \subseteq H^s$ is dense, H^t continuously embeds into H^s for $s < t$ (i.e. we can embed into a space with lower regularity requirements), and $\nabla^\alpha : H^{s+|\alpha|} \to H^s$ is bounded (i.e. derivative decreases regularity).

Question 4 (b), (e)

We have that $H^s(\mathbb{R}^n)' = H^{-s}(\mathbb{R}^n)$, with the pairing

$$\langle f, g \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi) \mathrm{d}\xi$$

Furthermore, we have that $\delta_x \in H^s$ if $s < -n/2$, i.e. $\delta_x \in (H^s)'$ if $s > n/2$, i.e. when we can choose a continuous representative.

Question 7

This is the definition of the trace. That is, we define $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^{n-1})$ by

$$Tu(x') = u(x', 0)$$

Then we show that it can be extended to a continuous bounded linear operator $T : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$.

4 Differential geometry

4.1 Sheet 1

Questions 6 and 7

An immersion is a smooth map $f : X \to Y$ such that $\mathrm{d}f_p$ is injective for all p, and a submersion is a smooth map $f : X \to Y$ such that $\mathrm{d}f_p$ is surjective for all p. Then we can choose local coordinates such that f is the inclusion onto the first coordinates, or projection onto the first coordinates respectively.

Question 8

We have that if y is a regular value of $f : X \to Y$, then $T_p f^{-1}(y) = \ker(\mathrm{d}f_p)$. To see this, notice that if α is a curve on $f^{-1}(y)$ with $\alpha(0) = 0$, then $f(\alpha(t)) = y$ for all t. So we must have that $\mathrm{d}f_p(\alpha'(0)) = 0$. So $T_p f^{-1}(y) \subseteq \ker(\mathrm{d}f_p)$. By counting dimensions, equality must hold.

4.2 Sheet 2

Question 3

For a plane curve contained in a disc of radius r, we must have a point on the curve for which the curvature is at least $1/r$.

Question 7

$$H(p) = \frac{1}{\pi} \int_0^\pi k_n(\theta) \mathrm{d}\theta$$

where $k_n(\theta)$ is the normal curvature of p along a line with angle θ with a given direction. To show this, we just note that we can choose an orthonormal basis for $\mathrm{T}_p S$ such that $\mathrm{d}N_p$ is diagonal.

Question 8

Let S be the surface of revolution given by

$$\phi(u,v) = (f(v)\cos(u), f(v)\sin(u), g(v))$$

Then S has

$$K = -\frac{f''}{f} \quad \text{and} \quad H = \frac{1}{2}\left(\frac{f''}{g'} - \frac{g'}{f}\right)$$

Question 9

For a compact orientable surface S, the Gauss map $S \to S^2$ is surjective. To see this, for $w \in S^2$, let p be such that $\langle p, w\rangle$ is maximal. Then for an appropriate choice of normal, $N(p) = w$.

Question 10

Suppose every point on S is umbilic, i.e. $\kappa_1 = \kappa_2 = -\lambda$. Then by symmetry of mixed partial derivatives, we get that $\lambda_u = \lambda_v = 0$. So λ is constant. By considering the cases $\lambda = 0$ where we get a plane, and $\lambda \neq 0$ where we get a sphere, we get the required result.

4.3 Sheet 3

Question 3

The covariant derivative only depends on the Christoffel symbols, which themselves only depend on the first fundamental form. Therefore, if V is a parallel vector field along α, then $W = \mathrm{d}f \circ V$ is a parallel vector field along $f \circ \alpha$. In particular, we have that

$$\frac{\mathrm{D}W}{\mathrm{d}t} = \mathrm{d}f_{\alpha}(t)\left(\frac{\mathrm{D}V}{\mathrm{d}t}\right)$$

A corollary is that isometries send geodesics to geodesics.

Question 4

We have the equivalent form for the geodesic equations, which is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) &= \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2) \\
\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) &= \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)
\end{aligned}$$

Question 5

Any minimal surface, i.e. if $H = 0$ identically, must have $K \leq 0$. Hence there are no compact minimal surfaces in $\mathbb{R}^3$.

Question 11

The most important thing from this question is that if we are in a geodesic normal neighbourhood, then

$$(\sqrt{G})_{rr} = -K\sqrt{G}$$

Therefore, knowing K determines G and vice versa. Intuitively, if we consider geodesics eminating from a point p, $G(r,\theta)$ determines how far apart the two geodesics are at (r,θ). In particular, this says that if $K > 0$, then $(\sqrt{G})_{rr} < 0$, so the geodesics are getting closer together. On the other hand, if $K < 0$, then the geodesics are getting further apart.

4.4 Sheet 4

Question 1

For a matrix Lie group, we have a concrete description of the exponential map, that is,

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

Question 4

Isometries are rigid, that is, if we know $f(p)$ and $\mathrm{d}f_p$, then we know all of f.

Question 5

Geodesics are local minimisers of length.

