Abelian Varieties

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Lecture 1

Contents

Recall that an *elliptic curve* E over a field k, which is a proper curve which is also a group. In particular, they are projective (in fact, we can write it as a nonsingular plane cubic), and that the group is commutative. If k is algebraically closed, then $E(k)$ is divisible, and for all $n \in \mathbb{N}$,

$$E[n] = \ker(n\cdot : E(k) \to E(k))$$

is finite. In particular, if n is invertible in k, then

$$E[n] = (\mathbb{Z}/n\mathbb{Z})^2$$

On the other hand, if $\operatorname{char}(k) = p > 0$, then either

$$E[p^r] = \mathbb{Z}/p^r\mathbb{Z}$$

for all $r \geq 1$, or

$$E[p^r] = 0$$

for all $r \geq 1$.

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The *Abel-Jacobi* theorem gives a homomorphism

$$E(k) \to \mathrm{Cl}^0(E) = \frac{\{\text{degree 0 divisors}\}}{\{\text{principal divisors}\}}$$

Given by $P \mapsto (P) - (0)$. In fact, by Riemann-Roch, this is an isomorphism.

Abelian varieties are "higher dimensional analogues of an elliptic curve". That is, they are a proper variety over k which is a group. If X is an abelian variety, we will see:

- X is projective and nonsingular,
- X is commutative,
- if k is algebraically closed, then $X(k)$ is divisible, with

  $$X[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$$

  where $g = \dim(X)$, n invertible in k. If $\mathrm{char}(k) = p > 0$, then

  $$X[p^r] = (\mathbb{Z}/p^r\mathbb{Z})^t$$

  with $0 \le t \le g$.
- and an analogue of the Abel-Jacobi theorem, there exists another abelian variety $\widehat{X}$, with $\dim(\widehat{X}) = g$, called the dual abelian variety, and an isomorphism

  $$\widehat{X}(k) \cong \mathrm{Pic}^0(X) \subseteq \mathrm{Pic}(X)$$

  where $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X, which is in turn, isomorphic to the divisor class group. Moreover, we have a morphism $X \to \widehat{X}$, which is surjective with finite kernel. Note that we don't need to have $X \cong \widehat{X}$.

In contrast to the one-dimensional case, we need a lot of algebraic geometry, especially the cohomology of coherent sheaves to prove these. One reason is that unlike elliptic curves, we don't have nice equations for them.

Conventions for this course: Every ring will be commutative unital, and ring homomorphisms will be unital. If $f : A \to B$ is a ring homomorphism, we say B is an A-algebra. We'll write ab for $f(a)b$. An A-module is *finite* if it is finitely generated. An A-algebra B is *finite* if it is finite as an A-module, and it is of *finite type* if it is finitely generated as an A-algebra.

A *family* $(x_i)_{i\in I}$ of elements of a set S is just $x_i \in S$, indexed by a set I. This is the same as a function $I \to S$. We'll write $\#S$ for the size of a set S, $\subset$ and $\subseteq$ will be used interchangably. Finally,

$$\mathbb{N} = \{0, 1, \dots\}$$

1 Kähler differentials

Let $\varphi : A \to B$ be a ring homomorphism. We'll define a B-module $\Omega_{B/A}$, which is the module of (Kähler) differentials, with an additive map $\mathrm{d} = \mathrm{d}_{B/A} : B \to \Omega_{B/A}$, such that

$$\mathrm{d}(b_1b_2) = b_1\mathrm{d}b_2 + b_2\mathrm{d}b_1$$
$$\mathrm{d}a = \mathrm{d}(\varphi(a)) = 0 \text{ for all } a \in A$$

In particular, d is an A-linear map.

