Linear analysis

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Contents

1 Inequalities

Proposition 1.1 (Young's inequality). Let $p, q \in (1, \infty)$ be conjugate indices. Then for all $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proposition 1.2 (Hölder). Let $p, q \in [1, \infty]$ be conjugate, then

$$\sum_k |x_k y_k| \leq \|x\|_p \|y\|_q$$

Proposition 1.3 (Minkowski). Let $p \in [1, \infty]$. Then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

2 Normed vector spaces

2.1 Topological and normed vector spaces

In this case, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 (normed vector space)
A norm $\|\cdot\|$ on a vector space V is a map $V \to \mathbb{R}_{\geq 0}$ such that

(i) $\|v\| = 0$ if and only if $v = 0$,

(ii) $\|\lambda v\| = |\lambda| \|v\|$,

(iii) $\|v + w\| \leq \|v\| + \|w\|$.

A pair $(V, \|\cdot\|)$ is called a normed vector space.

Proposition 2.2. $d(x, y) = \|x - y\|$ defines a metric on V. With respect to this metric,

1. $(+) : V \times V \to V$ and $(\cdot) : \mathbb{F} \times V \to V$ are continuous,
2. translation $x \mapsto x + v$ is a homeomorphism $V \to V$,
3. dilataion $x \mapsto \lambda x$ is continuous, and it is a homeomorphism if $\lambda \neq 0$.

Definition 2.3 (topological vector space)
A topological vector space V over a field $\mathbb{F}$ is a vector space with a topology which makes addition and scalar multiplication continuous, and V is T_1[a].

[a] That is, $\{x\}$ is closed for all $x \in V$.

Definition 2.4 (locally convex)
A topological vector space V is locally convex if every neighbourhood of 0 contains a convex neighbourhood of 0.

Definition 2.5 (bounded)
A subset $B \subseteq V$ is bounded if for any open set U containing 0, there exists $t_0 > 0$ such that for any $t > t_0$, $B \subseteq tU$.

Definition 2.6 (locally bounded)
A topological vector space V is locally bounded if there exists $U \subseteq V$ open bounded neighbourhood of 0.

Theorem 2.7. Let $(V, \mathcal{T})$ be a topological vector space, such that there is a bounded convex neighbourhood C of 0. Then V is normable. That is, there exists a norm on V which induces the same topology.

Proof. **Step 1: There exists a bounded balanced convex neighbourodd $\tilde{C} \subseteq C$ of 0.** We say $\tilde{C}$ is bounded if $\lambda\tilde{C} \subseteq \tilde{C}$ for all $|\lambda| \leq 1$. Since $(\cdot) : \mathbb{F} \times V \to V$ is continuous, $(\cdot)^{-1}(C)$ is a neighbourhood of $(0,0)$. So there exists an open ball $B_\varepsilon(0) \subseteq \mathbb{F}$, and an open neighbourhood $U \subseteq V$ of 0 such that

$$(\cdot)(B_\varepsilon(0) \times V) \subseteq C$$

Define $\tilde{C}$ to be the convex hull of $(\cdot)(B_\varepsilon(0) \times V)$. Then $\tilde{C} \subseteq C$ as C is convex, and so it is bounded. $\tilde{C}$ is balanced since $\lambda B_\varepsilon(0) = B_{|\lambda|\varepsilon}(0) \subseteq B_\varepsilon(0)$ for all $|\lambda| \leq 1$. Therefore $\lambda\tilde{C} \subseteq \tilde{C}$.

Step 2: Minkowski gauge Define the Minkowski gauge for $\tilde{C}$ as $\mu_{\tilde{C}} : V \to R_{\geq 0}$, by

$$\mu_{\tilde{C}}(v) = \inf\left\{t \geq 0 \mid v \in t\tilde{C}\right\}$$

To show that $\mu_{\tilde{C}}$ is well defined, it is clear that the set is bounded below, and that all elements are nonnegative, so all we need to show is that the set is nonempty. But by continuity of $(\cdot)$, $t^{-1}v \to 0$ as $t \to \infty$, so $t^{-1}v \in \tilde{C}$ for t large enough.

Step 3: $\mu_{\tilde{C}}$ is a norm

(i) $\mu_{\tilde{C}} \geq 0$ is true by construction, and it is clear that $\mu_{\tilde{C}}(0) = 0$. Now suppose if $\mu_{\tilde{C}}(v) = 0$, $v \neq 0$. Then there exists U open, $0 \in U$ with $v \notin U$[1]. Since $\tilde{C}$ is bounded, there exists $t_0 \geq 0$ such that $\tilde{C} \subseteq t_0 U$. Since $\mu_{\tilde{C}}(v) = 0$, there exists $t_1 < t_0^{-1}$ such that $v \in t_2\tilde{C}$. Then

$$v \in t_1\tilde{C} \subseteq t_0^{-1}\tilde{C} \subseteq U$$

Contradiction. Notice we needed $\tilde{C}$ balanced for the above.

(ii) For $\lambda = 0$ this is trivial. Now suppose $\lambda \neq 0$. Let t be such that $\lambda v \in t\tilde{C}$. Then

$$\frac{\lambda}{|\lambda|}v \in \frac{t}{|\lambda|}\tilde{C}$$

But $\tilde{C}$ is balanced, so we have that $v \in \frac{t}{|\lambda|}\tilde{C}$. Which means that

$$\mu_{\tilde{C}}(v) \leq \frac{1}{\lambda}\mu_{\tilde{C}}(\lambda v)$$

But $\lambda \neq 0$, so running the same argument with λ^{-1} instead we get equality.

(iii) Given $t_1, t_2 > 0$ such that $v_1 \in t_1\tilde{C}$ and $v_2 \in t_2\tilde{C}$, we have that

$$v_1 + v_2 \in t_1\tilde{C} + t_2\tilde{C} = (t_1 + t_2)\left(\frac{t_1}{t_1 + t_2}\tilde{C} + \frac{t_2}{t_1 + t_2}\tilde{C}\right) \subseteq (t_1 + t_2)\tilde{C}$$

by convexity. So $\mu_{\tilde{C}}(v_1 + v_2) \leq t_1 + t_2$. Taking the infimum over the right hand side, we get that $\mu_{\tilde{C}}(v_1 + v_2) \leq \mu_{\tilde{C}}(v_1) + \mu_{\tilde{C}}(v_2)$.

Step 4: $\mu_{\tilde{C}}$ induces the same topology. Consider an open ball $B_\varepsilon(v_0)$ with the $\mu_{\tilde{C}}$ norm. We will show this is open in $\mathcal{T}$. Let $v \in B_\varepsilon(v_0)$, then $B_{\varepsilon'}(v) \subseteq B_{\varepsilon'}(v_0)$, where $\varepsilon' = \varepsilon - \mu_{\tilde{C}}(v)$. But by definition of $\mu_{\tilde{C}}$, $B_{\varepsilon'}(v) \supseteq v + \frac{\varepsilon'}{2}\tilde{C}$, which is a $\mathcal{T}$-neighbourhood of v as translation and dilation are continuous. Hence $B_\varepsilon(v_0)$ contain an $\mathcal{T}$-open neighbourhood of every point in $B_\varepsilon(v_0)$, so it is $\mathcal{T}$-open.

Conversely if U is $\mathcal{T}$ open, wlog $0 \in U$. Then as $\tilde{C}$ is bounded, there exists $\varepsilon > 0$ such that $\tilde{C} \subseteq \varepsilon^{-1}U$. So $\varepsilon\tilde{C} \subseteq U$, and $\delta\tilde{C} \subseteq U$ for all $\delta < \varepsilon$. So $B_\varepsilon(0) \subseteq U$. □

Definition 2.8 (Banach space)

A normed vector space V is a Banach space if it is complete with respect to the metric induced by the norm.

[1] As V is T_1, $V \setminus \{v\}$ works

Proposition 2.9. V is a Banach space if and only if every series $\sum_n x_n$ with $\sum_n \|x_n\| < \infty$ is convergent.

Proof. Suppose V is a Banach space, and let x_n be such that $\sum_n \|x_n\|$ converges. Consider the partial sums

$$S_N = \sum_{n \le N} x_n$$

Then for $M \le N$, as $M, N \to \infty$, we have that

$$\|S_N - S_M\| = \left\| \sum_{M<n\le N} x_n \right\| \le \sum_{M<n\le N} \|x_n\| \to 0$$

as the series for the norm converges. Hence (S_N) is a Cauchy sequence, so converges.

Conversely, suppose (v_n) is a Cauchy sequence, then by passing to a subsequence, we can assume without loss of generality that for all $n \ge m$,

$$\|v_n - v_m\| \le 2^{-m}$$

Now define $x_1 = v_1$, and $x_i = v_i - v_{i-1}$ for all $i > 1$. So $x_1 + \cdots + x_n = v_n$. Then $\sum_n \|x_i\| < \infty$ as the geometric series converges, and so $\sum_n x_n$, and thus v_n converges. □

2.2 Examples

Definition 2.10 (ℓ^p)

Define the ℓ^p sequence spaces

$$\ell^p = \left\{ (x_n) \mid \|x\|_p < \infty \right\}$$

Proposition 2.11. ℓ^p are all Banach spaces.

Proposition 2.12. Let X be a topological space, $C_{\mathbb{F},b}(X)$ be the space of continuous bounded functions with pointwise operations, then $C_{\mathbb{F},b}(X)$ is a Banach space with the supremum norm.