Question 10

If ϕ is an orthogonal parametrisation about p, $\alpha : [0, \ell] \to \phi(U)$ a smooth simple closed curve parametrised by arc length, enclosing a domain R. Let $w_0 \in T_{\alpha(0)}S$, and W the parallel transport of w_0 along α. Let $\psi(t)$ be a smooth determination of angle between ϕ_u and W. Then

$$\psi(\ell) - \psi(0) = \int_R K \mathrm{d}A$$

Question 11

A curve with constant curvature and torsion is a circle (if $\tau = 0$) or a helix (if $\tau \neq 0$).

5 Galois theory

5.1 Sheet 1

Question 3

It's the Vandermonde determinant.

Question 4

If L/K is a quadratic extension, then if $\mathrm{char}(K) \neq 2$, $L = K(\sqrt{a})$ for some $a \in K$. If $\mathrm{char}(K) = 2$, then either $L = K(\sqrt{a})$, or $L = K(x)$ where $x^2 + x \in K$.

5.2 Sheet 2

Question 4

If $f \in K[X]$ is a degree of polynomial n, and L/K is a splitting field for f. Then $[L : K] \leq n!$.

Question 6

First, suppose K is a field of characteristic p, where every element is a p-th power. Let $f \in K[t]$ be irreducible. Then f is inseparable if and only if $f = g(t^p)$ for some polynomial g, i.e.

$$f(t) = a_n t^{np} + a_{n-1} t^{(n-1)p} + \cdots + a_0$$

But each element is a p-th power, say $a_i = b_i^p$. Define

$$h(t) = b_n t^n + \cdots + b_0$$

Then $h(t)^p = f(t)$, so f can't be irreducible. Hence any irreducible polynomial must be separable.

A field K is perfect if for all L/K finite, L is separable over K. We have that every characteristic 0 field is perfect. From the above, if every element in K is a p-th power (e.g. in finite fields, by Lagrange's theorem), then K is perfect.

Furthermore, the converse is also true. Suppose $x \in K$ is not a p-th power. Consider $f = t^p - x \in K[t]$, and let L/K be a splitting field for f. Then in $L[t]$, $f = (t-\alpha)^p$, so f is inseparable. Remains to show f is irreducible, since this shows $L = K(\alpha)$ is inseparable over K. Any proper factor of f must be $g = (t-\alpha)^m$ with $1 \leq m < p$. But the t^{m-1} coefficient of g is $-m\alpha$, m coprime to p, so it is invertible. Hence if $g \in K[t]$ then $\alpha \in K$, i.e. x is a p-th power. But we assumed this is not the case. Hence f must be irreducible.

Question 7

Suppose K is a field with $\text{char}(K) = p > 0$. Then x is inseparable over K if and only if $K(x) \neq K(x^p)$. Let f be the minimal polynomial of x over K.

Suppose $K(x) = K(x^p)$. Then $x = \frac{P(x^p)}{Q(x^p)}$ for some $p, q \in K[t]$ coprime. Rearranging, we find that $g(x) = 0$, where $g(t) = Q(t^p)t - P(t^p)$. So $f \mid g$. But $g' = Q(t^p)$, so $(g, g') = (P(t^p), Q(t^p))$. But P, Q coprime, so $(g, g') = 1$. Hence g is separable, and so is f.

Now suppose if $K(x) \neq K(x^p)$. Then as in question 6, $t^p - x^p$ is irreducible, so it is the minimal polynomial of x over $K(x^p)$. Hence $(t^p - x^p) = (t-x)^p \mid f$. So f is inseparable.

Question 8

If $M/L/K$ are finite extensions, then M/K is separable if and only if M/L is separable and L/K is separable. One direction is clear. For the other direction, we are done by counting embeddings.

Question 17

If x is algebraic over K, then we have finitely many intermediate fields $K \leq F \leq K(x)$, since the minimal polynomial of x/F is a factor of the minimal polynomial of x/K. The converse is also true if K is infinite (and the K finite case is trivial).

5.3 Sheet 3

Question 1

The transitive subgroups of S_4 are S_4, A_4, D_8, V_4, C_4 and their conjugates.

Question 2

If p is prime, then any transitive subgroup of S_p contains a p-cycle.

Question 7

We can compute the discriminant using the derivative, where

$$f = \prod_{i=1}^{n}(t - x_i) \implies \text{Disc}(f) = (-1)^{n(n-1)/2}\prod_{i=1}^{n} f'(x_i)$$

Question 13

Let K be a field of characteristic $p > 0$, $a \in K$, and

$$f(t) = t^p - t + a$$

Then noticing that $f(t+1) = f(t)$ for all t, we can see that if L/K is a splitting field for f, then $L = K(x)$ where $f(x) = 0$. Furthermore, f is separable, so L/K is Galois, with $\text{Gal}(L/K) \simeq \mathbb{Z}/p\mathbb{Z}$. L/K is called an Artin-Schreier extension.