Definition 1.1
We define $\Omega_{B/A} = \mathcal{P}/\mathcal{Q}$, where

$$\mathcal{P} = \text{free } B\text{-module on symbols } [b] \text{ for } b \in B$$
$$\mathcal{Q} = \text{submodule generated by } [a], [b_1+b_2]-[b_1]-[b_2], [b_1b_2]-b_1[b_2]-b_2[b_1]$$

and

$$\mathrm{d}b = [b] \mod \mathcal{Q}$$

An *A-derivation of B into a B-module M* is an additive map $\mathrm{D} : B \to M$, such that

$$\mathrm{D}(A) = 0$$
$$\mathrm{D}(b_1 b_2) = b_1 D(b_2) + b_2 D(b_1)$$

We will write $\mathrm{Der}_A(B, M)$ for the set of A-derivations $B \to M$. This is a B-module, with $(b'\mathrm{D})(b) = b'\mathrm{D}(b)$. Moreover, $\mathrm{Der}_A(B, \cdot)$ is a functor. That is, if we have a B-module homomorphism $f : M \to M'$, then we have a map

$$\mathrm{Der}(A, M) \to \mathrm{Der}(A, M')$$

given by $\mathrm{D} \mapsto f \circ \mathrm{D}$. In particular, $(\Omega_{B/A}, \mathrm{d}_{B/A})$ is the *universal* derivation.

Proposition 1.2. Suppose M is a B-module, then there is an isomorphism of B-modules,

$$\mathrm{Hom}_B(\Omega_{B/A}, M) \cong \mathrm{Der}_A(B, M)$$
$$\psi \mapsto \psi \circ \mathrm{d}_{B/A}$$

Proof. The fact that it is a B-module homomorphism is trivial. Let $\mathrm{D} \in \mathrm{Der}_A(B, M)$. Set $\widetilde{\psi} : \mathcal{P} \to M$ to be B-linear, with $\widetilde{\psi}([b]) = D(b)$. Since D is a derivation, $\widetilde{\psi}(Q) = 0$, so we have a map $\psi : \mathcal{P}/\mathcal{Q} = \Omega_{B/A} \to M$, with $\psi \circ \mathrm{d} = D$.

On the other hand, if $\psi \circ \mathrm{d} = 0$, then $\psi = 0$ since $\Omega_{B/A}$ is generated by the image of d. □

Lecture 2

Another characterisation of $\Omega_{B/A}$ is as follows. Consider the map $\mu : B \otimes_A B \to B$, given by $\mu(b_1 \otimes b_2) = b_1 b_2$. In particular, μ is a B-algebra homomorphism, for each of the B-algebra structues on $B \otimes_A B$. Let $J = \ker(\mu)$. Then J/J^2 is a $B \otimes_A B$-module, killed by J, so it is a B-module. In particular, all of the B-module structures on it agree.

Let

$$\mathrm{d}' : B \to J/J^2$$
$$b \mapsto (1 \otimes b - b \otimes 1) \mod J^2$$

Then d' is an A-derivation. To see this,

$$\mathrm{d}'(a) = 1 \otimes a - a \otimes 1 = 0$$

for all $a \in A$. Next,

$$b_1 \mathrm{d}'(b_2) = (b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1) \mod J^2$$
$$= b_1 \otimes b_2 - b_1 b_2 \otimes 1 \mod J^2$$

and by symmetry,

$$b_2 \mathrm{d}'(b_1) = 1 \otimes b_1 b_2 - b_1 \otimes b_2 \mod J^2$$

Adding these together gives the Leibniz rule. By the universal property, there exists a unique B-linear map $\psi : \Omega_{B/A} \to J/J^2$, such that $\psi \circ \mathrm{d} = \mathrm{d}'$.

Proposition 1.3. ψ is an isomorphism.

So we could have defined $(\Omega_{B/A}, \mathrm{d})$ as $(J/J^2, \mathrm{d}')$ instead.

