Riemann surfaces

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Contents

Notation 0.1. Throughout, we will fix the notation that

(i) $\mathbb{C}_\infty$ is the Riemann sphere,

(ii) $\mathbb{C}_* = \mathbb{C} \setminus 0$ is the punctured complex plane

(iii) $D(z, r)$ is the open disc centred at $z \in \mathbb{C}$ with radius r,

(iv) $D_*(z, r) = D(z, r) \setminus z$ is the punctured disc,

(v) $\mathbb{D} = D(0,1)$ is the open unit disc,

(vi) $\mathbb{T} = \partial\mathbb{D} = S^1$ is the unit circle

1 Analytic functions in $\mathbb{C}$

1.1 Analytic and meromorphic functions

Everything in this subsection except Casaroti-Weierstrass is from IB Complex analysis, so proofs have been omitted.

Definition 1.1 (domain)
A domain is an open connected subset of $\mathbb{C}$.

Definition 1.2 (holomorphic, analytic)
Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ is holomorphic, or analytic if f is $\mathbb{C}$-differentiable at every $z_0 \in D$. Equivalently, for any $z_0 \in D$, there exists $r > 0$ such that f has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for any $z \in D(z_0, r) \subseteq D$.

Proposition 1.3 (principle of isolated zeroes). Let $f : D \to \mathbb{C}$ be an analytic function. If $f(z_0) = 0$, then either f is identically zero on a neighbourhood of z_0, or f is nonzero on a punctured neighbourhood of z_0.

Corollary 1.4 (identity principle). Let $f, g : D \to \mathbb{C}$ be analytic functions. Either

(i) $\{z \in D \mid f(z) = g(z)\}$ is discrete,

(ii) or $f = g$ on D.

Definition 1.5 (isolated singularity)
An analytic function $f : D_*(z_0, r) \to \mathbb{C}$ has an isolated singularity at z_0.

Proposition 1.6 (Laurent series). If an analytic function f has an isolated singularity at z_0, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on $D_*(z_0, r)$ for some $r > 0$.

Definition 1.7 (classification of singularities)
Suppose f has Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Then

(i) if $a_n = 0$ for $n < 0$, then z_0 is a removable singularity. In this case, f can be extended to an analytic function $g(z)$ on a neighbourhood of z_0,

(ii) if there exists $m > 0$ such that $a_m = 0$ for $n < -m$, $a_{-m} \neq 0$, then f has a pole of order m at z_0, and

$$f(z) = (z - z_0)^{-m} g(z)$$

for some analytic function g defined on a neighbourhood of z_0, and with $g(z_0) \neq 0$,

(iii) otherwise (i.e. $a_n \neq 0$ for infinitely many $n < 0$), then we say f has an essential singularity at z_0.

Theorem 1.8. An analytic function f has a removable singularity at z_0 if and only if f is bounded on some punctured disc $D_*(z_0, r)$.

Theorem 1.9 (Casaroti-Weierstrass). An analytic function $f : D \to \mathbb{C}$ has an essential singularity at z_0 if and only if $f(D_*(z_0, r))$ is dense in $\mathbb{C}$, for any $r > 0$ such that $D_*(z_0, r) \subseteq D$.

Proof. ($\Longleftarrow$) First suppose z_0 was removable. Then $f(D_*(z_0, r))$ is bounded for some $r > 0$, so it can't be dense. Now suppose z_0 was a pole of order m. Then

$$f(z) = (z - z_0)^{-m} g(z)$$

where g is analytic and nonzero on a neighbourhood of z_0. Fix $\varepsilon > 0$ such that

$$|g(z)| \geq \varepsilon > 0$$

on some punctured disc $D_*(z_0, r)$. Therefore, on that disc,

$$|f(z)| \geq \frac{|g(z)|}{|z - z_0|^m} \geq \frac{\varepsilon}{r^m}$$

so f is bounded away from 0 on $D_*(z_0, r)$, so it is not dense.

Conversely, suppose $f(D_*(z_0, r))$ was not dense. Then there exists an open disc $D(w_0, \varepsilon)$ disjoint from $f(D_*(z_0, r))$. Now define

$$h(z) = \frac{1}{f(z) - w_0}$$

defined on $D_*(z_0, r)$. Since $|f(z) - w_0| \geq \varepsilon$ for all $z \in D_*(z_0, r)$, we have that $|h(z)| \leq \frac{1}{\varepsilon}$. Hence h has a removable singularity at z_0. Then we have that

$$f(z) = \frac{1}{h(z)} + w_0$$

so if $h(z_0) \neq 0$ then f has a removable singularity at z_0, and if $h(z_0) = 0$ then f has a pole at z_0. ∎

Definition 1.10 (meromorphic function)

Let $D \subseteq \mathbb{C}$ be a domain, $A \subseteq D$ discrete, $f : D \setminus A \to \mathbb{C}$ holomorphic, f has a pole at each $z \in A$, then f is a meromorphic function on D.

1.2 Analytic continuation

Definition 1.11 (function element)
Let $D \subseteq \mathbb{C}$ be a domain. A function element on D is a pair (f, U), where U is a subdomain of D and f is an analytic function on U.

Definition 1.12 (direct analytic continuation)
Suppose $(f, U), (g, V)$ are function elements on D, then say that (g, V) is a direct analytic continuation of (f, U), written $(f, U) \sim (g, V)$ if $U \cap V \neq \varnothing$ and $f|_{U\cap V} = g|_{U\cap V}$.

Definition 1.13 (analytic continuation)
If there exists a finite sequence of direct analytic continuations

$$(f, U) = (f_1, U_1) \sim \cdots \sim (f_{n-1}, U_{n-1}) \sim (f_n, U_n) = (g, V)$$

then we say that (g, V) is an analytic continuation of (f, U), and we write $(f, U) \approx (g, V)$.

Proposition 1.14. $\approx$ is an equivalence relation.

Definition 1.15 (complete analytic function)
A $\approx$-equivalence class $\mathcal{F}$ of function elements on D is called a complete analytic function on D.

Definition 1.16 (analytic continuation along a path)
Let (f, U) be a function element on a domain D, and consider an analytic continuation $(f, U) \approx (g, V)$, given by

$$(f, U) = (f_1, U_1) \sim \cdots \sim (f_{n-1}, U_{n-1}) \sim (f_n, U_n) = (g, V)$$

Let $\gamma : [0, 1] \to D$ be a continuous path, if there exists a dissection

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

such that $\gamma([t_{i-1}, t_i]) \subseteq U_i$ for each $1 \leq i \leq n$, then (g, V) is an analytic continuation of (f, U) along γ. We write $(f, U) \approx_\gamma (g, V)$.

1.3 Natural boundary

Throughout, consider wlog a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with radius of convergence 1. In particular, it converges absolutely and uniformly on any closed disc contained in $\mathbb{D}$.

Definition 1.17 ({regular, singular} point)
A point $z_0 \in \mathbb{T}$ is a regular point for f if there exists an open neighbourhood U of z_0 and an analytic function g on U such that $g = f$ on $U \cap D$. Otherwise, z_0 is a singular point for f.

Proposition 1.18. The set of regular points is open, and the set of singular points is closed.

Proposition 1.19. There exists $z \in \mathbb{T}$ which is a singular point for f.

Proof. Suppose not. Then for each $z \in \mathbb{T}$ there exists $\varepsilon_z > 0$ such that f extends analytically over $D(z, \varepsilon_z)$. By compactness, we can cover $\mathbb{T}$ with finitely many of these, so f extends analytically over some $D(0, 1+\delta)$ for some $\delta > 0$. But this implies that the radius of convergence is at least $1 + \delta$[1], contradiction. □

Definition 1.20 (natural boundary)
If every $z \in \mathbb{T}$ is a singular point for f, then we say that $\mathbb{T}$ is a natural boundary for f.

1.4 Complex logarithm

In this subsection, we construct the complete analytic function for the logarithm.

Define the domains

$$U_n = \{re^{i\theta} \mid r > 0, (n-1)\pi/2 < \theta < (n+1)\pi/2\}$$

with corresponding functions $f_n : U_n \to \mathbb{C}$ given by

$$f_n(re^{i\theta}) = \log(r) + i\theta \quad \text{where} \quad \frac{(n-1)\pi}{2} < \theta < \frac{(n+1)\pi}{2}$$

Thse give us function elements $F_n = (f_n, U_n)$. By considering the separate cases for $m - n \mod 4$, we see that $F_m \sim F_n \iff |m-n| \leq 1$. With this, all the F_n are in the same $\approx$-equivalence class, so give us a complete analytic function, which is the complex logarithm.

2 Riemann surfaces

2.1 Covering maps

Definition 2.1 (covering map)
Let $X, \tilde{X}$ be path connected Hausdorff topological spaces, a covering map $\pi : \tilde{X} \to X$ is a local homeomorphism, that is, each $\tilde{x} \in \tilde{X}$ has a neighbourhood $\tilde{U}$ such that $\pi(\tilde{U})$ is open, and $\pi|_{\tilde{U}}$ is a homeomorphism onto its image.