Question 14

If f is irreducible, then $f \in \mathbb{F}_q[t]$ divides $t^{q^n} - t$ if and only if $\deg(f) \mid n$. Therefore, $t^{q^n} - t$ is the product of all irreducible monic polynomials with degree dividing n.

5.4 Sheet 4

Question 2

If $\mathrm{Gal}(L/K)$ cyclic of prime order p, generated by σ, and suppose $y \in L$ has $\mathrm{Tr}(y) = 0$. By linear independence of field embeddings, choose $z \in L$ with $\mathrm{Tr}(z) \neq 0$. Then set

$$x = \frac{1}{\mathrm{Tr}(y)}(y\sigma(\theta) + \cdots + (y + \cdots + \sigma^{n-2}(y))\sigma^{n-1}(\theta))$$

Then $x - \sigma(x) = y$.

Question 11

If $L = K(x_1, \ldots, x_n)$, where the x_i are algebraically independent, let $G = S_n$ act on L by permuting the x_i. Then we have that $L^G = K(s_1, \ldots, s_n)$, where the s_i are the elementary symmetric polynomials in the x_j. Then consider

$$f = (t - x_1)\cdots(t - x_n) = t^n - s_1 t^{n-1} + \cdots + (-1)^n s_n \in L^G[t]$$

The Galois group of f/L^G is precisely S_n.

6 Linear analysis

6.1 Sheet 1

Question 2

There exists a discontinuous linear map $X \to X$ if and only if X is infinite dimensional. It is easy to see any linear map from a finite dimensional NVS is continuous. Conversely, let $(e_i)_{i \in I}$ be an algebraic basis for X, and choose a countable subset $J = \{j_1, j_2, \ldots\} \subseteq I$. Then define

$$T(e_i) = \begin{cases} k & \text{if } i = j_k \\ 0 & \text{otherwise} \end{cases}$$

Then T is an unbounded linear map from X to itself.

Question 3

The intersection of dense subspaces doesn't have to be dense. For example, take $V = C([0, 1])$. Then piecewise linear functions and polynomials are both dense, but the intersection is the linear functions, which are not dense.

Question 9

If $p \leq q$, then $\ell^p \leq \ell^q$, and the inclusion is continuous.

Question 10

$(\ell^p)' = \ell^q$ when p, q conjugate, $1 < p < \infty$.

6.2 Sheet 2

Question 7

Fix $\varepsilon > 0$, and define

$$E_n = \{x \geq 1 \mid |f(mx)| \leq \varepsilon \text{ for all } m \geq n\}$$

Then each E_n is closed, as it is an intersection of the preimages of closed sets, and

$$\{x \geq 1\} = \bigcup_n E_n$$

So by Baire, there exists $n \in \mathbb{N}$ and an open interval $I \subseteq E_n$. Finally, we can see that

$$\bigcup_k kI \supseteq [\ell, \infty)$$

for some $\ell > 1$.

Question 8

The key idea in this question is to consider the oscillation, i.e.

$$D_\varepsilon = \{x \mid \forall \delta > 0, \exists y, z \in B_\delta(x) \text{ s.t. } |f(y) - f(z)| \geq \delta\}$$

Each D_ε is nowhere dense, i.e. for any open interval I, we can find a sub-open-interval J such that $D_\varepsilon \cap J = \varnothing$.

Question 10

In fact, any Banach space must have uncountable algebraic dimension, by Baire.

6.3 Sheet 3

Question 4

K metrisable implies $C(K)$ separable follows by considering an ε-net for K, for countably many $\varepsilon \to 0$, and the fact that continuous functions on a compact space is uniformly continuous. For the converse, if (f_k) is a countable dense subset of $C(K)$, then we can check that $T : K \to \ell^1$ defined by

$$T(x)_n = \frac{f_n(x)}{2^n \|f_n\|}$$

is injective, as $C(K)$ separates points by Urysohn's lemma. The pullback then defines a metric on K.

6.4 Sheet 4

Question 2

If f is a rational function, with no poles in $\sigma(T)$, then

$$\sigma(f(T)) = f(\sigma(T))$$

Question 3

If F is a closed subspace of H, then $F^{\perp\perp} = F$, since we have that

$$H = F \oplus F^\perp$$

$F \subseteq F^{\perp\perp}$ is clear, and if $x \in F^{\perp\perp}$, then $x = y + z$ for some $y \in F$ and $z \in F^\perp$. But $\langle x, z \rangle = 0$ implies $z = 0$. So $F^{\perp\perp} \subseteq F$.

Therefore, we have that $\overline{X} = X^{\perp\perp}$ for any subspace X of H.

Question 6

Any unitary operator has norm 1, so by applying question 2 and $\sigma(T) \subseteq B_{\|T\|}(0)$ to U and U^{-1}, we get the required result.