Proof. Consider the map

$$\varphi : B \otimes_A B \to \Omega_{B/A}$$
$$\varphi(b_1 \otimes b_2) = b_1 \,\mathrm{d}(b_2)$$

Now

$$J = \left\{ \sum b_i \otimes b_i' \;\middle|\; \sum b_i b_i' = 0 \right\}$$

We claim that J is generated (as a B-module) by elements of the form $1 \otimes b - b \otimes 1$. To see this,

$$\sum b_i \otimes b_i' = \sum (b_i \otimes 1)(1 \otimes b_i' - b_i' \otimes 1) + \sum b_i b_i' \otimes 1$$

and the last term is zero. So it is a B-module under the $b \otimes 1$ action. Now

$$\varphi((1 \otimes b - b \otimes 1)(1 \otimes b' - b' \otimes 1)) = 0$$

and so φ vanishes on J^2, and it factors through $(B \otimes_A B)/J^2$. It remains to check that $\varphi \circ \psi = \mathrm{id}$ and $\psi \circ \varphi|_{J/J^2} = \mathrm{id}$. We leave this as an exercise. □

Remark 1.4. Let $C = (B \otimes_A B)/J^2$, then we have

$$\begin{array}{ccccc} B & \xrightarrow{\cong} & B \otimes_A A & \hookrightarrow & C \\ & \searrow^{\mathrm{id}} & & & \downarrow \mu \\ & & & & B \end{array}$$

This gives us an isomorphism of $B = B \otimes_A A$-algebras,

$$B \oplus J/J^2 \cong C$$

The ring structure on the left hand side is given by

$$(b, f) \cdot (b', f') = (bb', bf' + b'f)$$

From the definition, any commutative square of rings

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

induces a map

$$\Omega_{B/A} \to \Omega_{B'/A'}$$

which is B-linear, commuting with d. Moreover, this is transitive for $B \to B' \to B''$.

Proposition 1.5. (i) If $B' = B \otimes_A A'$, then

$$\Omega_{B/A} \otimes_A A' \cong \Omega_{B/A} \otimes_B B' \cong \Omega_{B'/A'}$$

(ii) if $A = A'$, and $B' = S^{-1}B$ is a localisation of B, then

$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A}$$

Proof. Exercise. □

Example 1.6

If $B = A[t_1, \ldots, t_n]$ is the polynomial algebra, then

$$\Omega_{B/A} = \oplus_{i=1}^n B \mathrm{d}t_i$$

is free. To see this,

$$B \otimes_A B = A[\{t_i \otimes 1, 1 \otimes t_j\}] = A[\{t_i \otimes 1, z_i\}] = B[\{z_i\}]$$

where $z_i = 1 \otimes t_i - t_i \otimes 1$. Using this, the map $\mu : B \otimes_A B \to B$ is just $z_i \mapsto 0$ and the identity map on B. So we have that

$$J = \langle z_1, \ldots, z_n \rangle$$

and

$$J^2 = \langle z_i z_j \rangle$$

Thus, J/J^2 is the free module

$$\bigoplus_i B \cdot (z_i \mod J^2)$$

But $z_i \mod J^2$ is just $\mathrm{d}'(z_i) = \mathrm{d}t_i$.

We have two exact sequences for Ω:

Proposition 1.7 (transitivity). if $A \to B \to C$ are ring homomorphisms, then

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

is exact.

Proposition 1.8 (second exact sequence). If $A \to B \twoheadrightarrow C$ are ring homomorphisms, with say $C = B/I$, then

$$I/I^2 \longrightarrow \Omega_{B/A} \otimes C \longrightarrow \Omega_{C/A} \longrightarrow 0$$

is exact, where $f \mod I^2 \mapsto \mathrm{d}f \otimes 1$. This is an exact sequence of C-modules.

Proof of both have been left as exercises.

Corollary 1.9. If $B = A[\{x_i\}]$, and $C = B/\langle \{f_j\} \rangle$, then

$$\Omega_{C/A} = \frac{\bigoplus_i C\mathrm{d}x_i}{\sum C\mathrm{d}f_j}$$

In this case,

$$\mathrm{d}f_j = \sum_i \frac{\partial f_j}{\partial x_i} \mathrm{d}x_i$$

1.1 Sheafification

Let $f : X \to Y$ be a morphism of schemes. We would like to define a quasicoherent $\mathcal{O}_X$-module $\Omega_{X/Y}$, and a map

$$\mathrm{d} : \mathcal{O}_X \to \Omega_{X/Y}$$

which is $f^{-1}\mathcal{O}_Y$-linear, and satisfies the Leibniz rule for sections. This is called the *sheaf of relative differentials*, or the *cotangent sheaf*. There are two constructions, let's see the 'dirty' one first.