Definition 2.13 (L^p norm)

Let $(E, \mathcal{E}, \mu)$ be a measure space, then the L^p norm on measurable functions is defined by

$$\|f\|_p = \left(\int_E |f|^p \, \mathrm{d}\mu \right)^{1/p}$$

Proposition 2.14. The space of continuous functions is incomplete with respect to the L^p norm if $p < \infty$. The completion is the space L^p.

2.3 Bounded linear maps

Definition 2.15 (bounded linear map)

Let V, W be topological vector spaces. $T : V \to W$ is bounded if the image of a bounded set is bounded. We write $\mathcal{B}(V, W)$ for the space of bounded linear maps $V \to W$, and $\mathcal{B}(V) = \mathcal{B}(V, V)$ for the space of bounded linear maps on V.

Proposition 2.16. If V is a locally bounded topological vector space (such as a normed vector space), then bounded and continuous are equivalent.

Proof. **Bounded implies continuous.** Let U be an open neighbourhood of $0 \in W$, $\tilde{U}$ be an open bounded neighbourhood of $0 \in V$. Then as $T(\tilde{U})$ is bounded, $T(\tilde{U}) \subseteq tU$ for some $t > 0$. So $T^{-1}(U) \supseteq t^{-1}\tilde{U}$. Hence it is a neighbourhood of 0. As translation is a homeomorphism, T is continuous everywhere, so T is continuous.

Continuous implies bounded. Let $B \subseteq V$ be bounded, U an open neighbourhood of $0 \in W$. Then $T^{-1}(U)$ is an open neighbourhood of $0 \in U$, so there exists $t > 0$ such that $B \subseteq tT^{-1}(U)$, so $T(B) \subseteq tU$. Hence $T(B)$ is bounded, so T is bounded as B was arbitrary. □

Definition 2.17 (operator norm)

Let V, W be normed vector spaces, The operator norm of $T \in \mathcal{B}(V, W)$ is

$$\|T\| = \sup_{\|v\| \leq 1} \frac{\|Tv\|}{\|v\|} = \sup_{\|v\|=1} \|Tv\|$$

Proposition 2.18. $\mathcal{B}(V, W)$ is a normed vector space, with the operator norm.

Proof. Clearly the pointwise sum and scalar multiple of bounded operators are bounded, so $\mathcal{B}(V, W)$ is a vector space.

By definition $\|T\| \geq 0$, and $\|0\| = 0$. Suppose $\|T\| = 0$. Then $T = 0$ on $B_1(0)$. But then by homogeneity, $T = 0$. Homogeneity and triangle inequality for the operator norm are obvious. □

Proposition 2.19. For all $v \in V$ and $T \in \mathcal{B}(V, W)$, $\|Tv\| \leq \|T\| \|v\|$.

Proposition 2.20. Suppose V, W are normed vector spaces, W is complete. Then $\mathcal{B}(V, W)$ is also complete.

Proof. Given a Cauchy sequence (T_k) in $\mathcal{B}(V, W)$, for any $v \in V$, $(T_k(v))$ is a Cauchy sequence, so $T_k v \to Tv$ for some $Tv \in W$ as W is a Banach space. $T : V \to W$ is linear as pointwise limits are linear. Furthermore,

$$\|T_m(v) - Tv\| = \lim_{n\to\infty} \|T_m v - T_n v\| \leq \lim_{n\to\infty} \|v\| \|T_m - T_n\| \to 0$$

as $m \to \infty$. Furthermore, fix N, and for $\|v\| \leq 1$, we have that

$$\|Tv\| \leq \|T_N(v)\| + \lim_{m\to\infty} \|T_N(v) - T_m(v)\| \leq \|T_N\| + \lim_{m\to\infty} \|T_N - T_m\|$$

so T is bounded. Finally, we have that

$$\sup_{k \geq N} \|T_k - T\| \leq \sup_{m,n \geq N} \|T_m - T_n\| \to 0$$

as $N \to \infty$, so $T_k \to T$. □

Definition 2.21 (dual space)

The dual of a normed vector space V is $V^* = \mathcal{B}(V, \mathbb{F})$.

Corollary 2.22. V^* is a Banach space.

Definition 2.23 (adjoint)
Let V, W be normed vector spaces, $T \in \mathcal{B}(V, W)$. Then the adjoint of T is $T^* \in \mathcal{B}(W^*, V^*)$ given by

$$T^*(\psi)(v) = \psi(Tv)$$

Proposition 2.24. $\|T^*\| \leq \|T\|$.

Proof.

$$\begin{aligned}
\|T^*\| &= \sup_{\|\psi\| \leq 1} \|T\psi\| \\
&= \sup_{\|\psi\| \leq 1} \sup_{||v|| \leq 1} \|T\psi(v)\| \\
&= \sup_{||v|| \leq 1} \sup_{\|\psi\| \leq 1} \|\psi(Tv)\| \\
&\leq \sup_{||v|| \leq 1} \|Tv\| \\
&= \|T\|
\end{aligned}$$

□

Definition 2.25 (bidual)
The bidual of a normed vector space V is $V^{**} = (V^*)^*$.

Definition 2.26 (canonical embedding)
The canonical embedding $\Phi : V \to V^{**}$ is given by $\Phi(v) = \hat{v}$, where $\hat{v}(\psi) = \psi(v)$.

Proposition 2.27. $\Phi \in \mathcal{B}(V, V^{**})$ with $\|\Phi\| \leq 1$.

2.4 Finite dimensional normed vector spaces

Definition 2.28 (equivalence of norms)
Let V be a vector space, norms $||\cdot||$ and $||\cdot||'$ are equivalent if there are constants c, C such that for all $v \in V$,

$$c||v||' \leq ||v|| \leq C||v||'$$

Proposition 2.29. Equivalence of norms is an equivalence relation, and equivalent norms induce the same topology.

Proposition 2.30. All norms on a finite dimensional vector space V are equivalent.

Proof. Fix a basis $e_1, \dots, e_n$ for V. Let $\|\cdot\|$ be any norm on V. we will show that it is equivalent to the $\|\cdot\|_\infty$ norm. Now

$$\|v\| = \left\| \sum_{i=1}^n v_i e_i \right\| \leq \sum_{i=1}^n |v_i| \|e_i\| \leq \|v\|_\infty \underbrace{\sum_{i=1}^n \|e_i\|}_{=C}$$

Define $S = \{v \in V \mid \|v\|_\infty = 1\}$. This is a compact connected space, and $\|\cdot\| : S \to \mathbb{R}_{\geq 0}$ is continuous, so $\|S\|$ is a closed bounded interval. That is, there exists $v_0 \in S$ such that $\|v_0\|$ minimal. As $v_0 \neq 0$, set $c = \|v_0\|$. Then for any $v \neq 0$,

$$\frac{\|v\|}{\|v\|_\infty} \geq c$$

by homogeneity. So $\|v\| \geq \|v\|_\infty$. □

Proposition 2.31. Let V be a normed vector space. Then V is finite dimensional if and only if $\overline{B} = \overline{B}_1(0)$ is compact.

Proof. ($\implies$) is just the Heine-Borel theorem and equivalence of norms. For the converse, notice that

$$\overline{B} \subseteq \bigcup_{v \in \overline{B}} B_{1/2}(v)$$

so by compactness, we have a finite subcover $v_1, \dots, v_n$. Let $W = \operatorname{span}\{v_1, \dots, v_n\}$, then

$$\overline{B} \subseteq \bigcup_{i=1}^n (v_i + \frac{1}{2}\overline{B}) \subseteq W + \frac{1}{2}\overline{B}$$

Iterating this, we get that $\overline{B} \subseteq W + 2^{-k}\overline{B}$ for all k, hence

$$\overline{B} \subseteq \bigcap_k (W + 2^{-k}\overline{B}) \subseteq \overline{W} = W$$

So $V = W$. □

2.5 *Hahn-Banach*

Strictly speaking Hahn-Banach is not in the schedules for Linear analysis, but it is in the schedules for Analysis of functions. We include the statements but not the proofs here for completeness.

Definition 2.32 (seminorm)

Let V be a vector space. A function $p : V \to \mathbb{R}_{\geq 0}$ is called a seminorm if

(i) $p(v_1 + v_2) \leq p(v_1) + p(v_2)$,

(ii) $p(\lambda v) = |\lambda| p(v)$.

Theorem 2.33 (Hahn-Banach (seminorm)). Let V be a vector space, $p : V \to \mathbb{R}_{\geq 0}$ a seminorm, $W \leq V$ a subspace, $f : W \to \mathbb{F}$ linear such that $|f(w)| \leq p(w)$ for all $w \in W$. Then there exists a $\tilde{f} : V \to \mathbb{F}$ such that $\tilde{f}$ is linear, $\tilde{f}|_W = f$ and $\left|\tilde{f}(v)\right| \leq p(v)$ for all $v \in V$.

Definition 2.34 (sublinear)

Let V be a real vector space, a function $p : V \to \mathbb{R}_{\geq 0}$ is called sublinear if

(i) $p(v_1 + v_2) \leq p(v_1) + p(v_2)$,

(ii) $p(\lambda v) \leq \lambda p(v)$ for all $\lambda > 0$.

Theorem 2.35 (Hahn-Banach (sublinear)). Let V be a real vector space, $p : V \to \mathbb{R}_{\geq 0}$ sublinear, $W \leq V$ subspace, $f : W \to \mathbb{R}$ with $f \leq p$ on W. Then there exists $\tilde{f} : V \to \mathbb{R}$ such that $\tilde{f}$ is linear, $\tilde{f}|_W = f$ and $\tilde{f} \leq p$ on V.