Question 7

The set of invertible operators need not be dense. Let L and R be the left and right shift on ℓ^2. Suppose T has $\|R - T\| = 1$ and T is invertible. Then

$$\|I - LT\| = \|L(R - T)\| \leq \|L\| \|R - T\| < 1$$

Hence LT is invertible, so L is invertible. Contradiction.

Question 10

A is compact if and only if it is the limit of finite rank operators, and the adjoint of a finite rank operator has finite rank.

Question 11

For a compact operator A, we have $\sigma(A) \subseteq \sigma_p(A) \cup \{0\}$. Hence it is clear that if $\lambda \notin \sigma(A)$, then $A - \lambda$ is bounded below.

Question 13

In this case, notice that as U is unitary, $T = U - I$ is normal, so $\ker(T) = \operatorname{im}(T)^{\perp}$. Hence we have that

$$U = \ker(T) \oplus \ker(T)^{\perp} = \ker(T) \oplus \overline{\operatorname{im}(T)}$$

7 Number fields

7.1 Sheet 1

Question 9

If $K = \mathbb{Q}(\alpha)$ is a number field, $\alpha \in \mathcal{O}_K$, and $f \in \mathbb{Z}[x]$ minimal polynomial of f. If $\operatorname{Disc}(f)$ is squarefree, then so is $\Delta(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$, so $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

Question 11

The sign of the discriminant of a number field K is $(-1)^s$, where s is the number of pairs of complex conjugate embeddings.

Question 12

Suppose $f \in \mathbb{Q}[t]$ is an irreducible polynomial of degree n, then

$$\operatorname{Disc}(f) = (-1)^{n(n-1)/2} N_{K/\mathbb{Q}}(f'(\theta))$$

where $\theta \in \mathbb{C}$ is any root of f, $K = \mathbb{Q}(\theta)$.

7.2 Sheet 2

Question 1

If $\mathfrak{a}, \mathfrak{b}$ are ideals, then $\mathfrak{a} + \mathfrak{b}$ is the gcd of $\mathfrak{a}$ and $\mathfrak{b}$, and $\mathfrak{a} \cap \mathfrak{b}$ is the lcm of $\mathfrak{a}$ and $\mathfrak{b}$. Furthermore, if $\mathfrak{a} + \mathfrak{b} = \mathcal{O}_K$, i.e. they are coprime, then $\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$. In addition, we have the CRT

$$\frac{\mathcal{O}_K}{\mathfrak{a}\mathfrak{b}} = \frac{\mathcal{O}_K}{\mathfrak{a}} \times \frac{\mathcal{O}_K}{\mathfrak{b}}$$

Question 5

If p is an odd prime, $K = \mathbb{Q}(\zeta_p)$ is a cyclotomic extension, then $[K : \mathbb{Q}] = p - 1$, and we have that $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$. To show this, by standard norm and trace computations,

$$\mathbb{Z}[\zeta_p] \subseteq \mathcal{O}_K \subseteq \frac{1}{p}\mathbb{Z}[\zeta_p]$$

Now if we define $\pi = 1 - \zeta_p$, then $\pi^{p-1} = up$, where u is a unit. Furthermore, the natural map $\mathbb{Z} \to \mathcal{O}_K/(\pi)$ is surjective, i.e. $\mathcal{O}_K = \mathbb{Z} + \pi\mathcal{O}_K$. Repeating this, we get

$$\mathcal{O}_K = \mathbb{Z} + \pi\mathbb{Z} + \cdots + \pi^m\mathbb{Z} + \pi^{m+1}\mathcal{O}_K$$

for all $m \in \mathbb{N}$. Setting $m + 1 = p - 1$, we get

$$\mathcal{O}_K = \mathbb{Z} + \pi\mathbb{Z} + \cdots + \pi^{p-2}\mathbb{Z} + p\mathcal{O}_K \subseteq \mathbb{Z}[\pi] = \mathbb{Z}[\zeta_p]$$

Question 10

By question 1, we want to show that every ideal in $\mathcal{O}_K/\mathfrak{p}^n$ is principal. By the correspondence theorem, the nonzero ideals are precisely $\mathfrak{p}^r/\mathfrak{p}^n = (\mathfrak{p}/\mathfrak{p}^n)^r$ for $1 \leq r \leq n$. Therefore, suffices to show $\mathfrak{p}/\mathfrak{p}^n$ is principal.

If $\mathfrak{p}/\mathfrak{p}^n = \mathfrak{p}^2/\mathfrak{p}^n$, then $\mathfrak{p}/\mathfrak{p}^n = 0$. Otherwise, let $\alpha \in \mathfrak{p}/\mathfrak{p}^n \setminus \mathfrak{p}^2/\mathfrak{p}^n$. Then (α) is a nonzero ideal which is not contained in $\mathfrak{p}^k/\mathfrak{p}^n$ for $k \geq 2$, so we must have $\mathfrak{p}/\mathfrak{p}^n = (\alpha)$.