Definition 2.2 (regular covering map)
A covering map $\pi : \tilde{X} \to X$ is regular if for each $x \in X$, there is an open neighbourhood U of x and a discrete set Δ_x such that $\pi^{-1}(U)$ is homeomorphic to $U \times \Delta_x$, and the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\sim} & U \times \Delta_x \\ & \searrow^{\pi} & \downarrow \text{proj}_1 \\ & & U \end{array}$$

commutes.

[1]For example by the integral formula for Taylor series.

Remark 2.3. In Algebraic topology (or rather, the rest of maths), "covering maps" in this course are called local homeomorphisms, and "regular covering maps" are called covering maps.

Proposition 2.4. $\exp : \mathbb{C} \to \mathbb{C}_*$ is a regular covering map.

2.2 Riemann surfaces

Throughout, assume R is a connected Hausdorff topological space.

Definition 2.5 (chart, atlas)

A chart on R is a pair (ϕ, U), where $U \subseteq R$ open, $\phi : U \to D$ is a homeomorphism onto $D \subseteq \mathbb{C}$ open. A set $\mathcal{A}$ of charts is called an atlas on R if

1. $$\bigcup_{(\phi,U)\in\mathcal{A}} U = R$$

2. if $(\phi_1, U_1), (\phi_2, U_2) \in \mathcal{A}$, with $U_1 \cap U_2 \neq \varnothing$, then the transition function
$$\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$$
is analytic.

Definition 2.6 (conformal structure)

A conformal structure on R is an atlas $\mathcal{A}$ on R which is maximal. That is, if (ψ, V) is a chart on R such that for any $(\phi, U) \in \mathcal{A}$, the transition function $\phi \circ \psi^{-1}$ is analytic, then $(\psi, V) \in \mathcal{A}$.

Definition 2.7 (Riemann surface)

A Riemann surface is a pair $R = (R, \mathcal{A})$, where $\mathcal{A}$ is a conformal structure on R.

Lemma 2.8. Every atlas $\mathcal{A}$ is contained in a unique conformal structure $\hat{\mathcal{A}}$.

Proof. **Existence:** Define

$$\hat{A} = \left\{(\psi, V) \text{ chart on R s.t. } \psi \circ \phi^{-1} \text{ analytic for all } (\phi, U) \in \mathcal{A}\right\}$$

By definition this is maximal, so we just need to show that this is an atlas. Covering is clear as $\mathcal{A} \subseteq \hat{\mathcal{A}}$, so we only need to show that the transition functions are analytic. Choose $(\psi_1, V_1), (\psi_2, V_2) \in \hat{\mathcal{A}}$, and $p \in V_1 \cap V_2$. Since $\mathcal{A}$ is an atlas, there exists $(\phi, U) \in \mathcal{A}$ such that $p \in U$. Then

$$\psi_1 \circ \psi_2^{-1} = (\psi_1 \circ \phi^{-1}) \circ (\phi \circ \psi_2^{-1}) = (\psi_1 \circ \phi^{-1}) \circ (\psi_2 \circ \phi^{-1})^{-1}$$

is analytic at $\psi_2(p)$. But p was arbitrary so we are done.

Uniqueness: Suppose $\mathcal{A}'$ is any atlas containing $\mathcal{A}$, then $\hat{A}$ contains $\mathcal{A}'$ by definition. ☐

Lemma 2.9. Every open subset of a Riemann surface is a Riemann surface.

Proof. Just restrict the charts. ☐

Canonical charts on Riemann surfaces

Here we list the charts which we choose to be "canonical" for some Riemann surfaces.

Definition 2.10 ((open subsets of the) complex plane)
For $\mathbb{C}$, we take the chart $\mathrm{id} : \mathbb{C} \to \mathbb{C}$. For $U \subseteq \mathbb{C}$ open, we can take the inclusion map $\iota : U \hookrightarrow C$.

Definition 2.11 (Riemann sphere)
Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. we define an atlas with two charts, given by $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ and $z \in \mathbb{C}_\infty \setminus \{0\} \mapsto \frac{1}{z} \in \mathbb{C}$.

2.3 Analytic functions

Definition 2.12 (analytic maps)
Let R, S be Riemann surfaces, a continuous map $f : R \to S$ is analytic, or holomorphic if for all charts (ϕ, U) on R, (ψ, V) on S, the map $\psi \circ f \circ \phi^{-1}$ is analytic on $\phi(U \cap f^{-1}V)$.

Lemma 2.13. A continuous map $R \to S$ of Riemann surfaces is a analytic if and only if for each $p \in R$, there exists a chart (ϕ_p, U_p) on R with $p \in U_p$, and a chart (ϕ_p, V_p) on S with $f(p) \in V_p$, such that

$$\psi_p \circ f \circ \phi_p^{-1}$$

is analytic at $\phi_p(p)$.

Proof. ($\implies$) is clear. For the converse, notice that

$$\psi \circ f \circ \phi^{-1} = (\psi \circ \psi_p^{-1}) \circ (\psi_p \circ f \circ \phi_p^{-1}) \circ (\phi_p \circ \phi)$$

and the transition functions are analytic. □

Lemma 2.14. If $f : R \to S$ and $g : S \to T$ are analytic, then $g \circ f$ is analytic.

Proof. By the previous lemma we can just check this at $p \in R$. Then

$$\theta_{f(p)} \circ (g \circ f) \circ \phi_p^{-1} = (\theta_{f(p)} \circ g \circ \phi_{f(p)}^{-1}) \circ (\psi_{f(p)} \circ \psi_p^{-1}) \circ (\psi_p \circ f \circ \phi_p^{-1})$$

for appropriate choices of charts. □

Definition 2.15 (conformal equivalence)
A conformal equivalence of Riemann surfaces is an analytic bijection $f : R \to S$ with analytic inverse.

Lemma 2.16. If $\pi : \tilde{R} \to R$ is a covering map, R is a Riemann surface, then there is a unique conformal structure on $\tilde{R}$ such that π is analytic.

Proof. **Existence:** Each point $p \in \tilde{R}$ has a neighbourhood $\tilde{N}_p$ such that $\pi|_{\tilde{N}_p}$ is a homeomorphism onto its image. Since R is a Riemann surface, we have a chart (ϕ_p, U_p) about π_p. Set $\tilde{\phi}_p = \phi_p \circ \pi$, $\tilde{U}_p = \tilde{N}_p \cap \pi^{-1}(U_p)$, we get a chart $(\tilde{\phi}_p, \tilde{U}_p)$ about p. We want to show that

$$\tilde{\mathcal{A}} = \left\{(\tilde{\phi}_p, \tilde{U}_p)\right\}_{p \in \tilde{R}}$$

defines an atlas on $\tilde{R}$. The fact that the charts cover is clear by construction, so suffices to check that the transition functions are analytic. But (with suitable restrictions),

$$\tilde{\phi}_p \circ \tilde{\phi}_q^{-1} = (\phi_p \circ \pi) \circ (\pi^{-1} \circ \phi_q^{-1}) = \phi_p \circ \phi_q^{-1}$$

which is a transition function on R, so analytic. Hence $\tilde{A}$ is an atlas, so it defines a conformal structure. At $p \in \tilde{R}$, we can take charts $(\tilde{\phi}_p, \tilde{U}_p)$ and (ϕ_p, U_p), and we have that

$$\phi_p \circ \pi \circ \tilde{\varphi}_p^{-1} = \phi_p \circ \pi \circ \pi^{-1} \circ \phi_p^{-1} = \mathrm{id}$$

which is analytic at p, so π is analytic.

Uniqueness: Now suppose $\tilde{\mathcal{B}}$ is a conformal structure which makes π into an analytic map. Let $p \in \tilde{R}$ be arbitrary, $(\psi, V) \in \tilde{B}$ be any chart about p. Since π is analytic, the composition

$$\tilde{\phi}_p \circ \psi^{-1} = \phi_p \circ \pi \circ \psi^{-1}$$

is analytic, so $\tilde{\phi}_p$ has analytic transition functions with every chart in $\tilde{\mathcal{B}}$, so $\tilde{\mathcal{A}} = \tilde{\mathcal{B}}$. □

Definition 2.17 (analytic function)
An analytic function on a Riemann surface R is an analytic map $R \to \mathbb{C}$.

Proposition 2.18. Let f be a non-constant analytic function on a Riemann surface R, and let $p \in R$ be a zero of f. Then there exists a chart (ϕ, U) about p with $\phi(p) = 0$ such that

$$f \circ \phi^{-1}(z) = z^m$$

for some $m > 0$.