Theorem 2.36 (Geometric Hahn-Banach). Let V be a real vector space, A, B disjoint nonempty convex sets. Then

(i) If A is open, then there exists $f \in V^* \setminus 0$ and $\alpha \in \mathbb{R}$ such that

$$\sup_A f \leq \alpha \leq \inf_B f$$

(ii) If A is closed and B is compact, then there exists $f \in V^* \setminus 0$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\sup_A f < \alpha < \beta < \inf_B f$$

Proposition 2.37.

(i) Given a normed vector space V, $W \leq V$ a subspace, $f \in W^*$, then there exists $\tilde{f} \in V^*$ such that $\tilde{f}|_W = f$, $\left\|\tilde{f}\right\| = \|f\|$.

(ii) if $V \neq 0$, then $V^* \neq 0$,

(iii) if $V \neq 0$, $v \neq w$ then there exists $f \in V^*$ such that $f(v) \neq f(w)$.

Proof. (i) Apply Hahn-Banach with $p(v) = \|f\| \|v\|$. Then $|f| \leq p$ on W, so there exists $\tilde{f} \in V^*$ with $\left|\tilde{f}(v)\right| \leq p(v) = \|f\| \|v\|$, so $\left\|\tilde{f}\right\| \leq \|f\|$. But trivially we have $\|f\| \leq \left\|\tilde{f}\right\|$.

(ii) Fix $v_0 \in V$ nonzero, then define the support functional for v_0 by $f : \langle v_0 \rangle \to \mathbb{F}$, $f(v_0) = \|v_0\|$. By (i), we have an extension $\tilde{f} \in V^*$ such that $\tilde{f}(v_0) = \|v_0\|$. Note $\left\|\tilde{f}\right\| = \|f\| = 1$, since $|f(v_0)| = \|v_0\|$. In particular $\tilde{f}$ is nonzero.

(iii) Let $v_0 = v - w$, and let $\tilde{f}$ be as in (ii). Then $\tilde{f}(v) - \tilde{f}(w) = \|v_0\| \neq 0$. □

Proposition 2.38. Let V be a normed vector space. Then the bidual embedding $\Phi : V \to V^{**}$ is an isometry. In particular, $\|\Phi\| = 1$.

Proof. Given $v \in V$ nonzero, let f_v be a support functional for v. That is, $f_v(v) = \|v\|$ and $\|f_v\| = 1$. Then

$$\left\|\Phi(v)(f_v)\right\| = \left\|f_v(v)\right\| = \|v\|$$

therefore we have that $\left\|\Phi(v)\right\| \geq \|v\|$. But we have already shown the converse. □

Proposition 2.39. Given normed vector spaces V, W, $T : V \to W$ bounded linear map. Then $\left\|T^*\right\| = \left\|T\right\|$.

Proof. Suffices to show $\left\|T^*\right\| \geq \left\|T\right\|$. Let $v \in V$ with $\|v\| = 1$, $w = Tv \neq 0$. Let $g_w \in W^*$ be a support functional for w. Then

$$T^*(g_w)(v) = g_w(Tv) = g_w(w) = \|w\|$$

so $\left\|T^*(g_w)\right\| \geq \|w\|$. Thus, we have that

$$\left\|T^*\right\| = \sup_{\|g\|=1} \left\|T^*(g)\right\| \geq \left\|T^*(g_w)\right\| \geq \|w\| = \left\|Tv\right\| \geq \left\|T\right\|\|v\|$$

Taking the supremum over all v with $\|v\| = 1$, we have $\left\|T^*\right\| \geq \left\|T\right\|$. □

3 Baire category theorem

Definition 3.1 (rare, meagre)

Let X be a topological space, then

(i) $B \subseteq X$ is rare (or nowhere dense) if $\operatorname{Int}(\overline{B}) = \varnothing$. That is, for any $U \subseteq X$ open, $B \cap U$ is not dense in U.

(ii) $B \subseteq X$ is meagre if it can be written as a countable union of rare sets.

(iii) X is meagre if it is meagre as a subset of itself.

Remark 3.2. Alternative terminology is

1. first category := meagre,
2. second category := non-meagre.

Proposition 3.3. Given a topological space X, the following are equivalent.

(i) X is non-meagre,

(ii) for every countable collection $\{C_n\}_{n\in\mathbb{N}}$ of closed sets, with $\bigcup_n C_n = X$, at least one C_n has nonempty interior,

(iii) for every countable collection $\{U_n\}_{n\in\mathbb{N}}$ of open sets, U_n dense in X, then $\bigcap_n U_n \neq \varnothing$.

Proof. **Not (i) $\implies$ not (ii).** Suppose X is meagre. Then

$$X = \bigcup_n B_n = \bigcup_n \overline{B_n}$$

where B_n are rare, so $\operatorname{Int}(\overline{B_n}) = \varnothing$.

(i) $\implies$ (ii). If $X = \bigcup_n C_n$ with $\operatorname{Int}(C_n) = \varnothing$ for all n, then X is meagre as the C_n are rare.

(ii) $\implies$ (iii). Notice that $U_n^{\complement}$ is closed with empty interior, and $\bigcap_n U_n = \varnothing$ if and only if $\bigcup_n U_n^{\complement} = X$.

(iii) $\implies$ (ii). If all C_n have empty interiors, then taking $U_n = C_n^{\complement}$ we get a contradiction with (iii). So at least one has nonempty interior. □

Definition 3.4 (Baire space)
A topological space X is a Baire space if every countable intersection of dense open sets is dense.

Theorem 3.5 (Baire)**.** A complete metric space is Baire. In particular, it is non-meagre.

Proof. Let (U_n) be open dense sets, $V \subseteq X$ open. We want to show that $V \cap (\bigcap_n U_n) \neq \varnothing$. Since U_1 is dense, $U_1 \cap V$ is nonempty open. Choose $x_1 \in U_1 \cap V$, with $B_{r_1}(x_1) \subseteq U_1 \cap V$.

Now notice that $U_2 \cap B_{r_1/2}(x_1)$ is nonempty open, so we can choose x_2, r_2 such that $B_{r_2}(x_2) \subseteq U_2 \cap B_{r_1/2}(x_1)$. In general, we have that $(x_n), (r_n)$ such that $B_{r_{n+1}}(x_{n+1}) \subseteq U_{n+1} \cap B_{r_n/2}(x_n)$. The (x_n) are Cauchy, since for all $n \geq N$, $x_n \in B_{r_N/2}(x_N)$, and $r_{n+1} < r_n/2$, so $r_n \to 0$ and (x_n) is a Cauchy sequence. Hence by completeness, $x_k \to x$, with

$$x \in \overline{B}_{r_N/2}(x_N)$$

for all N, hence $x \in U_{n+1} \cap B_{r_k}(x)$ for all k. Thus, $x \in U \cap \left(\bigcap_n U_n\right)$. □

Theorem 3.6. A compact Hausdorff space is normal. That is, for all $C_1, C_2 \subseteq X$ disjoint closed sets, there exists disjoint open sets $U_1, U_2 \subseteq X$ such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$.

Proof. By Hausdorff, for each $x \in C_1$, $y \in C_2$, there exists disjoint open neighbourhoods $V_{x,y}, V'_{x,y}$ of x, y respectively. Now fix $y \in C_2$, then

$$C_1 \subseteq \bigcup_{x \in C_1} V_{x,y}$$

C_1 is a closed subspace of a compact space, so it is compact. Hence we have a finite subcover $x_1, \ldots, x_m \in C_1$ such that

$$C_1 \subseteq \bigcup_{i=1}^{m} V_{x_i,y}$$

Define

$$W_y = \bigcup_{i=1}^{m} U_{x_i,y} \quad \text{and} \quad W'_y = \bigcap_{i=1}^{m} U'_{x_i,y}$$

Then W_y, W'_y are disjoint open sets. Repeating the same argument with C_2 compact Hausdorff space, $\mathcal{A} \subseteq C_{\mathbb{R}}(K)$ be a subalgebra which separates points. Then either, we have $y_1, \ldots, y_n \in C_2$ such that

$$C_2 \subseteq \bigcup_{j=1}^{n} W'_{y_j}$$

Now define

$$U_1 = \bigcap_{j=1}^{n} W_{y_j} \quad \text{and} \quad U_2 = \bigcup_{j=1}^{n} W'_{y_j}$$

Then U_1, U_2 are open, disjoint with $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$. □

Theorem 3.7. A compact Hausdorff space X is a Baire space.