Finally, if $\mathfrak{a}$ is an ideal in $\mathcal{O}_K$, which is nonzero and not a unit. Choose $a \in \mathfrak{a}$ such that a is nonzero and not a unit. Then $\mathfrak{a}/a\mathcal{O}_K$ is principal. Let $b \in \mathcal{O}_K$ be such that $\mathfrak{a}/a\mathcal{O}_K = (b)$. Then $\mathfrak{a} = (a, b)$.

7.3 Sheet 3

Question 5

In $\mathbb{Q}(\sqrt{10})$, $\varepsilon_0 = 3+\sqrt{10}$ is the fundamental unit. Therefore, the units are all of the form $u = \pm\varepsilon_0^n$. In particular, $N(\pm\varepsilon_0^n) = N(\varepsilon_0)^n = (-1)^n$, so the solutions to $x^2 - 10y^2 = -1$ are precisely $x + y\sqrt{10} = \pm\varepsilon_0^{2n+1}$.

For the equation $x^2 - 10y^2 = 6$, we know that $(x + y\sqrt{10})$ is a principal ideal of norm 6. Using Dedekind we can find all such ideals, and then using the fundamental unit, we can find all solutions.

Question 6

Rewriting the equation as $x^3 = y^2 + 13$, let $\omega = \sqrt{-13}$. Then let $K = \mathbb{Q}(\omega)$, $\mathcal{O}_K = \mathbb{Z}[\omega]$. Then the ideal equation

$$(x)^3 = (y^2 + 13) = (y - \omega)(y + \omega)$$

factorises uniquely. First we show the ideals $(y - \omega)$ and $(y + \omega)$ are coprime. Suppose $\mathfrak{p}$ divides both. Then $2\omega \in \mathfrak{p}$, so $\mathfrak{p} \mid (2\omega)$, hence $N(\mathfrak{p})$ divides $N(2\omega) = 4 \times 13$.

Using Dedekind's criterion to factorise 2 and 13, we get $(2) = (2, \omega+1)^2$ and $(13) = (13, \omega)^2$. If $\mathfrak{p} = (13, \omega)$, then we get that $y^2 + 13 \in (13, \omega)$, so $13 \mid y$. But this can't happen, for example by considering the equation mod 13^2.

Similarly, if $\mathfrak{p} = (2, \omega + 1)$, we would have $y + \omega \in (2, \omega + 1)$, i.e. $y - 1 \in (2, \omega + 1)$. This then implies y is odd, which means x is even. But then x^2 is 0 mod 4, whereas $y^2 + 13$ is 2 mod 4. Contradiction. Hence $(y - \omega)$ and $(y + \omega)$ are coprime.

This means that $(y + \omega) = \mathfrak{a}^3$ for some ideal $\mathfrak{a}$. In this case, $D_K = -52$, so the Minkowski bound is

$$c_L = \frac{2\sqrt{52}}{\pi} < \frac{16}{\pi} < 7$$

So we need to check the primes 2, 3 and 5. We've already checked 2, so we only need to check 3 and 5. As it turns out, 3 and 5 are inert. Therefore, the class group is either trivial or C_2. In either case, as $\mathfrak{a}^3$ is principal, we must have that $\mathfrak{a}$ is itself principal. Say $\mathfrak{a} = (a + b\omega)$. Note the units in $\mathbb{Z}[\omega]$ are just ± 1. Furthermore,

$$(a + b\omega)^3 = a^3 + 3a^2b\omega - 39ab^2 - 13b^3\omega = (a^3 - 39ab^2) + (3a^2b - 13b^3)\omega$$

Solving the equation

$$3a^2b - 13b^3 = \pm 1$$

over $\mathbb{Z}$, we must have $b = \pm 1$, and $3a^2 - 13 = \pm 1$ has only solution $a = \pm 2$. So we have $(a, b) = (\pm 2, \pm 1)$, i.e. $y = \pm 70, x = 17$.

8 Probability and measure

8.1 Sheet 1

Question 1.4

A d-system which is also a π-system is a σ-algebra.

Question 1.8

Let $B \subseteq \mathbb{R}$ be a Borel set with finite measure. Then for all $\varepsilon > 0$, there exists a finite union of disjoint intervals $I = I_1 \cup \cdots \cup I_n$ such that $|B \triangle I| < \varepsilon$. This follows by regularity of the Lebesgue measure, defined as an outer measure.

Question 2.1

The sum, product, inf, sup, lim inf, lim sup of measureable functions is measurable. Furthermore, the set

$$\{x \mid f_n(x) \text{ converges as } n \to \infty\}$$

is measurable.

Question 2.2

The image measure is

$$\nu(A) = \mu(f^{-1}(A))$$

Question 2.3

Ranom variables X, Y are independent if and only if for all $x, y \in \mathbb{R}$,

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$$

Question 2.7

Let C_n be the n-th approximation to the Cantor set C, F_n the distribution function of a random variable uniformly distributed on C_n. Then C has measure zero. Moreover, define

$$F(x) = \lim_{n\to\infty} F_n(x)$$

the limit exists for all $x \in [0, 1]$. Then F is continuous, $F(0) = 0, F(1) = 1$, and for a.e. $x \in [0, 1]$ (i.e. $[0, 1] \setminus C$), F is differentiable with derivative 0.