Proof. Choose a chart (ψ, V) about p, where wlog $\psi(p) = 0$. Then expanding in Taylor series, we have that

$$f \circ \phi^{-1}(z) = z^m g(z)$$

where g is analytic, and $g(0) \neq 0$. By the identity principle for Riemann surfaces, $m > 0$ since f is non-constant. Now as $g(0) \neq 0$ and g is continous, there exists $\delta > 0$ such that $g(D(0, \delta)) \subseteq D(g(0), |g(0)|)$. But then there is an analytic m-th root on $D(g(0), |g(0)|)$, so we have an analytic function $\sqrt[m]{g(z)}$ in a neighbourhood of 0. Define $h(z) = z\sqrt[m]{g(z)}$. Then $h'(0) = \sqrt[m]{g(0)} \neq 0$, so by the inverse function theorem, h has an inverse on some $D(0, \varepsilon)$. Setting $\phi = h \circ \psi$ and $U = \phi^{-1}(D(0, \varepsilon))$ gives the desired chart. □

2.4 Complex Tori

In this subsection, we construct the complex tori.

Let $\tau_1, \tau_2 \in \mathbb{C}$ be $\mathbb{R}$-linearly independent. Let $\Lambda = \langle \tau_1, \tau_2 \rangle$ be the subgroup generated by them. Let $T = \mathbb{C}/\Lambda$ be the quotient group, $\pi : \mathbb{C} \to T$ be the quotient map.

We consider T with the quotient topology.

Definition 2.19 (fundamental parallelogram)
The fundamental parallelogram of T is $P \subseteq \mathbb{C}$, the parallelogram with vertices $0, \tau_1, \tau_2, \tau_1 + \tau_2$.

T is obtained (topologically) by gluing opposite edges of P together. Therefore, it is homeomorphic to $S^1 \times S^1$ and is compact and Hausdorff.

Lemma 2.20. $\pi : \mathbb{C} \to T$ is a regular covering map.

Proof. Choose $\varepsilon < \min\{|\lambda| \mid \lambda \in \Lambda \setminus 0\}/2$. Then for any $z \in \mathbb{C}$, $D(z, \varepsilon) \cap z + \Lambda = \{z\}$. Therefore, if $p = \pi(z_0) \in T$, then $U = \pi(D(z_0, \varepsilon))$ is open, with preimage

$$\pi^{-1}(U) = \bigsqcup_{\lambda \in \Lambda} D(z_0 + \lambda, \varepsilon)$$

□

Finally, we use π to construct a conformal structure on T. For any $z_0 \in \mathbb{C}$, let $U = \pi(D(z_0, \varepsilon))$ with ε as above. Then π restricts to a homeomorphism $D(z_0, \varepsilon) \to U$, so we can set $\phi = \pi|_{D(z_0,\varepsilon)}^{-1}$ to define a chart (φ, U) on T. Since z_0 is arbitrary, the set of all such charts covers T. Furthermore, if (ψ, V) is another chart of this form, the transition fucntion $\phi \circ \psi^{-1}$ is just translation by some $\lambda \in \Lambda$, which is analytic.

2.5 Open mapping theorem

Theorem 2.21 (Open mapping theorem)**.** Any non-constant analytic map of Riemann surfaces $f : R \to S$ is an open map.

Proof. Consider an open set $W \subseteq R$. Let $p \in W$ be arbitrary. Then choose charts (ϕ, U) about p and (ψ, V) about $f(p)$. By the identity principle for Riemann surfaces, f is not locally constant, so

$$\psi \circ f \circ \phi^{-1} : \phi(U \cap W \cap f^{-1}V) \to \psi(V)$$

is a non constant analytic map, so by the open mapping theorem for domains in $\mathbb{C}$, $\psi \circ f(U \cap W \cap f^{-1}V)$ is open. Since ψ is a homeomorphism onto it's image, $f(U \cap W \cap f^{-1}V)$ is an open neighbourhood of $f(p)$ in $f(W)$. As p was arbitrary, $f(W)$ is open. □

Corollary 2.22. Let $f : R \to S$ be a non-constant analytic map of Riemann surfaces. If R is compact, then f is surjective, and S is compact.

Proof. $f(R)$ is a compact subset of a Hausdorff space, so closed. On the other hand, $f(R)$ is open by the open mapping theorem. But S is connected, so f is surjective. □

2.6 Harmonic functions

Lemma 2.23. Let $D \subseteq \mathbb{C}$ be a disc. Then $u : D \to \mathbb{R}$ is harmonic if and only if $u = \text{Re}(f)$ for some $f : D \to \mathbb{C}$ analytic.

Proof. Omitted (ES1 Q11). □

Definition 2.24 (harmonic function)

A function $u : R \to \mathbb{R}$ on a Riemann surface R is harmonic if for any chart (ϕ, U) of R, $u \circ \phi^{-1}$ is harmonic.

Proposition 2.25 (identity principle for harmonic functions)**.** Let $u, v : R \to \mathbb{R}$ be harmonic. Either $u = v$, or the set

$$\{p \in R \mid u(p) = v(p)\}$$

has empty interior.

Proof. Omitted (ES1 Q12). □

Theorem 2.26 (open mapping theorem for harmonic functions)**.** Any non-constant harmonic function $u : R \to \mathbb{R}$ is an open map.

Proof. Suppose $W \subseteq R$ is open. Let $p \in W$ and choose a chart (ϕ, U) about p. Shrinking U if necessary, there is an analytic function $f : \phi(U) \to \mathbb{C}$ such that $u \circ \phi^{-1} = \text{Re}(f)$. If f is constant on $\phi(U)$, then u is constant on U, hence it is constant on R by the identity principle. Therefore, f is non-constant, so $f \circ \phi(U)$ is open by the open mapping theorem for Riemann surfaces.

Say $f \circ \phi(p) = a + ib$. Since the topology on $\mathbb{C}$ is the same as the one on $\mathbb{R} \times \mathbb{R}$, $f \circ \phi(U)$ contains an open set of the form

$$(a - \delta, a + \delta) + i(b - \varepsilon, b + \varepsilon)$$

But $a = u(p)$, hence $u(W)$ contains $(u(p) - \delta, u(p) + \delta)$. Since $p \in W$ was arbitrary, $u(W)$ is open. □

2.7 Meromorphic functions

Definition 2.27 (meromorphic function)

A meromorphic function on a Riemann surface R is an analytic map $f : R \to \mathbb{C}_\infty$ which is not identically ∞.

Proposition 2.28. Let $D \subseteq \mathbb{C}$ be a domain. A function $f : D \to \mathbb{C}$ is meromorphic if and only if there exists a discrete subset $A \subseteq D$ such that $f : D \setminus A \to \mathbb{C}$ is analytic, and f has a pole at each $a \in A$.

Proof. For ($\implies$), let $A = f^{-1}(\infty)$. By the identity principle for Riemann surfaces, A is discrete. Consider the standard chart $1/z$ on $\mathbb{C}_\infty$ about ∞, any $a \in A$ has a neighbourhood on which $\frac{1}{f(z)}$ is analytic, so we can write

$$\frac{1}{f(z)} = (z - a)^m g(z)$$

for some $m \geq 1$, and g analytic with $g(a) \neq 0$. Therefore, near a we have that

$$f(z) = (z - a)^{-m} \frac{1}{g(z)}$$

so a is a pole of f. Converse follows by reversing the above argument. □

2.8 Gluing Riemann surfaces

Definition 2.29 (gluing)

Let X, Y be topological space, suppose we have $X' \subseteq X$ and $Y' \subseteq Y$, and a homeomorphism $\Phi : X' \to Y'$. Then the quotient space

$$X \cup_\Phi Y = \frac{X \sqcup Y}{\sim}$$

where $\sim$ is the relation generated by $x \sim \Phi(x)$ for all $x \in X'$. We call this the result of gluing X and Y along Φ.

Proposition 2.30. Let R_1, R_2 be Riemann surfaces, suppose $S_j \subseteq R_j$ for $j = 1, 2$ are nonempty connected open subsets, $\Phi : S_1 \to S_2$ is a conformal equivalence of Riemann surfaces. Then there is a unique conformal structure on

$$R = R_1 \cup_\Phi R_2$$

such that the inclcuion maps $i_j : R_j \hookrightarrow R$ are analytic. In particular, if R is Hausdorff then it is a Riemann surface.

Proof. For $j = 1, 2$, every chart (ϕ_j, U_j) on R_j gives a chart $(\phi_j \circ i_j^{-1}, i_j(U))$ on R. By construction, these charts cover R. The transition functions betweens two charts coming from R_j are just the transition functions on R_j, hence analytic. Now if we have $(\phi_1, U_1), (\phi_2, U_2)$ are charts on R_1, R_2 respectively, the resulting transition fiunction is

$$\phi_2 \circ i_2^{-1} \circ i_1 \circ \phi_1^{-1} = \phi_2 \circ \Phi \circ \phi_1^{-1}$$

which is analytic as Φ is a conformal equivalence. For the uniqueness statement, suppose $\mathcal{A}$ is any conformal structure on R which makes i_j analytic. If (ϕ_j, U_j) is a chart in R_j, and $(\psi, V) \in \mathcal{A}$, then

$$\psi \circ i_j \circ \phi_j^{-1}$$

is analytic, so $\phi_j \circ i_j^{-1}$ has analytic transition functions with every chart in $\mathcal{A}$, so it is in $\mathcal{A}$ by maximality.