Proof. Let (U_n) be a collection of open dense subsets, $V \subseteq X$ nonempty open. We want to show that $V \cap \left(\bigcap_n U_n\right) \neq \varnothing$. Since U_1 is dense, there exists $x_1 \in U_1 \cap V$. As $\{x\}$ is disjoint from $(U_1 \cap V)^{\complement}$, by normality there exists W_1, W_1' disjoint open such that $x_1 \in W_1$ and $(U_1 \cap V)^{\complement} \subseteq W_1'$. Then we have that

$$\overline{W_1} \subseteq (W_1')^{\complement} \subseteq U_1 \cap V$$

Repeat this to get $x_n \in W_n \subseteq \overline{W_n} \subseteq U_n \cap W_{n-1}$. As $\bigcap_n \overline{W_n}$ is nonempty, as X is compact, choose $z \in \bigcap_k \overline{W_k}$. Then

$$z \in \bigcap_k \overline{W_k} \subseteq V \cap \left(\bigcap_n U_n\right)$$

□

3.1 Uniform boundedness principle

Theorem 3.8 (Uniform boundedness principle)**.** Let V, W be Banach spaces, $(T_i)_{i \in I}$ be a collection of bounded linear maps $V \to W$, which is locally bounded. That is, for any $v \in V$,

$$\sup_{i \in I} \left\| T_i(v) \right\| < \infty$$

Then

$$\sup_{i \in I} \left\| T_i \right\| < \infty$$

Proof. Let $C_n = \left\{ v \in V \mid \sup_{i \in I} \left\| T_i(v) \right\| \leq n \right\}$. Then C_n is a closed subspace, as we have that

$$C_n = \bigcap_{i \in I} T_i^{-1}([-n, n])$$

By local boundedness, $V = \bigcup_n C_n$. As V is a Baire space, there exists n such that $\mathrm{Int}(C_n) \neq \varnothing$. That is, there exists $v_0, \varepsilon > 0$ such that $B_\varepsilon(v_0) \subseteq C_n$. That is, for all $i \in I$, $v \in B_\varepsilon(v_0)$, $\left\| Tv \right\| \leq n$.

Then for all $v \in V$, with $||v|| < \varepsilon$,

$$\left\| T_i(v) \right\| \leq \left\| T_i(v + v_0) \right\| + \left\| T_i(v_0) \right\| \leq n + \sup_i \left\| T_i(v_0) \right\|$$

Therefore, we must have that

$$\sup_i \left\| T_i \right\| \leq \frac{1}{\varepsilon} \left(n + \sup_i \left\| T_i(v_0) \right\| \right) < \infty$$

□

Corollary 3.9. Let (T_n) be bounded linear maps $V \to W$, $T_n \to T$ pointwise, T linear. Then T is bounded with

$$\left\| T \right\| \leq \liminf_n \left\| T_n \right\|$$

Proof. By the uniform boundedness principle, $\sup_n \left\| T_n \right\| = c < \infty$. Then

$$\left\| Tv \right\| = \lim_n \left\| T_n v \right\| \leq \lim_n \left\| T_n \right\| ||v|| \leq c||v||$$

. So $\left\| Tv \right\| \leq c||v||$, which means that $\left\| T \right\| \leq c$. Given $\varepsilon > 0$, there exists $v \in V$ such that $||v|| = 1$, and $\left\| T \right\| \leq \left\| Tv \right\| + \varepsilon$ by definition of $\left\| T \right\|$ as a sup. Then as $T_n v \to Tv$, there exists N such that $\left\| T_n v - Tv \right\| < \varepsilon$ for $n \geq N$. So

$$\left\| T \right\| \leq \left\| Tv \right\| + \varepsilon \leq \left\| T_n \right\| v + 2\varepsilon \leq \left\| T_n \right\| + 2\varepsilon$$

for all $n \geq N$. Hence we must have that

$$\|T\| \leq \liminf_n \|T_n\| + 2\varepsilon$$

for all $\varepsilon > 0$, so $\|T\| \leq \liminf_n \|T_n\|$. □

Corollary 3.10. Let V be a Banach space. Then $B \subseteq V$ is bounded if and only if for all $f \in V^*$, $f(V) \subseteq \mathbb{R}$ is bounded.

Proof. Suppose B is bounded. Then for any $f \in V^*$, $f(B)$ is bounded since f is bounded. Now suppose $B \subseteq V$ has $f(B)$ bounded for every $f \in V^*.f(B)$ bounded implies that

$$\sup_{v \in B} |\Phi(v)(f)| = \sup_{v \in B} |f(v)| < \infty$$

for all $f \in V^*$, so by the uniform boundedness principle with $\{\Phi(v)\}_{v \in B}$, we have that $\sup_{v \in B} \|\phi(v)\| = \sup_{v \in B} ||v|| < \infty$, so B is bounded. □

Corollary 3.11. $B \subseteq V^*$ is bounded if and only if for all $v \in V$, $\Phi(v)(B) \subseteq \mathbb{R}$ is bounded.

Proof. Suppose B is bounded, then $\Phi(v)(B)$ is bounded as $\Phi(v)$ is bounded. Conversely, applying the uniform boundedness principle to B we get that $\sup_{f \in B} \|f\| < \infty$. □

3.2 Open mapping, inverse mapping and closed graph

Theorem 3.12 (Open mapping theorem)**.** Let V, W be Banach spaces, $T \in \mathcal{B}(V, W)$ be surjective. Then T is open.

Proof. Suffices to show there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq T(B_1(0))$. Since T is surjective,

$$W = \bigcup_{n \in \mathbb{N}} T(B_n(0)) = \bigcup_n \overline{T(B_n(0))}$$

Hence by Baire, there exists n such that $\text{Int}(\overline{T(B_n(0))}) \neq \varnothing$. Since dilation is a homeomorphism, without loss of generality $n = 1$. So there exists w_0 and $\varepsilon > 0$ such that

$$w_0 + B_{4\varepsilon}(0) \subseteq \overline{T(B_1(0))}$$

But then as $\overline{T(B_1(0))}$ is balanced and convex, we have that

$$B_{4\varepsilon}(0) \subseteq \frac{1}{2}(w_0 + B_{4\varepsilon}(0)) + \frac{1}{2}(-w_0 + B_{4\varepsilon}(0)) \subseteq \overline{T(B_1(0))}$$

So without loss of generality $w_0 = 0$. We will now show that $B_\varepsilon(0) \subseteq T(B_1(0))$. Let

$$w_1 \in B_\varepsilon(0) = \frac{1}{4} B_{4\varepsilon}(0) \subseteq \frac{1}{4}\overline{T(B_1(0))} = \overline{T(B_{1/4}(0))}$$

So there exists $v_1 \in B_{1/4}(0)$ such that $\|Tv_1 - w_1\| < \varepsilon/2$. Now define

$$w_2 = w_1 - Tv_1 \in B_{\varepsilon/2}(0) \subseteq \overline{T(B_{1/8}(0))}$$

and so we have $v_2 \in B_{1/8}(0)$ such that $\|w_2 - Tv_2\| < \varepsilon/4$. Repeating this, we have sequences $(w_k), (v_k)$ with

$$w_k = w_{k-1} - Tv_{k-1} \in B_{\varepsilon/2^k}(0) \subseteq \overline{T(B_{1/2^{k+1}}(0))} \quad \text{and} \quad v_k \in B_{1/2^{k+1}}(0) \text{ with } \|w_k - Tv_k\| < \frac{\varepsilon}{2^k}$$

In particular, $w_k \to 0$, and $v = \sum_k v_k$ converges, since V is complete and the series converges in norm. In particular, $||v|| \leq \frac{1}{2} < 1$, and

$$w_k = w_1 - T\left(\sum_{i=1}^{k-1} v_i\right) \to 0$$

So $w_1 \in T(B_1(0))$. □

Theorem 3.13 (inverse mapping)**.** Suppose $T \in \mathcal{B}(V, W)$ is bijective. Then $T^{-1} : W \to V$ is also bounded.

Proof. By the open mapping theorem T is open, so T^{-1} is continuous, and hence bounded. □

Theorem 3.14 (closed graph)**.** Suppose $T : V \to W$ is linear, then T is bounded if and only if the graph of T

$$\Gamma_T = \{(v, Tv) \mid v \in V\} \subseteq V \times W$$

is closed.

Proof. First suppose T is bounded, and $(v_n, Tv_n) \to (v, w)$. Then $v_n \to v$ and $Tv_n \to w$. But T is continuous, so $Tv_n \to Tv$. Hence $w = Tv$, so $(v, w) = (v, Tv) \in \Gamma_T$.

Conversely, suppose Γ_T is a closed subspace of the Banach space $V \times W$[2]. Then it is also a Banach space. Then the projection $\pi_V : V \times W \to V$ is continuous, and restricts to a bijective bounded linear map $\pi_V : \Gamma_T \to V$. So π_V^{-1} is bounded by the inverse mapping theorem, so there exists $C > 0$ such that

$$\|v\| + \left\|Tv\right\| \leq C\|v\|$$

for all $v \in V$. □

Corollary 3.15. Suppose for all sequences (v_n) such that $v_n \to v$ and $Tv_n \to w$, we have that $Tv = w$, then T is bounded.

4 Topology of $C(K)$

4.1 Tietze extension theorem

Definition 4.1 (normal)

A topological space X is normal if for all C_1, C_2 disjoint closed subsets of X, there exists disjoint open subsets U_1, U_2 such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$.

Lemma 4.2 (Urysohn)**.** Let X be a topological space. Then X is normal if and only if for all closed subsets C_1, C_2 of X, there exists $f : X \to [0, 1]$ continuous with $f|_{C_1} = 0$ and $f|_{C_2} = 1$.

Proof. Suppose such an f exists. Then $U_1 = f^{-1}([0, 1/2))$ and $U_2 = f^{-1}((1/2, 1])$ are disjoint open sets such that $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$.

For the converse, first we note that by normality, there exists U_0, U_1 open such that $C_1 \subseteq U_0$ and $C_2 \subseteq \overline{U_1}$. Without loss of generality, we may assume C_2 is nonempty. Then suffices to define f such that $f = 0$ on $\overline{U_0}$ and $f = 1$ on C_2.

Step 1: Given $U_0 \subseteq U_1 \subsetneq X$ nonempty open sets, with $\overline{U_0} \subseteq U_1$, there exists $U_{1/2}$ open such that

$$U_0 \subseteq \overline{U_0} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq U_1$$

[2]With the ℓ^1 norm on the product structure.