If we have open affines $U = \mathrm{Spec}(B) \subseteq X$, $V = \mathrm{Spec}(A) \subseteq Y$, with $f(U) \subseteq V$, then we can define

$$\Omega_{X/Y}|_U = \widetilde{\Omega_{B/A}}$$

where $\widetilde{\Omega_{B/A}}$ is the quasicoherent $\mathcal{O}_U$-module attached to the B-module $\Omega_{B/A}$. The map d is induced by

$$\begin{array}{ccc} \mathcal{O}_X(U) & \xrightarrow{\ \mathrm{d}\ } & \Omega_{X/Y}(U) \\ \cong \downarrow & & \downarrow \cong \\ B & \xrightarrow[\ \mathrm{d}\ ]{} & \Omega_{B/A} \end{array}$$

Functoriality means that we can glue to get a quasicoherent sheaf on X. We will omit the details.

For the second construction, consider the diagonal map $\Delta : X \to X \times_Y X$. This map is an immersion, and so it factors as

$$\begin{array}{ccc} X & \longrightarrow & X \times_Y X \\ & {\scriptstyle \text{closed}} \searrow \quad \nearrow {\scriptstyle \text{open}} & \\ & U & \end{array}$$

Let $i : X \hookrightarrow U$ be the closed immersion. Define

$$\Omega_{X/Y} = i^* \left(\mathcal{I}_{X/U}/\mathcal{I}_{X/U}^2\right)$$

where $\mathcal{I}_{X/U}$ is the ideal sheaf. When $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$, this is just $\widetilde{\Omega_{B/A}}$.

Lecture 3

Note that we also have that $\Omega_{X/Y} = i^*\mathcal{I}_{X/U}$, since

$$i^*\mathcal{I}_{X/U} = \mathcal{O}_X \otimes i^{-1}\mathcal{I}_{X/U} = i^{-1}(\mathcal{O}_U/\mathcal{I}_{X/U}) \otimes i^{-1}\mathcal{I}_{X/U}$$

If $X \to Y$ is separated, then we can take $U = X \times_Y X$.

We will define $\mathrm{d} : \mathcal{O}_X \to \Omega_{X/Y}$ by: if we have open affines $V = \mathrm{Spec}(B) \subseteq X$, $U = \mathrm{Spec}(A) \subseteq Y$, with $f(V) \subseteq U$, then

$$\mathrm{d} : \mathcal{O}_X(V) = B \to \Omega_{B/A} = \Omega_{X/Y}(V)$$

is defined. Moreover, this is compatible with restriction to smaller open affines. The set of open affines $\mathrm{Spec}(B) \subseteq X$ such that the image is contained in an open affine in Y, forms a basis for the topology on X. Thus, this extends to a map of sheaves.

Amusing fact: There exists an immersion $X \hookrightarrow Z$ (i.e. a closed immersion then an open immersion), which cannot be factored as an open immersion, then a closed immersion.

We can rewrite the exact sequences, in the setting of sheaves.

- if we have morphisms of schemes $f : X \to Y, g : Y \to S$, then we have

$$f^*\Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

- If $Z \hookrightarrow X$ is a closed immersion, with ideal sheaf $\mathcal{I}$, and $f : X \to Y$ a morphism of sheaves, then we have the exact sequence

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow i^*\Omega_{X/Y} \longrightarrow \Omega_{Z/Y} \longrightarrow 0$$

By construction, $\Omega_{X/Y}$ is quasicoherent.

Proposition 1.10. If $f : X \to Y$ is of finite type, and X and Y are Noetherian, then $\Omega_{X/Y}$ is coherent[a].

[a] i.e. affine locally it is the sheaf associated to a finite module.