Finally, as R_1, R_2 are path connected, S_1, S_2 nonempty, R is path connected. Therefore, if R is Hausdorff, then it is a Riemann surface. □

3 Covering spaces, monodromy and analytic continuation

3.1 Covering spaces

Definition 3.1 (lift)

Suppose $\pi : \tilde{X} \to X$ is a covering map, $\gamma : [0, 1] \to X$ a path. Then a lift of γ along π is a path $\tilde{\gamma} : [0, 1] \to \tilde{X}$ such that $\pi \circ \tilde{\gamma} = \gamma$.

Proposition 3.2 (uniqueness of lifts)**.** Suppose $\tilde{\gamma}_1, \tilde{\gamma}_2$ are lifts of γ along a covering map π. If $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$, then $\tilde{\gamma}_1 = \tilde{\gamma}_2$.

Proof. Let

$$I = \{t \in [0, 1] \mid \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)\}$$

Then $I = (\gamma_1 \times \gamma_2)^{-1}(\Delta_X)$ is closed as X Hausdorff implies the diagonal is closed. By looking in an open set $\tilde{U}$ such that $\pi|_{\tilde{U}}$ is a homeomorphism, I is open. □

Proposition 3.3 (path lifting lemma)**.** Let $\pi : \tilde{X} \to X$ be a regular covering map, $\gamma : [0, 1] \to X$ be a path, $\pi(\tilde{x}) = \gamma(0)$. Then there exists a unique lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = \tilde{x}$.

Proof. Omitted, see algebraic topology. □

Notation 3.4. We write $\alpha \simeq \beta$ for paths α, β being path homotopic, i.e. rel end points.

Definition 3.5 (simply connected)

A topological space X is simply connected if X is path connected and every pair α, β of paths with the same end point are homotopic.

Theorem 3.6 (monodromy theorem)**.** Let $\pi : \tilde{X} \to X$ be a covering map, α, β paths in X. Suppose

(i) $\alpha \simeq \beta$ in X,

(ii) there are lifts $\tilde{\alpha}$ of α and $\tilde{\beta}$ of β, such that $\tilde{\alpha}(0) = \tilde{\beta}(0)$,

(iii) every path γ in X with $\gamma(0) = \alpha(0) = \beta(0)$ has a lift to $\tilde{X}$ with $\tilde{\gamma}(0) = \tilde{\alpha}(0) = \tilde{\beta}(0)$.

Then $\tilde{\alpha} = \tilde{\beta}$. In particular, $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

Proof. Omitted. It's the homotopy lifting lemma from algebraic topology. □

3.2 Monodromy group

Let $\pi : \tilde{X} \to X$ be a regular covering map, and fix a base point $x_0 \in X$.

For a loop γ based at x_0 and $\tilde{x} \in \pi^{-1}(x_0)$, let $\tilde{\gamma}_{\tilde{x}}$ be the unique lift of γ at $\tilde{x}$. Notice

$$\pi(\tilde{\gamma}_{\tilde{x}}(1)) = \gamma(1) = x_0$$

so $\tilde{\gamma}_{\tilde{x}}(1) \in \pi^{-1}(x_0)$. Define a map $\sigma_\gamma : \pi^{-1}(x_0) \to \pi^{-1}(x_0)$ by

$$\sigma_\gamma(\tilde{x}) = \tilde{\gamma}_{\tilde{x}}(1)$$

Lemma 3.7.

(i) σ_γ only depends on the homotopy class of γ,

(ii) the constant map lifts to the identity map,

(iii) if $\overline{\gamma}(t) = \gamma(1-t)$, then $\sigma_{\overline{\gamma}} = \sigma_\gamma^{-1}$.

(iv) $\sigma_{\alpha\beta} = \sigma_\beta \circ \sigma_\alpha$

In particular, the set of all σ_γ forms a group.

Proof. (i) follows from the monodromy theorem, (ii)-(iv) follow from uniqueness of lifts. □

Definition 3.8 (monodromy group)

The monodromy group of the regular covering map based at x_0 is

$$H_{x_0} \leq \mathrm{Sym}(\pi^{-1}(x_0))$$

which is the subgroup formed of all σ_γ.

Proposition 3.9. The isomorphism class of H_{x_0} is independent of the choice of base point.

Proof. $\gamma \mapsto \alpha\gamma\overline{\alpha}$ defines a homomorphism of monodromy groups. □

3.3 Space of germs

Notation 3.10. Let (f, U) and (g, V) be function elements on D, For any $z \in D \cap E$, we write

$$(f, U) \equiv_z (g, V)$$

if $f = g$ on a neighbourhood of z.

Definition 3.11 (germ)
Let (f, U) be a function element and $z \in U$. The $\equiv_z$-equivalence class of (f, U) is called the germ of f at z, denoted by $[f]_z$.

We say $[f]_z = [g]_w$ if and only if $z = w$ and $f = g$ on a neighbourhood of $z = w$.

Definition 3.12 (space of germs)
The space of germs over D is

$$\mathcal{G} = \{[f]_z \mid z \in D, (f, U) \text{ function element with } z \in U\}$$

Notation 3.13. For a function element (f, U) on D, we write

$$[f]_U = \{[f]_z \mid z \in U\}$$

Lemma 3.14. Unions of sets of the form $[f]_U$ defines a topology on $\mathcal{G}$.

Proof. Taking the empty union shows $\varnothing$ is open. By definition, each $[f]_z \in [f]_U$ for some U, so $\mathcal{G}$ is also open. By definition the topology is closed under unions, so we only need to check that it is closed under finite intersections. By some basic set manipulation, suffices to show any set of the form $[f]_U \cap [g]_V$ is open. Consider any germ $[h]_z \in [f]_U \cap [g]_V$. Then $z \in U \cap V$ and h agrees with f and g on a neighbourhood W of z. So $[h]_W$ is an open neighbourhood of $[h]_z$ in $[f]_U \cap [g]_V$. □

Lemma 3.15. $\mathcal{G}$ is Hausdorff.

Proof. Consider distinct germs $[f]_z \neq [g]_w$, and choose representative function elements (f, U) and (g, V). If $z \neq w$, then we by shrinking U, V we can assume U, V are disjoint. So $[f]_U \cap [g]_V = \varnothing$.

The case $z = w$ is all thar remains. In this case, choose function elements (f, U) and (g, U) for U connected. Suppose $[h]_x \in [f]_U \cap [g]_U$. Then x has a neighbourhood W in U in which $f = g = h$. By the identity principle, $f = g$ on U. So $[f]_U = [g]_U$, and so $[f]_z = [g]_z$. Contradiction. □

Definition 3.16 (forgetful map)
Let $\mathcal{G}$ be the space of germs over a domain D. The forgetful map $\pi : \mathcal{G} \to D$ is

$$\pi([f]_z) = z$$

Lemma 3.17. For each component $G \subseteq \mathcal{G}$, the restriction of the forgetful map $\pi : G \to D$ is a covering map.

Proof. Let $U \subseteq D$ be open. Then

$$\pi^{-1}(U) = \bigcup_{V \subseteq U} [f]_V$$

where we take the union over all function elements on U. In particular, π is continuous. The restriction of π to any open set of the form $[f]_U$ has an inverse, namely $z \mapsto [f]_z$. Furthermore, the inverse is also continuous, as the preimage of an open set $[f]_V$ is the open set $V \cap U$. Since the sets $[f]_U$ cover $\mathcal{G}$, π is a local homeomorphism, and so its restriction to any component is also a covering map. □

Corollary 3.18. Each component of $\mathcal{G}$ has an atlas which makes π into an analytic map. In fact, we can write down an atlas, where the charts are of the form $(\pi, [f]_U)$.

Definition 3.19 (evaluation map)
The evaluation map $\mathcal{E} : \mathcal{G} \to \mathbb{C}$ is defined by

$$\mathcal{E}([f]_z) = f(z)$$

Proposition 3.20. $\mathcal{E}$ restricted to each component of $\mathcal{G}$ is an analytic map.

3.4 Analytic continuation

Theorem 3.21. Let (f, U) and (g, V) be function elements on a domain $D \subseteq \mathbb{C}$, and let $\gamma : [0, 1] \to D$ be a path starting in U and ending in V. Then $(f, U) \approx_\gamma (g, V)$ if and only if the lift $\tilde{\gamma}$ to a component of $\mathcal{G}$ starting at $[f]_{\gamma(0)}$ exists, and ends at $[g]_{\gamma(1)}$.