To see this, let $C_1 = \overline{U_0}$ and $C_2 = U_1^\complement$. Then C_2 is nonempty. As X is normal, thete exists $U_{1/2}, V_{1/2}$ disjoint open such that $C_1 \subseteq U_{1/2}$, $C_2 \subseteq V_{1/2}$. Then $\overline{U_0} = C_1 \subseteq U_{1/2}$, $(V_{1/2}^\complement \subseteq C_2^\complement = U_1$. Since $V_{1/2}^\complement$ is closed, $U_{1/2} \subseteq \overline{U_{1/2}} \subseteq V_{1/2}^\complement \subseteq U_1$.

Step 2: Dyadic induction. By induction on k, we can define U_r for all $r = m/2^k \in [0, 1]$, such that if $r < s$, then we have that

$$U_r \subseteq \overline{U_r} \subseteq U_s$$

Step 3: Defining f. Let $\mathcal{D}$ denote the set of dyadic rationals in $[0, 1]$. Then define

$$f(x) = \begin{cases} \inf \{r \in \mathcal{D} \mid x \in U_r\} & \text{if } x \in U_1 \\ 1 & \text{if } x \in C_2 = U_1^\complement \end{cases}$$

Notice that $f|_{\overline{U_0}} = 0$ and $f|_{C_2} = 1$.

Step 4: f is continuous. This is equivalent to showing that for all $a \in [0, 1)$, $f^{-1}((a, 1])$ is open. Fix such an a, then $x \in f^{-1}((a, 1])$ if and only if $f(x) > a$, which is true if and only if there exists $r, s \in \mathcal{D}$ such that $f(x) > r > s > a$ by density of dyadic rationals. So $x \in U_r^\complement$. As $\overline{U_s} \subseteq U_r$, $x \in (\overline{U_s})^\complement$ open. But by definition of f as an infimum, we have that

$$x \in (\overline{U_s})^\complement \subseteq f^{-1}((a, 1])$$

□

Corollary 4.3. If K is a normal and T_1 topological space, then $C(K)$ separates points.

Proof. As K is T_1, $\{x\}$ and $\{y\}$ are closed, so we can apply Urysohn's lemma. □

Theorem 4.4 (Tietze extension)**.** Let X be a normal topological space, $C \subseteq X$ nonempty closed, $f : C \to \mathbb{R}$ continuous bounded. Then there exists $\tilde{f} : X \to \mathbb{R}$ continuous such that $\tilde{f}|_C = f$, $\sup_X \left|\tilde{f}\right| = \sup_C |f|$.

Proof. If f is constant the result is trivial. Otherwise, by replacing f with

$$\frac{f - \inf f}{\sup f - \inf f}$$

We can assume $f : C \to [0, 1]$ with $\inf f = 0$ and $\sup f = 1$. Define

$$C_1 = f^{-1}([0, 1/3]) \quad \text{and} \quad C_2 = f^{-1}([2/3, 1])$$

Then by Urysohn's lemma, there exists $g_1 : X \to [0, 1/3]$ continuous such that $g_1|_{C_1} = 0$ and $g_1|_{C_2} = 1/3$. Then if we set $f_1 = f$, $f_2 = f_1 - g_1|_C : C \to [0, 2/3]$. Repeating this, we get $f_k : C \to [0, (2/3)^{k-1}]$ and $g_k : X \to [0, 1/3 \cdot (2/3)^{k-1}]$ so that $f_{k+1} = f_k - g_k|_C : C \to [0, (2/3)^k]$. Furthermore, we can choose the g_k and f_k so that the sup and inf are attained. Therefore, as

$$\sum_k \|g_k\|_\infty < \infty$$

$C_{\mathbb{R},b}(X)$ is a Banach space, so the limit $\tilde{f} = \sum_k g_k \in C_{\mathbb{R},b}(X)$ exists. Furthermore,

$$\sup_C \left|\sum_{k=1}^n g_k - f\right| = \sup_C |f_{n+1}| \leq \left(\frac{2}{3}\right)^n \to 0$$

so $\tilde{f}|_C = f$. □

4.2 Arzelà-Ascoli

Definition 4.5 (totally bounded)
Let (X, d) be a metric space, $Y \subseteq X$ is totally bounded if for all $\varepsilon > 0$, there exists a finite ε-net $N = \{x_1, \dots, x_n\} \subseteq X$ such that

$$Y \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

Definition 4.6 (relatively compact)
Let X be a topological space, $Y \subseteq X$ is relatively compact if $\overline{Y}$ is compact.

Proposition 4.7. Let X be a complete metric space, $Y \subseteq X$. Then Y is relatively compact if and only if Y is totally bounded.

Proof. From IB Analysis and Topology, we have that a metric space Z is compact if and only if it is complete and totally bounded. As $\overline{Y}$ is a closed subspace of a complete space, it is complete. So $\overline{Y}$ is compact if and only if it is totally bounded. □

Definition 4.8 (equi{bounded, continuous} {on K, at $x \in K$})
Let K be a compact Hausdorff space, $\mathcal{F} \subseteq C(K)$. Then

(i) $\mathcal{F}$ is equibounded at $x \in K$ if $\sup_{f \in \mathcal{F}} |f(x)| < \infty$,

(ii) $\mathcal{F}$ is equibounded on K if it is equibounded at all $x \in K$,

(iii) $\mathcal{F}$ is equicontinuous at $x \in K$ if for all $\varepsilon > 0$, there exists an open neighbourhood U of x such that

$$\sup_{y \in U} \sup_{f \in \mathcal{F}} |f(x) - f(y)| < \varepsilon$$

(iv) $\mathcal{F}$ is equicontinuous on K if it is equicontinuous at all $x \in K$.

Theorem 4.9 (Arzelà-Ascoli). Let K be a compact Hausdorff space, $\mathcal{F} \subseteq C(K)$ is relatively compact if and only if it is equibounded and equicontinuous on K.

Proof. Note that $C(K)$ is complete. Suppose $\mathcal{F}$ is relatively compact, then $\mathcal{F}$ is totally bounded, so it is bounded with respect to the supremum norm, so it is equibounded. For equicontinuity, given $x \in K$, $\varepsilon > 0$, consider an ε-net for $\mathcal{F}$. So there exists $f_1, \dots, f_m \in C(K)$ such that

$$\mathcal{F} \subseteq \bigcup_{i=1}^{m} B_\varepsilon(f_i)$$

Since each f_i is continuous at x, there exists U_i open neighbourhood of x such that $f_i(U_i) \subseteq B_\varepsilon(f_i(x))$. Then $U = U_1 \cap \cdots \cap U_m$ is an open neighbourhood of x, and for any $y \in U$, $f \in \mathcal{F}$, let f_j be such that $\left\|f_j - f\right\|_\infty < \varepsilon$, then we have that

$$|f(y) - f(x)| \leq \left|f(y) - f_j(y)\right| + \left|f_j(y) - f_j(x)\right| + \left|f_j(x) - f(x)\right| \leq 3\varepsilon$$

Conversely, fix $\varepsilon > 0$. Then for each $x \in K$, we have U_x open, $x \in U_x$ such that $f(U_x) \subseteq B_\varepsilon(f(x))$ for all $f \in \mathcal{F}$ by equicontinuity. As K is compact, we can choose a finite subcover $x_1, \dots, x_n \in K$ such that

$$K = \bigcup_{i=1}^{n} U_{x_i}$$

Define $A = \{(f(x_1), \dots, f(x_n)) \mid f \in \mathcal{F}\} \subseteq \mathbb{F}^n$. As $\mathcal{F}$ is equibounded, A is bounded, so $\overline{A}$ is closed and bounded, hence compact by Heine-Borel. This means that $\overline{A}$ is totally bounded, so A is totally bounded. Hence we have an ε-net for A. That is, $N = \{f_1, \dots, f_m\} \subseteq \mathcal{F}$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_\varepsilon((f_j(x_1), \dots, f_j(x_n)))$$

In fact, N is a 3ε-net for $\mathcal{F}$. Given $f \in \mathcal{F}, x \in K$, then there exists x_i such that $x \in U_{x_i}$ and f_j such that $f(x_1), \dots, f(x_n) \in B_\varepsilon((f_j(x)1), \dots, f_j(x_n))$. Then

$$\left|f(x) - f_j(x)\right| \leq \left|f(x) - f(x_i)\right| + \left|f(x_i) - f_j(x_i)\right| + \left|f_j(x_i) - f_j(x)\right| < 3\varepsilon$$

□

4.3 Stone-Weierstrass

Definition 4.10 (algebra)

A vector space V over $\mathbb{F}$ is an alebra if V has a multiplication such that

(i) $(\lambda v)(w) = \lambda(vw) = v(\lambda w)$

(ii) $u(v + w) = uv + uw$, $(v + w)u = vu + wu$.

Definition 4.11 (normed algebra, Banach algebra)

An algebra V is a normed algebra if V is also a normed vector space with $\|vw\| \leq \|v\|\|w\|$ for all $v, w \in V$. If V is a Banach space, we call V a Banach algebra.

Definition 4.12 (commutative algebra)

V is a commutative algebra if V is an algebra and $vu = uv$ for all $u, v \in V$.

Definition 4.13 (unital algebra)

V is a unital algebra if V is an algebra and there exists $1 \in V$ such that $1v = v1 = v$ for all $v \in V$.