Proof. Locally, we can cover X with $\mathrm{Spec}(B)$, such that $f(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A)$, and this makes B a finite type A-algebra. That is,

$$B = \frac{A[t_1, \dots, t_m]}{I}$$

where $I = \langle f_1, \dots, f_n \rangle$. The second exact sequence is

$$I/I^2 \longrightarrow \Omega_{A[\mathbf{t}]/A} \otimes_{A[\mathbf{t}]} B \longrightarrow \Omega_{B/A} \longrightarrow 0$$

But note that

$$\Omega_{A[\mathbf{t}]/A} \otimes_{A[\mathbf{t}]} B = \bigoplus_i B\mathrm{d}t_i$$

and so $\Omega_{B/A}$ is a finite B-module, and so $\Omega_{X/Y}$ is coherent. □

For field extensions, Ω tells us whether the extension is separable.

Example 1.11

Let L/K be a finite field extension. Then L/K is separable if and only if $\Omega_{L/K} = 0$. To see this, factor the extension as $K \subseteq K_1 \subseteq L$, with K_1/K separable, and L/K_1 purely inseparable. That is, K_1 is the set of all elements of L which is separable over K.

By the primitive element theorem, $K_1 = K(\alpha) = K[t]/\langle g \rangle$, where $g \in K[t]$ is irreducible, $g(\alpha) = 0$ and $g'(\alpha) \neq 0$. So

$$\Omega_{K_1/K} = \frac{K_1\mathrm{d}\alpha}{K_1 g'(\alpha)\mathrm{d}\alpha}$$

by corollary 1.9. But this is zero as $g'(\alpha) \neq 0$. In this case, transitivity gives us $\Omega_{L/K} = \Omega_{L/K_1}$. If $L = K_1$, i.e. L/K is separable, then $\Omega_{L/K_1} = 0$.

Conversely, if L/K is inseparable, then there exists K_2 with $K_1 \subseteq K_2 \subseteq L$, such that $L = K_2(\beta) = K_2[t]/\langle t^p - b\rangle$, with $p = \mathrm{char}(K) > 0$, $b \in K_2$ is not a p-th power. In this case.

$$\Omega_{L/K_2} = L\mathrm{d}\beta$$

since in this case, $\frac{\mathrm{d}}{\mathrm{d}t}(t^p - b) = 0$. Thus, by transitivity again, $\Omega_{L/K} \neq 0$.

1.2 Tangent and cotangent spaces

Let X be a k-scheme, locally of finite type. That is, X has an open cover by affines $\mathrm{Spec}(A)$, where A is a finite type k-algebra. Let $|X|$ denote the set of closed points of X. If $x \in |X|$, then the residue field at k,

$$\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$$

is a finite extension of k. Define

$$\Omega_{X/k}(x) = \Omega_{X/k,x} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

This is a finite dimensional $\kappa(x)$ vector space. This follows from the fact that $\Omega_{X/k}$ is coherent. We call this the *cotangent space of X at x*.

The corresponding *tangent space* is

$$\mathrm{T}_{X,x} = \mathrm{Hom}_{\kappa(x)}(\Omega_{X/k}(x), \kappa(x))$$

Proposition 1.12. If $\kappa(x) = k$, then there exists a canonical isomorphism

$$\Omega_{X/k}(x) \cong \mathfrak{m}_x/\mathfrak{m}_x^2$$

of k-vector spaces.

Proof. In this case,

$$\Omega_{X/k}(x) = \Omega_{\mathcal{O}_{X,x}/k} \otimes_{\mathcal{O}_{X,x}} k$$

The second exact sequence gives us that

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \Omega_{\mathcal{O}_{X,x}/k} \otimes k \longrightarrow \Omega_{k/k} = 0 \longrightarrow 0$$

It suffices to show that the first map is injective, which in turn, means that it suffices to show that the dual

$$\mathrm{Hom}_{\mathcal{O}_{X,x}/k}(\Omega_{\mathcal{O}_{X,x}/k}, k) \to \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$$

is surjective. The left hand side can be identified with

$$\mathrm{Der}_k(\Omega_{X,x}, k)$$

and so the map is given by

$$\mathrm{D} \mapsto \left(f + \mathfrak{m}_x^2 \mapsto \mathrm{D}(f) \mod \mathfrak{m}\right)$$