Proof. Suppose $(f, U) \approx_\gamma (g, V)$. That is, we have a sequence of direct analytic continuations

$$(f, U) = (f_1, U_1) \sim \cdots \sim (f_{n-1}, U_{n-1}) \sim (f_n, U_n) = (g, V)$$

a continuius path $\gamma : [0, 1] \to D$, and a dissection

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

such that $\gamma([t_{i-1}, t_i]) \subseteq U_i$ for $1 \leq i \leq n$. Define a lift by

$$\tilde{\gamma}(t) = [f_i]_{\gamma(t)}$$

whenever $t \in [t_{i-1}, t_i]$, which is well defined since $[f_i]_{\gamma(t_i)} = [f_{i+1}]_{\gamma(t_i)}$ for each $0 < i < n$. For continuity, notice that $\tilde{\gamma}$ is continuous on each interval on the dissection, so by the gluing lemma it is continuous on $[0, 1]$.

Conversely, suppose there is a lift $\tilde{\gamma}$ of γ to $\mathcal{G}$ such that $\tilde{\gamma}(0) = [f]_{\gamma(0)}$ and $\tilde{\gamma}(1) = [g]_{\gamma(1)}$. By the compactness of $[0, 1]$, there is a finite sequence of function elements (f_i, U_i) for $1 \leq i \leq n$, and a dissection

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

such that $\tilde{\gamma}([t_{i-1}, t_i]) \subseteq [f_i]_{U_i}$ for $1 \leq i \leq n$. Indeed, we can assume U_i is an open disc in $\mathbb{C}$. Applying the forgetful map π, we have that $\gamma([t_{i-1}, t_i]) \subseteq U_i$, so it remains to show $(f_{i-1}, U_{i-1}) \sim (f_i, U_i)$ for all i. But for each such i,

$$[f_{i-1}]_{\gamma(t_{i-1})} = \tilde{\gamma}(t_{i-1}) = [f_i]_{\gamma(t_{i-1})}$$

so f_i, f_{i-1} agree on a neighbourhood of $\gamma(t_{i-1}) \in U_{i-1} \cap U_i$. Since the U_i are discs, $U_{i-1} \cap U_i$ connected, by the identity principle $f_{i-1} = f_i$ on $U_{i-1} \cap U_i$. So $(f_{i-1}, U_{i-1}) \sim (f_i, U_i)$ for all i, and we have a sequence of direct analytic continuations. □

Corollary 3.22. Let $\mathcal{F}$ be a complete analytic function on a domain $D \subseteq \mathbb{C}$, then

$$\mathcal{G}_\mathcal{F} = \bigcup_{(f,U)\in\mathcal{F}} [f]_U$$

is a path component of $\mathcal{G}$.

Definition 3.23 (Riemann surface associated with a complete analytic function)
We call $\mathcal{G}_{\mathcal{F}}$ the Riemann surface associated with $\mathcal{F}$.

Theorem 3.24 (classicial monodromy). Let $D \subseteq \mathbb{C}$ be a domain, and suppose (f, U) be a function element in D, which can be continued along any path in D starting in U. If $(f, U) \approx_\alpha (g_1, V)$ and $(f, U) \approx_\beta (g_2, V)$ and $\alpha \simeq \beta$, then $g_1 = g_2$ on V.

Proof. Let $\tilde{\alpha}, \tilde{\beta}$ be the lifts to $\mathcal{G}$ of α, β respectively, starting at $[f]_{\alpha(0)} = [f]_{\beta(0)}$. Then by the previous theorem and the monodromy theorem,

$$[g_1]_{\alpha(1)} = \tilde{\alpha}(1) = \tilde{\beta}(1) = [g_2]_{\beta(1)}$$

so $g_1 = g_2$ on a neighbourhood of $\alpha(1) = \beta(1)$, so $g_1 = g_2$ on V by the identity principle. □

Corollary 3.25. Let D be a simply connected domain, (f, U) a function element on D. If (f, U) can be analytically continued along every path in D starting in U, then (f, U) extends uniquely to an analytic function $f : D \to \mathbb{C}$.

4 Branching

4.1 Branching

Definition 4.1 (multiplicity)
Let $f : R \to S$ be an analytic map of Riemann surfaces, $p \in R$. Choose charts $(\phi, U), (\psi, V)$ about $p, f(p)$ respectively, with $\phi(p) = 0$ such that

$$\psi \circ f \circ \phi^{-1}(z) = z^{m_f(p)}$$

for some integer $m_f(p) \geq 0$. We call $m_f(p)$ the multiplicity of f at p.

Proposition 4.2. $m_f(p)$ is the number of preimages of points in a sufficiently small punctured neighbourhood of $f(p)$, so it is independent of the choice of charts.

Definition 4.3 ({ramification, branch} point, ramification index)
If $m_f(p) > 1$ then we call p a ramification point of f, and $f(p)$ a branch point of f. In this case, we call $m_f(p)$ the ramification index at p.

Lemma 4.4. Let $f : R \to \mathbb{C}$ be a non-constant analytic function, $p \in R$ and (ϕ, U) be any chart about p with $\phi(p) = z_0$. Then p is a ramification point if and only if $F'(z_0) = 0$, where $F = f \circ \phi^{-1}$.

Proof. We have that

$$F = f \circ \phi^{-1}(z) = (z - z_0)^{m_f(p)} g(z)$$

where g analytic with$g(z_0) \neq 0$. Hence by the product rule,

$$F'(z) = (m_f(p)g(z) + (z - z_0)g'(z))(z - z_0)^{m_f(p)-1}$$

So

$$F'(z_0) = \begin{cases} g(z_0) \neq 0 & \text{if } m_f(p) = 1 \\ 0 & \text{if } m_f(p) > 1 \end{cases}$$

□

Lemma 4.5. If $f : R \to S$ and $g : S \to T$ are analytic functions of Riemann surfaces, then

$$m_{g \circ f}(p) = m_g(f(p)) m_f(p)$$

for any $p \in R$.

Proof. Fix any chart (θ, W) about $g(f(p))$, with $\theta(g(f(p))) = 0$. Then choosing a chart (ψ, V) about $f(p)$ such that

$$\theta \circ g \circ \psi^{-1}(z) = z^{m_g(f(p))}$$

on a neighbourhood of 0. Likewise, we have a chart (ϕ, U) about p such that

$$\psi \circ f \circ \phi^{-1}(z) = z^{m_f(p)}$$

Hence

$$\theta \circ (g \circ f) \circ \phi^{-1}(z) = (\theta \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1})(z) = z^{m_g(f(p)) m_f(p)}$$

□

4.2 Valency theorem

Theorem 4.6 (valency). Suppose $f : R \to S$ is a non-constant analytic map between compact Riemann surfaces. Then the function $n : S \to \mathbb{N}$ defined by

$$n(q) := \sum_{p \in f^{-1}(q)} m_f(p)$$

is constant on S.

Proof. Since R is compact, each $q \in S$ has only finitely many preimages in R, by the identity principle[2]. So $n(q)$ is finite.

Since R is connected, suffices to show that n is locally constant. Therefore, fix $q_0 \in S$, $n_0 = n(q_0)$. Then it suffices to find an open neighbourhood of q_0 on which $n(q) = n_0$. Let

$$f^{-1}(q_0) = \{p_1, \dots, p_k\}$$

and fix a chart (ψ, V) about q_0. Then we have charts (ϕ_i, U_i) about each p_i, such that

$$\psi \circ f \circ \phi_i^{-1}(z) = z^{m_f(p_i)}$$

on U_i. By passing to smaller charts, we may assume that the $\{U_i\}$ are disjoint. Now $R \setminus \bigcup_i U_i$ is closed, so compact. Therefore, the image $f(R \setminus \bigcup_i U_i)$ is compact, so closed. Therefore, there is a connected open neighbourhood $V' \subseteq V$ of q_0 which is disjoint from K. Therefore,

$$f^{-1}(V') \subseteq R \setminus f^{-1}(K) \subseteq R \setminus (R \setminus \bigcup_i U_i) = \bigcup_i U_i$$

Setting $U_i' = f^{-1}(V') \cap U_i$, we obtain charts (ϕ_i, U_i') about p_i and ψ, V' about q such that f is a power map centered on some p_i everywhere on the preimage of V'. So $n(q) = n_0$ for all $q \in V'$. □

[2] A closed discrete subset of a compact space is finite.

Definition 4.7 (degree)
For a non-constant analytic map f, then number $n = n(q)$ from the theorem above is called the degree or valency of f, written as $\deg(f)$.

4.3 Riemann-Hurwitz

Note that the content about trinagulations of surfaces and Euler characteristic has been omitted.

Theorem 4.8 (Riemann-Hurwitz). Let $f : R \to S$ be any non-constant analytic map of compact Riemann surfaces, then

$$\chi_R = \deg(f)\chi(S) - \sum_{p \in R}(m_f(p) - 1)$$

Sketch proof. First of all, $m_f(p) > 1$ if and only if p is a ramification point, but R is compact and ramification points are isolated, so there are only finitely many of them, so the sum is finite.

From the proof of the valency theorem, each $q \in S$ has a neighbourhood U such that f is a power map on each component of the preimage of U. These neighbourhoods form an open cover of S, so by compactness we have a finite subcover $U_1, \ldots, U_k$, where U_i is the neighbourhood associated to the point q_i. The only point of U_i that can be a branch point is a q_i itself, so there are only finitely many branch points.