Theorem 4.14 (Stone-Weierstrass). Let K be a compact Hausdorff space, $\mathcal{A} \subseteq C_{\mathbb{R}}(K)$ be a subalgebra which separates points. Then either

(i) $\overline{\mathcal{A}} = C_{\mathbb{R}}(K)$,

(ii) or there exist $x_0 \in K$ such that

$$\overline{\mathcal{A}} = \{f \in C_{\mathbb{R}}(K) \mid f(x_0) = 0\}$$

Proof. **Step 1: $\overline{\mathcal{A}}$ is a subalgebra.** This is immediate since if $f_k \to f$ and $g_k \to g$, then $f_k g_k \to fg$. Therefore, from now on, we can assume without loss of generality that $\mathcal{A}$ is closed.

Step 2: $\mathcal{A}$ is closed under finite min **and** max. Suffices to show if $f \in \mathcal{A}$ then $|f| \in \mathcal{A}$ since we can write min and max in terms of absolute values. By scaling, wlog $\|f\| \leq 1$. Then for $\varepsilon > 0$, define $\phi_\varepsilon(r) = \sqrt{\varepsilon^2 + r}$ for $r \in [0, 1]$. Then we have that

$$\left|\phi_\varepsilon(r) - \sqrt{r}\right| = \left|\frac{\varepsilon^2 + r - r}{\sqrt{\varepsilon^2 + r} + \sqrt{r}}\right| \leq \varepsilon$$

and ϕ_ε is real analytic on $[0, 1]$, so if we expand in Taylor series about $r = 1/2$, we have that

$$\phi_\varepsilon(r) = \sum_{k=0}^{N} a_{k,\varepsilon} \left(r - \frac{1}{2}\right)^k + R_{N,\varepsilon}(r)$$

such that $\sup |R_{N,\varepsilon}(t)| \to 0$ as $N \to \infty$. Define

$$G_{N,\varepsilon}(r) = \sum_{k=0}^{N} a_{k,\varepsilon} \left(r - \frac{1}{2}\right)^k$$

Then $G_{N,\varepsilon}(0) \to \phi_\varepsilon(0) = \varepsilon$ as $N \to \infty$. Hence given $f \in \mathcal{A}$, in the limit $N \to \infty$,

$$\begin{aligned}
|f| &= \left(|f| - \phi_\varepsilon(f^2)\right) + \phi_\varepsilon(f^2) \\
&= \left(|f| - \phi_\varepsilon(f^2)\right) + G_{N,\varepsilon}(f^2) + R_{N,\varepsilon}(f^2) \\
&= \underbrace{\left(|f| - \phi_\varepsilon(f^2)\right)}_{|\cdot| \leq \varepsilon} + \underbrace{G_{N,\varepsilon}(0)}_{|\cdot| \leq 2\varepsilon} + \underbrace{R_{N,\varepsilon}(f^2)}_{|\cdot| \leq \varepsilon} + \left(G_{N,\varepsilon}(f^2) - G_{N,\varepsilon}(0)\right) \\
&= \underbrace{\left(G_{N,\varepsilon}(f^2) - G_{N,\varepsilon}(0)\right)}_{\in \mathcal{A}} + \mathcal{O}(4\varepsilon) \\
&\in \overline{\mathcal{A}} = \mathcal{A}
\end{aligned}$$

Step 3: Suppose g satisfies that for any $x, y \in K$, $\varepsilon > 0$, there exists $f \in \mathcal{A}$ such that $|f(x) - g(x)| < \varepsilon$ and $|f(y) - g(y)| < \varepsilon$, then $g \in \mathcal{A}$. For $x, y \in K$, choose $f_{x,y} \in \mathcal{A}$ such that $\left|f_{x,y}(x) - g(x)\right| < \varepsilon$ and $\left|f_{x,y}(y) - g(y)\right| < \varepsilon$. Then by continuity, there exists open neighbourhoods $U_{x,y}, V_{x,y}$ of x, y respectively such that $\left|f_{x,y} - g\right| \leq 2\varepsilon$ on $U_{x,y}$ and $V_{x,y}$. Since K is compact, $K \subseteq \bigcup_y V_{x,y}$, so we have $y_1, \dots, y_n$ such that

$$K = \bigcup_{i=1}^{n} V_{x,y_i}$$

Now define

$$\tilde{U}_x = \bigcap_{i=1}^{n} U_{x,y_i} \quad \text{and} \quad f_x = \min\{f_{x,y_1}, \dots, f_{x,y_n}\} \in \mathcal{A}$$

U_x is an open neighbourhood of x, and f_x satisfies $f_x(z) < g(z) + \varepsilon$ for $z \in K$ and $f_x(z) > g(z) - \varepsilon$ for $z \in \tilde{U}_x$. Again by compactness we have $x_1, \dots, x_m$ such that

$$K = \bigcup_{i=1}^{m} \tilde{U}_{x_i}$$

Then define $f = \max\{f_{x_1}, \dots, f_{x_n}\} \in \mathcal{L}$. Then

$$g(z) - \varepsilon < f(z) < g(z) + \varepsilon$$

for all $z \in K$. Taking $\varepsilon \to 0$, $g \in \overline{\mathcal{A}} = \mathcal{A}$.

Step 4 case 1: Suppose for all $x \in K$, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. For $x, y \in K$ distinct, we have $f_x, f_y, f_{x,y} \in \mathcal{A}$ such that $f_x(x) \neq 0, f_y(y) \neq 0, f_{x,y}(x) \neq f_{x,y}(y)$. Then there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\tilde{f} = f_x + \alpha f_y + \beta f_{x,y}$$

satisfies $\tilde{f}(x) \neq 0$, $\tilde{f}(y) \neq 0$ and $\tilde{f}(x) \neq \tilde{f}(y)$. Then $\tilde{f}, \tilde{f}^2 \in \mathcal{A}$, and

$$\text{span}\left\{(\tilde{f}(x), \tilde{f}(y)), (\tilde{f}(x)^2, \tilde{f}(y)^2)\right\} = \mathbb{R}^2$$

Now given any $g \in C_{\mathbb{R}}(K)$, there exists a linear combination of $\tilde{f}$ and $\tilde{f}^2$ which agrees with g at x, y, so by step 3, $g \in \mathcal{A}$. Hence $\mathcal{A} = C_{\mathbb{R}}(K)$.

Step 4 case 2: There exists $x_0 \in K$ such that for all $f \in \mathcal{A}$, $f(x_0) = 0$. In this case, let 1 denote the constant 1 function, then $\overline{\mathcal{A} \oplus \mathbb{R} \cdot 1}$ is a closed subalgebra which satisfies case 1. So $\overline{\mathcal{A} \oplus \mathbb{R} \cdot 1} = C_{\mathbb{R}}(K)$. Fix $g \in C_{\mathbb{R}}(K)$ with $g(x_0) = 0$. Then for any $\varepsilon > 0$, there exists $f \in \mathcal{A}$, $\lambda \in \mathbb{R}$ such that

$$\left\| g - (f + \lambda) \right\|_\infty < \varepsilon$$

Then $g(x_0) = 0$, $(f + \lambda)(x_0) = \lambda$, and so $|\lambda| < \varepsilon$, and $|g - f| < 2\varepsilon$ and $g \in \overline{\mathcal{A}} = \mathcal{A}$. □

Theorem 4.15 (Stone-Weierstrass for complex algebras)**.** Let K be compact Hausdorff, $A \subseteq C_{\mathbb{R}}(K)$ a subalgebra which separates points and is closed under complex conjugation. Then either

(i) $\overline{\mathcal{A}} = C_{\mathbb{R}}(K)$

(ii) or there exists $x_0 \in K$ such that

$$\overline{A} = \{f \in C_{\mathbb{C}}(K) \mid f(x_0) = 0\}$$

Proof. Note that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are in $\mathcal{A}$ if and only if $f \in \mathcal{A}$, so if we define $\mathcal{A}_{\mathbb{R}} = \{\mathrm{Re}(f), \mathrm{Im}(f) \mid f \in \mathcal{A}\}$, and apply the real version of the theorem, we get the required result. □

Theorem 4.16 (Weierstrass approximation)**.** The set of real polynomials is dense in $C_{\mathbb{R}}[0, 1]$ and the set of complex polynomials is dense in $C_{\mathbb{C}}[0, 1]$.

5 Inner product spaces

Note in this course we take the inner products to be linear in the first argument, and conjugate linear in the second argument.

Proposition 5.1 (Cauchy-Schwarz)**.** Let V be an inner product space on $\mathbb{F}$, then for any $v_1, v_2 \in V$,

$$|\langle v_1, v_2 \rangle| \leq \sqrt{\langle v_1, v_1 \rangle \langle v_2, v_2 \rangle}$$

Definition 5.2 (Euclidean space)

An inner product space V with norm defined by

$$||v|| = \sqrt{\langle v, v \rangle}$$

is called a Euclidean space.

Proposition 5.3 (polarisation)**.** If V is a Euclidean space, then we have the polarisation identities:

- $\mathbb{F} = \mathbb{R}$,
$$\langle v, w \rangle = \frac{1}{4} \left(||v + w||^2 - ||v - w||^2 \right)$$
- $\mathbb{F} = \mathbb{C}$,
$$\langle v, w \rangle = \frac{1}{4} \left(||v + w||^2 - ||v - w||^2 + i\left\|v + iw\right\|^2 - i\left\|v - iw\right\|^2 \right)$$

Proof. Expand. □

Theorem 5.4 (Jordan-von Neumann)**.** Let V be a normed vector space. Then V is Euclidean if and only if it satisfies the parallelogram identity

$$||v+w||^2 + ||v-w||^2 = 2||v||^2 + 2||w||^2$$

Proof. Suppose V is Euclidean. Then expanding the norm in terms of the inner product gives the parallelogram law. For the converse, we can define the inner product using the polarisation identities, and notice that we can reduce the complex case to the real case as $\langle iv, w\rangle = i\langle v, w\rangle$ by polarisation identities.