8.2 Sheet 2

Question 3.2

If μ, ν are finite measures on $\mathbb{R}$, with $\mu(f) = \nu(f)$ for all continuous bounded f. Then $\mu = \nu$. By uniqueness of measures, suffices to show that this holds for an interval, since the Borel σ-algebra is generated by the π-system of clopen intervals $(a, b]$. But we can approximate the indicator function of an interval by piecewise linear functions, then taking a limit gives the required result.

Question 3.5

For $f(x) = x^{-\alpha}$ is integrable on $(0, 1]$ if and only if $\alpha < 1$, and it is integrable on $[1, \infty)$ if and only if $\alpha > 1$.

Question 3.6

If $u, v \in \mathbb{C}^1(\mathbb{R})$, with $uv \to 0$ as $|x| \to \infty$. Then

$$\int_{\mathbb{R}} uv'\mathrm{d}x = -\int_{\mathbb{R}} u'v\mathrm{d}x$$

Question 3.9

If $\nu = \mu \circ f^{-1}$ is the image measure, then for all g, $\nu(g) = \mu(g \circ f)$ for all nonnegative measurable function g.

Question 3.11

Suppose $X_1, \dots, X_n$ are random variables, with densities $f_1, \dots, f_n$. Furthermore, if $X = (X_1, \dots, X_n)$ has density f. Then $X_1, \dots, X_n$ are independent if and only if

$$f(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n) \text{ a.e.}$$

8.3 Sheet 3

Question 4.1

Suppose $f_n \to f$ a.e. and $\|f_n\|_{L^1} \to \|f\|_{L^1}$. Then $\|f_n - f\|_{L^1} \to 0$. To see this, notice by Minkowski, $g_n = |f_n| + |f| - |f_n - f| \geq 0$, and $g_n \to 2|f|$ a.e. Therefore, by Fatou's lemma,

$$2\|f\|_{L^1} = 2\int |f| = 2\int \liminf_n g_n \leq 2\liminf_n \int g_n = 2\|f\|_{L^1} - \limsup_n \|f_n - f\|_{L^1}$$

As $\|f\|_{L^1} < \infty$, this means that

$$\limsup_n \|f_n - f\|_{L^1} \leq 0 \implies \|f_n - f\|_{L^1} \to 0$$

Question 4.5

Simple functions are dense in $L^p(\mathbb{R}^n)$ for $p < \infty$. Similarly, so is $C_c(\mathbb{R}^n)$.

Question 5.1

The question is true in any Hilbert space. If H is a Hilbert space, V_n increasing sequence of closed subspaces, $x \in H$ and $P_n : H \to V_n$ is the orthogonal projection.

Now notice that

$$P_{n+1}(x) - P_n(x) \in V_n^\perp \quad \text{and} \quad P_{m+1}(x) - P_m(x) \in V_n \text{ for m < n}$$

We have that the sequence $P_1(x), P_2(x) - P_1(x), \dots$ is orthogonal. Moreover,

$$P_n(x) = (P_n(x) - P_{n-1}(x)) + \cdots + (P_2(x) - P_1(x)) + P_1(x)$$

Therefore, by Pythagoras, we have that

$$||x||^2 \geq \|P_n(x)\|^2 = \|P_n(x) - P_{n-1}(x)\|^2 + \cdots + \|P_2(x) - P_1(x)\|^2 + \|P_1(x)\|^2$$

In particular, this means that $\sum_{n=1}^\infty \|P_n(x) - P_{n-1}(x)\|^2 < \infty$. But for $m > n$, we have that

$$\|P_m(x) - P_n(x)\|^2 = \|P_m(x) - P_{m-1}(x)\|^2 + \cdots + \|P_{n+1}(x) - P_n(x)\|^2$$

which converges to zero as $n \to \infty$. Hence $(P_n(x))$ is a Cauchy sequence, so converges by completeness.

8.4 Sheet 4

Question 7.8

In $\mathbb{R}$, weak convergence of random variables and convergence in distribution are equivalent, since we can approximate $1_{(-\infty,x]}$ by continuous functions.

Question 7.9

To prove weak convergence (which is in fact weak-$*$ convergence on $C_b(\mathbb{R})'$), we can just show that pointwise convergence on a dense subspace, e.g. to show weak convergence of Borel probability measures, suffices to show weak-$*$ convergence on $C_c^\infty(\mathbb{R}^n)$.

Question 9.1

If $(E, \mathcal{E}, \mu)$ is a measure space, $\tau : E \to E$ is measure preserving, then

$$\mathcal{E}_\tau = \{A \in \mathcal{E} \mid \tau^{-1}(A) = A\}$$

is a σ-algebra, and $f : E \to \mathbb{R}$ is $\mathcal{E}_\tau$ measurable if and only if it is invariant, i.e. $f \circ \tau = f$.