Take a triangulation of S. By a subdivision, we can assume each branch point is a vertex in the triangulation, and that each triangle is contained in a U_i. In particular, the preimage of the triangles in S form a triangulation of R. Let $n = \deg(f)$, V_S, E_S, F_S (resp. V_R, E_R, F_R) be the number of vertices, edges and faces of S (resp. R). Then

1. each triangle in S has exactly n preimages in R, so $F_R = nF_S$,
2. each face in S has exactly n preimages in R, so $E_R = nE_S$,
3. each vertex $q \in S$ has
$$n - \sum_{p \in f^{-1}(q)}(m_f(p) - 1)$$
preimages in R, so
$$V_R = nV_S - \sum_{q \in S}\sum_{p \in f^{-1}(q)}(m_f(p) - 1) = nV_S - \sum_{p \in R}(m_f(p) - 1)$$

Therefore,

$$\chi_R = F_R - E_R + V_R = nF_S - nE_S + nV_S - \sum_{p \in R}(m_f(p) - 1) = n\chi(S) - \sum_{p \in R}(m_f(p) - 1)$$

□

Corollary 4.9. If R and S have genus g_R, g_S respectively, then

$$2g_R - 2 = n(2g_S - 2) + \sum_{p \in R}(m_f(p) - 1)$$

Proof. $\chi(R) = 2 - 2g_R$ and $\chi(S) = 2 - 2g_S$. □

Corollary 4.10. The correction term

$$C = \sum_{p \in R}(m_f(p) - 1)$$

is even.

Proof. All other terms in Riemann-Hurwitz are even. □

Corollary 4.11. Suppose $f : R \to S$ is a covering map (i.e. it is unramified). Then $C = 0$, and $g_R - 1 = n(g_S - 1)$. In particular,

(i) if $g_S = 0$, then $n = 1$ and $g_R = 0$, so f is a conformal equivalence,

(ii) if $g_S = 1$, then $g_R = 1$ and we have no constraints on n,

(iii) if $g_S > 1$, then either $n = 1$, where f is a conformal equivalence, or $g_R > g_S$.

5 Rational and periodic functions

5.1 Rational functions

Proposition 5.1. Every meromorphic function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a rational function. That is, it is of the form

$$f(z) = c\frac{(z-a_1)\cdots(z-a_m)}{(z-b_1)\cdots(z-b_n)}$$

for $m, n \geq 0$ and $a_i, b_j, c \in \mathbb{C}$.

Proof. Without loss of generality, f is non constant, and $f(\infty) \in \mathbb{C}$. Now $f^{-1}(\infty)$ is a finite set of poles $b_1, \ldots, b_{n'} \in \mathbb{C}$, and f takes the form

$$f(z) = \sum_{l=-k_j}^{\infty} c_{j,l}(z-b_j)^l$$

in a punctured neighbourhood of each b_j. Let Q_j be the principal part, i.e.

$$Q_j(z) = \sum_{l=-k_j} -1 c_{j,l}(z-b_j)^l$$

we see that all singularities of

$$g(z) = f(z) - \sum_{j=1}^{n'} Q_j(z)$$

are removable. Hence $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is not surjective, so it is constant. Since the Q_j are rational functions, we are done. □

Corollary 5.2. For a rational function

$$f(z) = c\frac{(z-a_1)\cdots(z-a_m)}{(z-b_1)\cdots(z-b_n)}$$

where the a_is are all distinct from the b_js, i.e. m, n minimal, then $\deg(f) = \max\{m, n\}$.

5.2 Simply and doubly periodic functions

Definition 5.3 (period)

Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be meromorphic. A period pf f is $\omega \in \mathbb{C}$ such that $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$.

Lemma 5.4. Let Ω be the set of periods of a meromorphic function f on $\mathbb{C}$, then one of the following holds:

(i) $\Omega = \{0\}$,

(ii) $\Omega = \mathbb{Z}\omega \simeq \mathbb{Z}$ for some $\omega \neq 0$,

(iii) $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \simeq \mathbb{Z}^2$ for some ω_1, ω_2 $\mathbb{R}$-linearly independent.

(iv) $\Omega = \mathbb{C}$.

Proof. Omitted. □

Definition 5.5 (simply periodic)
A meromorphic function f on $\mathbb{C}$ for which the group of periods is isomorphic to $\mathbb{Z}$ is called simply periodic.

Proposition 5.6. If f is a meromorphic function on $\mathbb{C}$, and the periods of f contain an infinite cyclic group $\mathbb{Z}\omega$, then there is a unique meromorphic function $\overline{f}$ on $\mathbb{C}_*$ such that

$$f(z) = \overline{f} \circ \exp((2\pi i/\omega)z)$$

Proof. On a small open neighbourhood of any point in $\mathbb{C}_*$, choose a branch of the complex logarithm and define

$$\overline{f}(w) = f((\omega/2\pi i)\log(w))$$

which is a locally defined analytic function, with $f(z) = \overline{f} \circ \exp((2\pi i/\omega)z)$. Furthermore, this definition is independent of the choice of branch, since

$$f((\omega/2\pi i)(\log(w) + 2\pi i n)) = f((omega/2\pi i)\log(w) + n\omega) = f((\omega/2\pi i)\log(w)) = \overline{f}(w)$$

since $n\omega \in \mathbb{Z}\omega$ is a period of f. □

Morally, simply periodic functions on $\mathbb{C}$ are the same as functions on $\mathbb{C}_*$.

Definition 5.7 (doubly periodic/elliptic)
A meromorphic function f on $\mathbb{C}$ with periods $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \simeq \mathbb{Z}^2$ is called doubly periodic or elliptic.

Proposition 5.8. If f is a meromorphic fucntion on $\mathbb{C}$ and the periods of f contain a lattice Λ, then there is a unique meromorphic function $\overline{f}$ on $\mathbb{C}/\Lambda$ such that

$$f(z) = \overline{f}(\pi(z))$$

for all $z \in \mathbb{C}$, where $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is the quotient covering map.

Proof. Same as in the simply periodic case. □

Corollary 5.9. Any analytic function f on $\mathbb{C}$ which is doubly periodic is constant.

Proof. Since $\mathbb{C}/\Lambda$ is compact, any analytic function $\mathbb{C}/\Lambda \to \mathbb{C}$ is constant. □

Definition 5.10 (degree)
For a doubly periodic function f, define $\deg(f) = \deg(\overline{f})$, where $\overline{f}$ is the associated function on a complex torus.

Corollary 5.11. If f is a doubly periodic non constant meromorphic function, then $\deg(f) \geq 2$.

Proof. Suppose $\deg(1) = 1$. Then $m_f(p) = 1$ for all p. In this case, the Riemann-Hurwitz theorem gives

$$(2 \times 1 - 2) = 1 \times (2 \times 0 - 2)$$

□

Definition 5.12 (period parallelogram)
A period parallelogram of a doubly periodic function f with $\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is

$$\mathcal{P} = \{z_0 + t_1\omega_1 + t_2\omega_2 \mid t_1, t_2 \in [0, 1]\}$$

Alternative proof of corollary 5.11. Let $\mathcal{P}$ be a period parallelogram, such that there are no zeros or poles of f on $\partial\mathcal{P}$. Then by the residue theorem, we have that

$$\sum_z \mathrm{Res}_f(z) = \frac{1}{2\pi i}\int_{\partial\mathcal{P}} f(z)\mathrm{d}z$$

where we sum over the poles of f in $\mathcal{P}$, and $\mathrm{Res}_f(z)$ is the residue of f at z. But as f is doubly periodic, the integrals along parallel edges in $\partial\mathcal{P}$ cancel. Therefore the right hand side is zero, so the sum of the residues is zero. Since f is non-constant, it must have at least one pole, therefore it must have at least two, counted with multiplicity. □

5.3 Weierstrass $\wp$-function

Definition 5.13 (Weierstrass $\wp$-function)
Let Λ be a lattice in $\mathbb{C}$. The associated Weierstrass $\wp$-function is

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

We write $\wp = \wp_\Lambda$ if Λ is clear from context.

Lemma 5.14. Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ and $t \in \mathbb{R}$. Then the sum

$$\sum_{\omega\in\Lambda\setminus 0}\frac{1}{|\omega|^t}$$

converges if and only if $t > 2$.