To show the result is an inner product, use the parallelogram law to show that $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$, and use this to show that $\langle nv, w\rangle = n\langle v, w\rangle$ for all $n \in \mathbb{Z}$. This then gives us the case for rational scalars, then by continuity we get the result for real scalars. □

Definition 5.5 (orthogonal)

Let V be an inner product space, v, w are orthogonal, written $v \perp w$ if $\langle v, w\rangle = 0$. If $S \subseteq V$, its orthogonal space is

$$S^\perp = \{v \in V \mid \forall w \in S, v \perp w\}$$

Proposition 5.6. Given a Euclidean space V, $S \subseteq V$, then $S^\perp$ is a subspace, and

$$S^\perp = \left(\overline{\mathrm{span}(S)}\right)^\perp$$

Proposition 5.7. Subspace is clear by linearity of the inner product, and by the order reversing property, clearly we have that $\left(\overline{\mathrm{span}(S)}\right)^\perp \subseteq S^\perp$. Now let $v \in S^\perp$ be arbitrary. For any $w \in \overline{\mathrm{span}(S)}$, there exists a sequence (w_n) in $\mathrm{span}(S)$ with $w_n \to w$. Then $\langle v, w_n\rangle = 0$ for all n, so $\langle v, w\rangle = 0$ by continuity.

Theorem 5.8 (Bessel)**.** Let V be a Euclidean space, (e_n) be an orthonormal sequence, then for $v \in V$,

$$||v||^2 = \left\| v - \sum_{n=1}^{N} \langle v, e_k\rangle e_k \right\|^2 + \sum_{n=1}^{N} |\langle v, e_k\rangle|^2$$

and we have Bessel's inequality,

$$||v||^2 \geq \sum_{n=1}^{\infty} |\langle v, e_k\rangle|^2$$

with equality if and only if $\sum_{n=1}^{N} \langle v, e_n\rangle e_n \to v$ as $N \to \infty$.

Proof. The first formula follows by Pythagoras, and Bessel's inequality follows by taking $N \to \infty$, which also gives us the equality condition. □

5.1 Hilbert spaces

Theorem 5.9 (completion of normed vector spaces)**.** Let V be a normed vector space. Then there exists a Banach space $\overline{V}$, with $\Phi : V \to \overline{V}$ a linear isometry, $\overline{V} = \overline{\Phi(V)}$. $\overline{V}$ is unique up to isometric isomorphism.

Proof. Let $V^*\ast$ be the bidual of V, $\Phi : V \to V^{**}$ the bidual embedding, then Φ is an isometry, and $\overline{V} = \overline{\Phi(V)}$ is a closed subspace of a Banach space, so it is a Banach space.

For uniqueness, let $\overline{V}_1, \overline{V}_2$ be Banach spaces, $\Phi_i : V \to \overline{V}_i$ isometries with $\overline{V}_i = \overline{\Phi_i(V)}$. Define $\Psi = \Phi_2 \circ \Phi_1^{-1} : \Phi_1(V) \to \Phi_2(V)$. Then Ψ is a linear isometry, so it is bounded. Therefore there exists a unique continuous linear extension $\tilde{\Psi} : \overline{V}_1 \to \overline{V}_2$.

Now let (y_n) be a sequence in $\tilde{\Psi}(\overline{V}_1)$, $y_n \to y$ in $\overline{V}_2$. Then let x_n be the sequence such that $y_n = \Phi(x_n)$. Then (y_n) is Cauchy, so (x_n) is also Cauchy. By completeness, $x_n \to x$, so $\tilde{\Psi}(x) = y$. Hence $\tilde{\Psi}(\overline{V}_1) = \overline{V}_2$. □

Definition 5.10 (Hilbert space)

A complete Euclidean space is called a Hilbert space.

Theorem 5.11 (Hilbert basis). Let H be an infinite dimensional separable Hilbert space, then there exists (e_n) orthonormal, such that

$$\overline{\operatorname{span}\{e_n \mid n \in \mathbb{N}\}} = H$$

We call (e_n) a Hilbert basis.

Proof. Let (y_n) be a countable dense subset of H. By passing to a subsequence, we may assume the (y_n) are linearly independent, not necessarily dense but with

$$\overline{\operatorname{span}\{y_n \mid n \in \mathbb{N}\}} = H$$

Applying Gram-Schmidt we get the required result. □

Corollary 5.12 (Parseval). If $v_n = \langle v, e_n \rangle$ and $w_n = \langle w, e_n \rangle$, then

$$\langle v, w \rangle = \sum_n v_n \overline{w_n}$$

Proof. By Bessel's inequality. □

Corollary 5.13. The map $\Psi : H \to \ell^2$ given by $\Psi(v)_n = v_n = \langle v, e_n \rangle$ is an isometric isomorphism.

5.2 Projections

Proposition 5.14. Let V be a Euclidean space, $C \subseteq V$ be convex, nonempty and complete. Then

(i) For all $v \in V$, there exists a unique $P_C(v) \in C$ such that

$$d(v, C) = \inf_{z \in C} \|v - z\| = \left\|v - P_C(v)\right\|$$

(ii) for all $z \in C$,

$$\operatorname{Re} \langle z - P_C(v), v - P_C(v) \rangle \leq 0$$

That is, the angle between them is at least $\pi/2$.

(iii) $P_C : V \to C$ is 1-Lipschitz.

Proof. (i) For existence, if $v \in C$, then $P_C(v) = v$. Otherwise, let (w_n) be a sequence such that $\left\|v - w_n\right\|^2 \leq d(v, C)^2 + \frac{1}{n}$. Then by the parallelogram law,

$$\left\|w_n - w_m\right\|^2 + \left\|w_m + w_n - 2v\right\|^2 = 2\left\|w_m - v\right\|^2 + 2\left\|w_n - v\right\|^2$$

So we get that

$$\begin{aligned}
\frac{\left\|w_n - w_m\right\|^2}{2} &= \left\|w_m - v\right\|^2 + \left\|w_n - v\right\|^2 - 2\left\|v - \frac{w_n + w_m}{2}\right\|^2 \\
&\leq 2d(v, C)^2 + \frac{1}{m} + \frac{1}{n} - 2d(v, C)^2 \\
&= \frac{1}{m} + \frac{1}{n}
\end{aligned}$$

So (w_n) is Cauchy. Hence by completeness, $w_n \to w$. Define $P_C(v) = w$. For uniqueness, we have that

$$\frac{\left\|w_1 - w_2\right\|^2}{2} = \left\|w_1 - v\right\|^2 + \left\|w_2 - v\right\|^2 \ - 2\left\|v - \frac{w_1 + w_2}{2}\right\|^2 \leq d(v, C)^2 + d(v, C)^2 - 2d(v, C)^2 = 0$$

So $w_1 = w_2$.

(ii) Define $\phi(\lambda) = \left\|\lambda z + (1 - \lambda)P_C(v) - v\right\|^2 - \left\|P_C(v) - v\right\|^2$. Then $\phi(0) = 0$ and $\phi(\lambda \geq 0)$, so $\phi'(0) \geq 0$. Hence we have that

$$\phi'(0) = 2\operatorname{Re}\langle z - P_c(v), P_c(v) - v\rangle \geq 0$$

Note in fact the converse is also true in this case.

(iii) Follows by (ii) and Cauchy-Schwarz. □

Theorem 5.15. Let V be Euclidean, $W \leq V$ complete. Then $V = W \oplus W^\perp$, given by

$$v = P_W(v) + (v - P_W(v))$$

That is, we have

(i) P_W linear,

(ii) $P_W|_W = \mathrm{id}$,

(iii) $P_W|_{W^\perp} = 0$,

(iv) $P_W^2 = P_W$

Proof. All the properties except for (iii) are easy to check. For (iii), notice that for $v \in V, z' \in W$, we have that

$$\operatorname{Re}\left\langle z' - P_W(v), v - P_W(v)\right\rangle \leq 0$$

Letting $z' = \pm z + P_W(v)$ and $z' = \pm iz + P_W(v)$, we get that

$$\pm\operatorname{Re}\langle z, v - P_W(v)\rangle \leq 0 \quad \text{and} \quad \mp\operatorname{Im}\langle z, v - P_w(v)\rangle \leq 0$$

So $\langle z, v - P_w(v)\rangle = 0$ for all z. Hence $v - P_W(v) \in W^\perp$. □

5.3 Riesz-Fréchet representation theorem

Theorem 5.16 (Riesz-Fréchet representation)**.** Given a Hilbert space H, the map $\phi : H \to H^*$ given by

$$\phi(v)(w) = \langle w, v\rangle$$

is a bijective isometric sesquilinear map. So $H \simeq H^*$ isometrically, and H^* is a Hilbert space.

Proof. Sesquilinearity is obvious. Now notice that $|\phi(v)(w)| = |\langle w, v\rangle| \leq \|v\|\|w\|$, so $\|\phi(v)\| \leq \|v\|$. But $|\phi(v)(v)| = \|v\|^2$, so equality holds, and ϕ is an isometry.