Question 9.2

As f is θ-invariant, $A_x = f^{-1}((-\infty, x)) \in \mathcal{E}_\theta$. Hence for all x, $\mu(A_x) = 0$ or $\mu(A_x^\complement) = 0$. Using this we can show f is a.e. constant.

Question 9.3

The baker map $\tau(x) = 2x \mod 1$ is measure preserving on $[0, 1)$, and it is ergodic, since by looking at the binary expansion, this corresponds exactly to the shift map.

Question 9.4

Now let $\tau(x) = x + a \mod 1$ be the rotation map on S^1, and suppose A is τ invariant. Let $f(x) = 1_A(x)$. Then $f(x) = f \circ \tau(x)$. Now computing the Fourier coefficients (i.e. Fourier *series*) for f, we find by a change of variables,

$$\hat{f}(k) = \int_0^1 f(x) e^{2\pi i k x} \mathrm{d}x = \int_0^1 f(x) e^{2\pi i k x} e^{-2\pi i k a} \mathrm{d}x = e^{-2\pi i k a} \hat{f}(k)$$

Therefore,

$$(1 - e^{-2\pi i k a}) \hat{f}(k) = 0$$

for all $k \in \mathbb{Z}$. Therefore, if a is irrational, then all of the Fourier coefficients vanish for $k \neq 0$, which means that f itself must be constant a.e.. As f is the indicator of the set A, this means that we must have $|A| = 0$ or $|A| = 1$.

9 Riemann surfaces

9.1 Sheet 1

Question 3

If $\pi : \tilde{X} \to X$ is a covering map, $f : \tilde{X} \to \tilde{X}$ a map such that $\pi circ f = \pi$, i.e. a covering transformation. Then f has a fixed point if and only if it is the identity.

Question 5

When it comes to finding the natural boundary for a power series, often it makes sense to consider the derivative. For example, if

$$f(z) = \sum_n \frac{z^{2^n}}{2^n}$$

then

$$z f'(z) = \sum_n z^{2^n}$$

and it is much easier to find the natural boundary of $zf'(z)$ than $f(z)$ in this case.

Question 7

Any injective analytic map $\mathbb{C} \to \mathbb{C}$ is of the form $z \mapsto az + b$, and any injective analytic map $\mathbb{C}_\infty \to \mathbb{C}_\infty$ is a Möbius transformation.

Question 10

The identity principle for Riemann surfaces follows from the identity principle for holomorphic functions, by doing local computations and using connectedness.

Question 11

Suppose $D \subseteq \mathbb{C}$ is an open disc, $u : D \to \mathbb{R}$ harmonic. Define $g = u_x - iu_y$. Then g is analytic. Moreover, if z_0 is the centre of the disc, defining

$$f(z) = u(z_0) + \int_{z_0}^{z} g(t)\mathrm{d}t$$

where we take the integral over the straight line segment, f is analytic with $f' = g$, and $u = \mathrm{Re}(f)$.

Question 12

The identity principle for harmonic functions is similar, except we have that the set of points where u and v agree have empty interior. That is, if u and v agree on an open set, then they agree everywhere.

Question 15

Let $f(z) = \sum_n a_n z^n$ be a power series with radius of convergence 1. Let $\rho(z$ be the radius of convergence of the power series for f centred at z. Then $\zeta \in \mathbb{T}$ is a regular point if and only if $\rho(\zeta/2) > 1/2$.

9.2 Sheet 2

Question 2

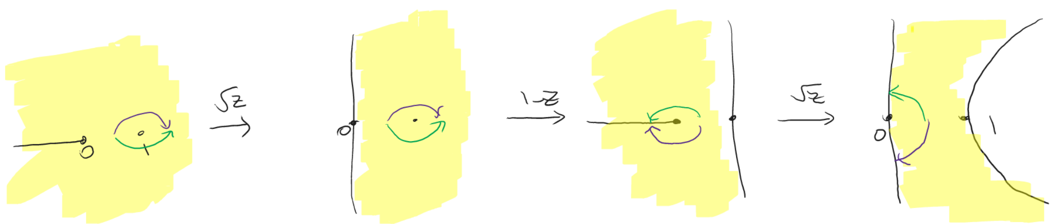
A regular covering map is surjective.

Question 3

That is, simply connected lifting. If $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is an analytic map, then we can define $F : \mathbb{C} \to \mathbb{C}$ such that $\pi_2 \circ F = f \circ \pi_1$, by noticing that if we fix $\mu \in \pi_2^{-1}(f(\pi_1(0)))$, then for any $w \in \mathbb{C}$, let γ be a path from 0 to w in $\mathbb{C}$. Then $f \circ \pi_1 \circ \gamma$ is a path in $\mathbb{C}/\Lambda_2$, so there exists a lift $\tilde{\gamma}$ on $\mathbb{C}$. Moreover, this lift is unique, so we can define $F(w) = \tilde{\gamma}(1)$. Notice that $F(w)$ is independent of the choice of γ by the monodromy theorem.