Proof. The set $\{(t_1, t_2) \in \mathbb{R}^2 \mid |t_1| + |t_2| = 1\}$ is compact, so the function $(t_1, t_2) \mapsto |t_1\omega_1 + t_2\omega_2|$ achieves its maximum M and minimum m. By linear independence, $m > 0$. Now let $(k, l) \in \mathbb{Z}^2$. Let $t_1 = k/(|k| + |l|)$ and $t_2 = l/(|k| + |l|)$, we get

$$m(|k| + |l|) \leq |k\omega_1 + l\omega_2| \leq M(|k| + |l|)$$

so the sum we are interested in is bounded above and below by constant multiples of

$$\sum_{(k,l)\in\mathbb{Z}^2\setminus 0} \frac{1}{(|k|+|l|)^t}$$

Set $n = |k| + |l|$, and noting there are exactly $4n$ pairs (k, l) with $|k| + |l| = n > 0$, we have that

$$\sum_{(k,l)\in\mathbb{Z}^2\setminus 0} \frac{1}{(|k|+|l|)^t} = \sum_{n=1}^{\infty} \frac{4n}{n^t} = 4\sum_{n=1}^{\infty} \frac{1}{n^{t-1}}$$

which converges if and only if $t > 2$. □

Theorem 5.15. $\wp_\Lambda$ is a well defined elliptic function with periods Λ. Moreover, $\wp_\Lambda$ is even and of degree 2.

Proof. First of all, we prove that $\wp_\Lambda(z)$ converges for all $z \in \mathbb{C}$.

$$\begin{aligned}
\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| &= \left|\frac{\omega^2 - (z-\omega)^2}{\omega^2(z-\omega^2)}\right| \\
&= \left|\frac{z(2\omega - z)}{\omega^2(z-\omega^2)}\right| \\
&= \left|\frac{z}{\omega^2}\right|\left|\frac{2\omega - z}{(z-\omega)^2}\right| \\
&\leq \frac{|z|}{|\omega|^2} \cdot \frac{2|\omega - z| + |z|}{|z-\omega|^2} \\
&= \frac{|z|}{|\omega|}\left(\frac{2}{|z-\omega|} + \frac{|z|}{|z-\omega|^2}\right)
\end{aligned}$$

But for all but finiely many $\omega \in \Lambda$, $|\omega| \geq 2|z|$, so $|\omega - z| \geq \frac{|\omega|}{2} \geq |z|$. Therefore, these terms in the sum are all bounded above by

$$\frac{|z|}{|\omega|^2}\left(|4||\omega|^2 + \frac{|z|}{|z||\omega|/2}\right) = \frac{6|z|}{|\omega|^3}$$

Therefore the sum defining $\wp_\Lambda$ converges absolutely and uniformly on compact sets. In addition, the definition immediately implies that $\wp_\Lambda$ is even. To show that $\wp_\Lambda$ is elliptic, we need to show that each $\omega_0 \in \Lambda$ is a period of $\wp_\Lambda$. Differentiating $\wp_\Lambda$, we see

$$\wp'_\Lambda(z) = \sum_{\omega\in\Lambda} \frac{-2}{(z-\omega)^3}$$

Therefore, ω_0 is a period of $\wp'_\Lambda$, and we have that

$$\wp_\Lambda(z+\omega_0) - \wp_\Lambda(z) = c$$

for some constant c, as it has derivative zero. Setting $z = -\omega_0/2$, and using the fact that $\wp_\Lambda$ is even gives

$$\wp_\Lambda(\omega_0/2) = c + \wp_\Lambda(\omega_0/2)$$

so $c = 0$, and ω_0 is a period of $\wp_\Lambda$. Finally, as the only poles of $\wp'_\Lambda$ are at the lattice points Λ, so the periods are precisely Λ. In particular, $\wp_\Lambda$ has a unique pole of order 2 on $\mathbb{C}/\Lambda$, so $\deg(\wp_\Lambda) = 2$. □

Corollary 5.16. $\wp_\Lambda$ is the unique function with the following properties:

(i) $\wp_\Lambda$ is meromorphic with periods Λ,

(ii) $\wp_\Lambda$ has poles only at Λ,

(iii) $\lim_{z\to 0}(\wp_\Lambda(z) - z^{-2}) = 0$.

Proposition 5.17.

(i) $\wp_\Lambda$ has poles precisely at the lattice points Λ,

(ii) $\deg(\wp_\Lambda) = 3$,

(iii) $\wp'_\Lambda$ is odd,

(iv) $\wp'_\Lambda$ has simple zeros at $\omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2$ in the fundamental parallelogram,

(v) the ramification points of $\wp_\Lambda$ in Λ are $0, \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2$, with corresponding branch points $\infty, e_1 = \wp_\Lambda(\omega_1/2), e_2 = \wp_\Lambda(\omega_2/2), e_3 = \wp_\Lambda((\omega_1+\omega_2)/2)$.

Proof. (i)-(iii) follow from the expression of the derivative as a series, that is

$$\wp'_\Lambda(z) = \sum_{\omega\in\Lambda} \frac{-2}{(z-\omega)^3}$$

For (iv), notice that for any $\omega \in \Lambda$,

$$\wp'_\Lambda(\omega/2) = -\wp'_\Lambda(-\omega/2) = -\wp'_\Lambda(\omega/2)$$

by oddness and periodicity, so $\wp'_\Lambda(\omega/2) = 0$. By the valency theorem, the zeroes at $\omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2$ are the only ones and they are simple.

For (v), recall that away from the poles, the ramification points are points where the derivative vanishes. Therefore, the ramification points are precisely $0, \omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2$. Finally, by the valency theorem again e_1, e_2, e_3 are distinct. □

Remark 5.18. Another way of seeing that there are exactly four ramification points is by Riemann-Hurwitz, which in this case says

$$0 = 2 \times (-2) + \sum_{p\in\mathbb{C}/\Lambda} (m_{\wp_\Lambda}(p) - 1)$$

5.4 Elliptic curves and elliptic functions

Proposition 5.19. There are constants $g_1, g_2 \in \mathbb{C}$ depending on Λ, such that

$$(\wp'_\Lambda)^2 = 4\wp_\Lambda^3 - g_2\wp_\Lambda - g_3$$

Proof. Since $\wp_\Lambda$ is even, every term in its Larent expansion about zero has an even exponent. Furthermore, the constant term is zero. Hence

$$\wp(z) = \frac{1}{z^2} + az^2 + o(z^4)$$

for some $a \in \mathbb{C}$. Cubing this, we get

$$(\wp(z))^3 = \frac{1}{z^6} + \frac{3a}{z^2} + f(z)$$

where f is analytic in a neighbourhood of zero. On the other hand, if we differentiate $\wp$, we get

$$\wp'(z) = \frac{-2}{z^3} + 2az + o(z^3)$$

Squaring this gives

$$(\wp'(z))^2 = \frac{4}{z^4} - \frac{8a}{z^2} + g(z)$$

where g is analytic in a neighbourhood of zero. Therefore,

$$(\wp'(z))^2 - 4(\wp(z))^3 = -\frac{20a}{z^2} - h(z)$$

for some analytic function h on a neighbourhood of zero. Set $g_2 = 20a$, then

$$(\wp'(z))^2 - 4(\wp(z))^3 + g_2\wp(z)$$

is periodic with the only poles being at Λ, but it is analytic in a neighbourhood of zero. Therefore it is constant. □

Corollary 5.20. The coefficients g_2, g_3 and e_1, e_2, e_3 are related by

$$4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$$

Proof. e_1, e_2, e_3 are the images of the zeros of $\wp'$, under $\wp$. □

Corollary 5.21. Let $\mathbb{C}/\Lambda$ be a complex torus. Then $\mathbb{C}/\Lambda$ is conformally equivalent to a one point compactification[a] of

$$X' = \left\{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2x - g_3\right\}$$

[a]Read: projective closure

Sketch proof. We will assume that X' can be compactified by adding one point at infinity, and with charts coming from the coordinate projections.

Define $F : \mathbb{C} \to X$ by $F(z) = (\wp(z), \wp'(z))$, then as each coordinate function is analytic, F is analyic. Furthermore, as $\wp$ is Λ-periodic, F gives us an analytic map

$$\Phi : \mathbb{C}/\Lambda \to X$$

As Φ is non-constant, $\mathbb{C}/\Lambda$ is compact, Φ must be surjective. For injectivity, consider a period parallelogram $\mathcal{P}$ with vertices $(\pm\omega_1 \pm \omega_2)/2$. Suppose $z, w \in \mathcal{P}$, with $F(z) = F(w)$. Then $\wp(z) = \wp(w)$ implies that $w = \pm z$, since $\wp$ is even and of degree 2. But then as $\wp'$ is odd, we must have $w = z$.

Therefore, away from the rammification points of $\wp$, Φ has degree 1 in $\mathbb{C}/\Lambda$. Hence by the valency theorem $\deg(\Phi) = 1$, so Φ is a conformal equivalence. □

Theorem 5.22. Let f be elliptic, with periods Λ. Then there exists rational functions Q_1, Q_2 such that

$$f(z) = Q_1(\wp(z)) + Q_2(\wp(z))\wp'(z)$$

Furthermore, if f is even then $Q_2 = 0$.

Proof. Furst suppose f is even. Since f and $\wp$ have finitely many branch points, choose c, d in $\mathbb{C}$ which are not branch points. Now consider

$$z \mapsto \frac{f(z) - c}{f(z) - d}$$

Hence we can assume wlog that the zeroes and poles of f are simple, and not ramification points of $\wp$. Say the zeroes of f are $\{\pm a_1, \dots, \pm a_m\}$, which are all distinct, and the poles are $\{\pm b_1, \dots, \pm b_n\}$.