Next, we need to show that ϕ is surjective. Let $\psi \in H^* \setminus 0$, $W = \ker(\psi) \leq H$ closed. Thus by the previous theorem, we have that

$$H = W \oplus W^\perp$$

and $W^\perp \neq 0$. Let $w_0 \in W^\perp \setminus 0$, then there exists $\alpha \in \mathbb{F}$ such that $\psi(\alpha w_0) = \|\alpha w_0\|^2$[3].
For any $w \in W$, $\psi(w) = 0$. On $W^\perp = \mathbb{F} \cdot w_0$, for any $\lambda \in \mathbb{F}$, we have that

$$\langle \lambda w_0, \alpha w_0\rangle = \overline{\alpha}\lambda\|w_0^2\| = \frac{\|\alpha^2\|}{\alpha}\lambda\|w_0\|^2 = \frac{\lambda}{\alpha}\psi(\alpha w_0) = \psi(\lambda w_0)$$

Hence we must have that $\psi = \phi(\alpha w_0)$. □

6 Spectral theory

6.1 Spectrum and resolvent

Definition 6.1 (resolvent set, spectrum)
Given a Hilbert space H, $T \in \mathcal{B}(H)$, the resolvent set of T is

$$\rho(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ invertible, i.e. bijective with bounded inverse}\}$$

and the spectrum of T is

$$\sigma(T) = \mathbb{C} \setminus \rho(T)$$

Definition 6.2 (resolvent map)
The resolvent map is $R_T : \rho(T) \to \mathcal{B}(H)$ given by

$$R_T(\lambda) = (T - \lambda I)^{-1}$$

Proposition 6.3. Let H be a Hilbert space, $T \in \mathcal{B}(H)$, then

(i) $\rho(T) \subseteq \mathbb{C}$ open. In particular, for every $\lambda_0 \in \rho(T)$,

$$B_{\|R_T(\lambda_0)\|^{-1}}(\lambda_0) \subseteq \rho(T)$$

(ii) $R_T : \rho(T) \to \mathcal{B}(H)$ is holomorphic, and locally given by an entire series,

(iii) $\sigma(T) \neq \varnothing$, and $\sigma(T) \subseteq B_{\|T\|}(0)$.

Proof. (i) If $U \in \mathcal{B}(H)$ with $\|U\| < 1$, then $I - U$ is invertible, with inverse given by the series

$$(I - U)^{-1} = \sum_{n\geq 0} U^n$$

[3] To see this, write $\alpha = re^{i\theta}$, then we have that

$$r^{-1}e^{i\theta} = \frac{\psi(w_0)}{\|w_0\|^2}$$

Now if $\lambda_0 \in \rho(T)$, $\lambda \in B_{\|R_T(\lambda_0)\|^{-1}}(\lambda_0)$, then

$$T - \lambda = (T - \lambda_0) - (\lambda - \lambda_0) = (T - \lambda_0)(I - (\lambda - \lambda_0)R_T(\lambda_0))$$

Now notice that $U = (\lambda - \lambda_0)R_T(\lambda_0)$ has $\|U\| < 1$, so $I - U$ is invertible. Hence $T - \lambda$ is invertible, and $\lambda \in \rho(T)$. Furthermore, we have that

$$R_T(\lambda) = (I - U)^{-1} R_T(\lambda_0) = R_T(\lambda_0) \sum_{n \geq 0} U^n = \sum_{n \geq 0} R_T(\lambda_0)^{n+1} (\lambda - \lambda_0)^n$$

Hence (ii) follows, and $R'_T(\lambda_0) = R_T(\lambda_0)^2$.

For (iii), if $|\lambda| > \|T\|$, then $T - \lambda = -\lambda(1 - \lambda^{-1}T)$, $\|\lambda^{-1}T\| < 1$, so $T - \lambda$ is invertible, with $\|(T - \lambda)^{-1}\| \leq (|\lambda| - \|T\|)^{-1}$ Finally, to show that $\sigma(T)$ is nonempty, we will use Liouville's theorem. Suppose $\sigma(T) = \varnothing$. Then for $v \in H, \phi \in H^*$, define $F_{v,\phi} : \mathbb{C} \to \mathbb{C}$ by

$$F_{v,\phi}(\lambda) = \phi(R_T(\lambda)v)$$

Then $F_{v,\phi}$ is holomorphic, and for $|\lambda| > \|T\|$, we have that

$$\left|F_{v,\phi}\right|(\lambda) \leq \|\phi\| \|R_T(\lambda)\| \|v\| \leq \|\phi\| \|v\| (|\lambda| - \|T\|)^{-1} \to 0$$

as $|\lambda| \to \infty$. Hence by Liouville, $F_{v,\phi}$ is constant, so by Hahn-Banach, $\lambda \mapsto R_T(\lambda)$ is constant. Contradiction. □

Definition 6.4 ({point, continuous, residual, approximate point} spectrum)
Let $T \in \mathcal{B}(H)$, then we have

1. The point spectrum is the set of eigenvalues, that is,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ not injective}\}$$

2. The continuous spectrum is

$$\sigma_c(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ injective, but not surjective, and with } (T - \lambda)(H) \text{ dense in } H\}$$

3. The residual spectrum is the complement of the above two, that is,

$$\sigma_r(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ injective, but } (T - \lambda)(H) \text{ is not dense in } H\}$$

4. The approximate point spectrum is

$$\sigma_{ap}(T) = \left\{\lambda \in \mathbb{C} \mid \text{there exists } (v_n) \text{ in } H \text{ such that } \|v_n\| = 1 \text{ for all } n \text{ and } (T - \lambda)v_n \to 0\right\}$$

Definition 6.5 (bounded below)
$T \in \mathcal{B}(H)$ is bounded below if there exists $C > 0$ such that $\|Tv\| \geq C\|v\|$ for all $v \in H$.

Proposition 6.6.

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not bounded below}\}$$

Furthermore, we have that $\sigma_p(T) \cup \sigma_c(T) \subseteq \sigma_{ap}(T)$. However the reverse inclusion is, in general, false.

Proof. Immediate from definitions. □

Proposition 6.7. If $T \in \mathcal{B}(H)$ is bounded below, then $T(H)$ is closed in H.

Proof. If $(T(v_n))$ is Cauchy, then as T is bounded below, so is (v_n). □

6.2 Spectral theorem

Proposition 6.8. For $T \in \mathcal{B}(H)$, there exists a unique $T^* \in \mathcal{B}(H)$ such that $\langle Tv, w\rangle = \langle v, T^* w\rangle$ for all $v, w \in H$.

Proof. Uniqueness follows from the fact that if $D \in \mathcal{B}(H)$ is such that $\langle v, Dw\rangle = 0$ for all $v, w \in H$, then $D = 0$.

Existence follows form the Riesz-Fréchet representation theorem. □

Definition 6.9 (normal, self-adjoint, unitary)
Let $T \in \mathcal{B}(H)$, then

(i) T is normal if $TT^* = T^*T$

(ii) T is self-adjoint if $T = T^*$

(iii) T is unitary if $T^* = T^{-1}$

Proposition 6.10. Let H be a Hilbert space, $T \in \mathcal{B}(H)$, then

(i) T unitary implies T normal,

(ii) T self adjoint implies T normal,

(iii) $(T^*)^* = T$,

(iv) $\ker(T) = \operatorname{im}(T^*)^{\perp}$,

(v) if T is normal, then $\|Tv\| = \|T^*v\|$ for all $v \in H$

(vi) if T is normal, then $\ker(T) = \ker(T^*) = \operatorname{im}(T)^{\perp} = \operatorname{im}(T^*)^{\perp}$,

(vii) if T is normal, then $\sigma_r(T) = \varnothing$, and the (approximate) eigenvalues of T^* are the complex conjugates of the (approximate) eigenvaleus of T,

(viii) if T is normal, eigenvectors with different eigenvalues are orthogonal,

(ix) if T is self-adjoing, then $\langle Tv, v\rangle \in \mathbb{R}$ for all $v \in H$, and $\sigma(T) \subseteq \mathbb{R}$.

Proof. As in the finite dimensional case from IB Linear algebra. □

Definition 6.11 (compact operator)
$T \in \mathcal{B}(H)$ is compact if $T(B_1(0))$ is relatively compact in H.

Proposition 6.12.

(i) T is compact if and only if it is the limit of finite rank operators,

(ii) T os compact if and only if T^* is compact,

(iii) If T is compact and $\lambda \in \sigma(T) \setminus 0$, then $\lambda \in \sigma_p(T)$, and

$$0 < \dim(\ker(T - \lambda)), \dim(\mathrm{im}(T - \lambda)) < \infty$$

Proof. Omitted. □

Theorem 6.13 (spectral theorem)**.** Let $T \in \mathcal{B}(H)$ be a compact self adjoint operator, then

(i) $\sigma(T) \setminus 0 \subseteq \sigma_p(T) \subseteq \mathbb{R}$,

(ii) $\sigma_p(T)$ is countable,

(iii) the only possible accumulation point of $\sigma_p(T)$ is 0,

(iv) $E_\lambda = \ker(T - \lambda)$ has finite dimension for all $\lambda \in \sigma_p(T) \setminus 0$,

(v) if $\sigma_p(T) \setminus 0 = \{\lambda_n \min n \geq 1\}$, then

$$H = \ker T \overset{\perp}{\oplus} \overline{\left(\overset{\perp}{\bigoplus_{n \geq 1}} E_{\lambda_n} \right)} \quad \text{and} \quad T = \sum_{n \geq 1} \lambda_n P_{E_{\lambda_n}}$$