Question 7

Let $f(z) = \sqrt{1 - \sqrt{z}}$. Then we can take homotopic paths in $\mathbb{C}_*$ from $1/2$ to $3/2$, but give different analytic continuations. One way to see this is as follows



Question 9

Suppose $f : R \to S$ is a non-constant analytic map between compact Riemann surfaces, $B \subseteq S$ the set of branch points. Then $f : R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map.

Let $q \in S \setminus B$. Fix a chart (ψ, W) at q, with $\psi(q) = 0$. Say $f^{-1}(q) = \{p_1, \dots, p_n\}$. For each i, let (ϕ_i, V_i) be a chart at p_i, such that

$$\psi \circ f \circ \phi_i(z) = z$$

This means that $f : V_i \cap f^{-1}(W) \to f(V_i) \cap W$ is a homeomorphism. By shrinking the V_i, we can assume wlog that they are disjoint. Let $V = \bigcup_i V_i$. Then V is open, so $R \setminus V$ is closed, which means $R \setminus V$ is compact. Hence $K = f(R \setminus V)$ is compact, so closed. Therefore, $S \setminus K$ open, so we can choose a connected open neighbourhood $\tilde{W}$ of q contained in $W \cap S \setminus K$. Then

$$f^{-1}(W) \subseteq R \setminus f^{-1}(K) \subseteq R \setminus (R \setminus V) = V$$

Setting $\tilde{V}_i = V_i \cap f^{-1}(W_i)$, we get that

$$f : \tilde{V}_i \to f(\tilde{V}_i)$$

is a homeomorphism (as $f(\tilde{V}_i) \subseteq \tilde{W} \subseteq W$).

9.3 Sheet 3

Questions 3 and 4

In both of these cases, we can assume wlog that f has no zeroes or poles on ∂P. Therefore, we can apply the argument principle or the residue theorem to a specific function. Furthermore, since f is periodic, the integral over ∂P has a nice form, which we can use to get the required result.

In particular,

$$\oint_{\partial P} \frac{f'(z)}{f(z)} \mathrm{d}z$$

gives that the number of zeroes and poles, with multiplicity, are the same. Similarly,

$$\oint_{\partial P} z \frac{f'(z)}{f(z)} \mathrm{d}z$$

gives that if $a_1, \ldots, a_n$ are the zeroes, and $b_1, \ldots, b_n$ are the poles, then

$$\sum_{j=1}^{n} a_j - \sum_{j=1}^{n} b_j \in \Lambda$$

Question 9

Suppose we have a conformal equivalence $f : \mathbb{C}_* \to \mathbb{C} \setminus \{p_1, \ldots, p_n\}$, where $n \geq 2$. A standard argument by Casaroti-Weierstrass and the open mapping theorem shows that f can't have an essential singularity at 0. Hence it must be removable, or a pole. The case where it is a pole can be considered by looking at $1/f$ instead.

Now if 0 is a removable singularity, then we must have $f(0) = p_j$ for some j. If not, then $f(0) = f(w)$ for some $w \in \mathbb{C}_*$ by surjectivity of f. Using the open mapping theorem, we find that this contradicts f being injective. Hence we must have that $f(0) = p_j$. By renumbering, wlog $f(0) = p_n$. Then we have a bijective holomorphic map, hence a conformal equivalence $f : \mathbb{C} \to \mathbb{C} \setminus \{p_1, \ldots, p_{n-1}\}$. As $n \geq 2$, the right hand side is not simply connected. Contradiction.

Therefore, no such conformal equivalence can exist. Hence by the uniformisation theorem, $\mathbb{C} \setminus \{p_1, \ldots, p_n\}$ must be uniformised by $\mathbb{D}$ for $n \geq 2$.

Question 11

Linear algebra shows that we can construct $P \in \mathbb{C}[Z, W]$ such that $P(f, g)$ has no poles. Hence it must be constant.

To see that such a P exists, suppose $T : \mathbb{C}^n \to \mathbb{C}^m$ is a linear map. Then by rank-nullity,

$$\dim(\ker(T)) = n - \dim(\operatorname{im}(T)) \geq n - m > 0$$

if $n > m$. Hence if we can represent the constraint that $P(f, g)$ has no poles as m-linear constraints, if we take a space of polynomials such that it has dimension $> m$, then there must be a nonzero polynomial satisfying the constraints.

Once we have that it is constant, we can make the constant zero by subtracting off a constant.

Question 12

Algebraic geometry is useful for constructing the charts to glue along. That is, we have

$$V = \mathbb{V}(x^d + y^d - 1)$$

Then the projective closure is $\mathbb{V}(x^d + y^d - z^d)$, so computing the points at infinity and transforming into a different affine gives the required charts.