Now let

$$g(z) = \frac{(\wp(z) - \wp(a_1)) \cdots (\wp(z) - \wp(a_m))}{(\wp(z) - \wp(b_1)) \cdots (\wp(z) - \wp(b_n))}$$

Then f and g have the same zeros and poles, so $f(z)/g(z)$ is a nonzero elliptic function with only removable singularities, so it is constant. Hence $f(z) = cg(z)$ for some $c \in \mathbb{C}$.

If f is odd, then $f(z)/\wp'(z)$ is even, so $f(z) = Q_2(\wp(z))\wp'(z)$ by above. Finally, for arbitrary f, we have that

$$f(z) = \underbrace{\frac{f(z) + f(-z)}{2}}_{\text{even}} + \underbrace{\frac{f(z) - f(-z)}{2}}_{\text{odd}}$$

□

6 Quotients and uniformisation

6.1 Quotients of Riemann surfaces

Definition 6.1 (properly discontinuous)
Let G act on a topological space X by homeomorphisms. The action is properly discontinuous if for every $K \subseteq X$ compact, the set

$$\{g \in G \mid g(K) \cap K \neq \varnothing\}$$

is finite.

Definition 6.2 (free)
Let G act on a topological space X by homeomorphisms. The action is free if for all $x \in X$, $\mathrm{Stab}_G(x) = 1$.

Lemma 6.3. Suppose R is a Riemann surface, G acts freely and properly discontinuously by homeomorphisms on R. Then the quotient $S = G \backslash R$[a] is Hausdorff, and the quotient map $\pi : R \to S$ is a regular covering map.

[a] It is a left action, not a right action.

Proof. **S is Hausdorff.** Let $x \in \pi^{-1}(p)$ and $y \in \pi^{-1}(q)$, p, q distinct. Let U, V be disjoint relatively compact neighbourhoods of x, y respectively. Let $K = \overline{U} \cup \overline{V}$, the set of $g \in G$ such that $\overline{U} \cap g(\overline{V}) \neq \varnothing$ is a finite set $\{g_1, \ldots, g_n\}$. Since $y \neq g_i(x)$ for any i, there exists disjoint open neighbourhoods U_i of x, V_i of $g_i(y)$. Let

$$U' = U \cap \left(\bigcap_i U_i\right) \quad \text{and} \quad V' = V \cap \left(\bigcap_i g_i^{-1}(V)\right)$$

Then $U' \cap GV' = \varnothing$. Hence $\pi(U'), \pi(V')$ are disjoint open neighbourhoods of p, q.

$\pi : R \to S$ is a covering map. Same as the proof for $\mathbb{C}/\Lambda$. □

Proposition 6.4. Let R be a Riemann surface, G acts freely and properly discontinuously on R by conformal equivalences. Then $S = G \backslash R$ is a Riemann surface, $\pi : R \to S$ is analytic and a regular covering map.

Proof. Since R is connected, $S = \pi(R)$ is as well. By the previous lemma, S is Hausdorff and π is a regular covering map. Finally, the construction of a conformal structure is the same as the construction for $\mathbb{C}/\Lambda$. □

Theorem 6.5 (Hurwitz)**.** Let R be a compact Riemann surface, with $g_R \geq 2$, and suppose G acts freely and properly discontinuously on R by conformal equivalences. Then G is finite, with $|G| \leq g_R - 1$.

Proof. The quotient $S = G\backslash R$ is a Riemann surface, and the quotient map is an analytic regular covering map. In particular, it is unramified. Furthermore, by construction $\deg(\pi) = |G|$. If g_S is the genus of S, then Riemann-Hurwitz gives

$$2g_R - 2 = |G|(2g_S - 2)$$

Since the left hand side is positive, $g_S \geq 2$, and so

$$|G| \leq |G|(g_S - 1) = g_R - 1$$

□

6.2 Uniformisation

Theorem 6.6 (Uniformisation)**.** Every simply connected Riemann surface is conformally equivalent to one of

(i) the Riemann sphere $\mathbb{C}_\infty$,

(ii) the complex plane $\mathbb{C}$,

(iii) the unit disc $\mathbb{D}$

Proof. The three Riemann surfaces listed are not conformally equivalent, since $\mathbb{C}_\infty$ is compact, and $\mathbb{C}$ and $\mathbb{D}$ are not constant, by Liouville's theorem.

The rest of the proof is omitted. □

Corollary 6.7. $\mathbb{C}_\infty$ is the unique conformal structure on S^2, up to conformal equivalence.

Proof. It's simply connected and compact. □

Theorem 6.8. Every Riemann surface R has a regular covering map $\pi : \tilde{R} \to R$ such that $\tilde{R}$ is simply connected. Furthermore, there is a group G acting freely and properly discontinuously on $\tilde{R}$, and π descends to a conformal equivalence

$$G\backslash R \simeq R$$

Sketch proof. The construction of $\tilde{R}$ is the existence of a universal cover, $G = \pi_1(R) = G_D(\pi)$ is the fundamental group of R, or the deck group of the universal cover, and so π descends to a homeomorphism $G\backslash\tilde{R} \to R$. Now there is a unique conformal structure on $\tilde{R}$ which makes π analytic. Finally, in standard local coordinates on $\tilde{R}$, each $g \in G$ acts as the identity, so G acts by conformal equivalences. The induced map $G\backslash\tilde{R} \to R$ is an analytic map of degree 1, so it is a conformal equivalence. □

Corollary 6.9. Every Riemann surface R is conformally equivalent to a quotient

$$R \simeq G\backslash\tilde{R}$$

where $\tilde{R}$ is one of $\mathbb{C}_\infty$, $\mathbb{C}$ or $\mathbb{D}$, and G is a properly discontinuous group of conformal equivalences on $\tilde{R}$. We say that R is uniformised by $\tilde{R}$.

Remark 6.10. In this course, the deck group acts by conformal equivalences, i.e.

$$G = \left\{ \phi : \tilde{R} \to \tilde{R} \mid \phi \text{ is a conformal equivalence, with } \pi \circ \phi = \pi \right\}$$

6.3 Classification of Riemann surfaces

Proposition 6.11. If R is uniformised by $\mathbb{C}_\infty$, then R is conformally equivalent to $\mathbb{C}_\infty$.

Proof. The group of conformal equivalences of $\mathbb{C}_\infty$ is the Möbius group $\mathrm{PSL}_2(\mathbb{C})$. But every Möbius transformation has fixed points, so $G = 1$. □

Proposition 6.12. If R is uniformised by $\mathbb{C}$, then

(i) $G = 1$ and $R \simeq \mathbb{C}$,

(ii) $G \simeq \mathbb{Z}$ and $R \simeq \mathbb{C}_*$,

(iii) $G \simeq \mathbb{Z}^2$ and $R \simeq \mathbb{C}/\Lambda$ for some lattice Λ.

Proof. The group of conformal equivalences of $\mathbb{C}$ is the group of affine linear maps, but as the action is free, they must only be translations. Then the result follows from classification of discrete subgroups of $\mathbb{C}$. □

Lemma 6.13. Let $f : R \to S$ be an analytic map of Riemann surfaces, R is simply connected, $\pi : \tilde{S} \to S$ be the uniformising map of S. Then there is an analytic map $F : R \to \tilde{S}$ such that $f = \pi \circ F$.

Proof. Omitted. □

Proposition 6.14. A Riemann surface R is uniformised by at most one of $\mathbb{C}_\infty$, $\mathbb{C}$ and $\mathbb{D}$.

Proof. By above and Liouville. □

Proposition 6.15. The group of conformal equivalences of $\mathbb{D}$ is the group

$$e^{i\theta} \frac{z - a}{1 - az}$$

of Möbius transformations.

Definition 6.16 (Fuchsian group)
A subgroup G of $\mathrm{PSL}_2(\mathbb{R})$ acting properly discontinuous on $\mathbb{H}$ is called a Fuchsian group.

6.4 Corollaries of uniformisation

Corollary 6.17. If R is a compact Riemann surface, with $g_R \geq 2$, then R is uniformised by $\mathbb{D}$.

Corollary 6.18 (Riemann mapping)**.** If $D \subsetneq \mathbb{C}$ is a simply connected proper subdomain of $\mathbb{C}$, and D is conformally equivalent to $\mathbb{D}$.

Sketch proof. D is not compact, so D is not conformally equivalent to $\mathbb{C}_\infty$. It is also not conformally equivalent to $\mathbb{C}$ by open mapping theorem and Casaroti-Weierstrass, so D is conformally equivalent to $\mathbb{D}$ by uniformisation. □

Corollary 6.19 (Picard)**.** Any analytic function $f : \mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$ is constant.

Proof. $\mathbb{C} \setminus \{0, 1\}$ is uniformised by D. So we have an analytic map $F : \mathbb{C} \to D$ with $f = \pi \circ F$. But F is constant, by Liouville's theorem. □