Since $\mathrm{D}(\mathfrak{m}_x^2) = 0$ by Leibniz. As $\mathcal{O}_{X,x}/\mathfrak{m}_x = k \hookrightarrow \mathcal{O}_{X,x}$, we must have that

$$\mathcal{O}_{X,x}/\mathfrak{m}^2 = k \oplus \mathfrak{m}_x/\mathfrak{m}_x^2$$

as k-vector spaces. Let $\pi : \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ be the projection. For $\varphi : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$, the map $\mathrm{D} = \varphi \circ \pi$ is a derivation which maps to φ.

To show that it is a derivation, recall the ring structure on $k \oplus \mathfrak{m}_x/\mathfrak{m}_x^2$ from before. In particular,

$$(a, b)(a', b') = (aa', ab' + a'b)$$

□

Lecture 4

Remark 1.13. THe statement holds more generally, if $\kappa(x)$ is separable over k.

This leads to a "geometric" interpretation of the tangent space.

Proposition 1.14. Let $x \in |X|$, X locally of finite type over k, and suppose $\kappa(x) = k$. Then

$$\mathrm{T}_{X,x} \cong \left\{ \text{morphisms of } k\text{-schemes Spec} \left(\frac{k[\varepsilon]}{\langle \varepsilon^2 \rangle} \right) \to X \text{ whose image is } X \right\}$$

The ring $k[\varepsilon]/\left\langle \varepsilon^2 \right\rangle$ is called the ring of *dual numbers*, and $\mathrm{Spec}(k[\varepsilon]/\left\langle \varepsilon^2 \right\rangle) = \{\langle \varepsilon \rangle\}$. The map $\mathrm{Spec}(k) \to \mathrm{Spec}(k[\varepsilon]/\left\langle \varepsilon^2 \right\rangle)$ is given by sending ε to 0. This is a closed immersion.

Morally, $\mathrm{Spec}(k[\varepsilon]/\left\langle \varepsilon^2 \right\rangle)$ is "a point with a direction".

Proof. Giving such a morphism is equivalent to giving a local homomorphism

$$\varphi : \mathcal{O}_{X,x} \to \frac{k[\varepsilon]}{\langle \varepsilon^2 \rangle}$$

of k-algebras. But $\mathcal{O}_{X,x} = k \oplus \mathfrak{m}_x$, since k is the residue field. So φ is determined by its restriction to $\mathfrak{m}$, which is a k-linear map $\mathfrak{m}_x \to k\varepsilon \subseteq k[\varepsilon]/\left\langle \varepsilon^2 \right\rangle$ (as it is a *local map*).

Since $\varepsilon^2 = 0$, φ is zero on $\mathfrak{m}_x^2$. So it is a k-linear map

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to k\varepsilon \cong k$$

So the set of such morphisms is isomorphic to

$$\mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k) \cong \mathrm{Hom}_k(\Omega_{X/k}(x), k) = \mathrm{T}_{X,x}$$

□

As above, this is also true when $\kappa(x)/k$ is separable.

Aside: In the proofs of the above, we used the fact that if $\kappa(x) = k$, then

$$\mathcal{O}_{X,x} = k \oplus \mathfrak{m}_x$$

If $\kappa(x) \neq k$, in general, $\mathcal{O}_{X,x}$ will not be a $\kappa(x)$-algebra. But it is the case (at least) when $\kappa(x)/k$ is separable. But

$$\frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^2} \cong \kappa(x) \oplus \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$$

That is, there is a field inside $\mathcal{O}_{X,x}/\mathfrak{m}_x^2$, which maps isomorphically onto $\kappa(x)$. The proof is just Hensel's lemma, write $\kappa(x) = k(\alpha)$, with $g(\alpha) = 0$. Lift $g(\alpha)$ to a solution of $g(t) \equiv 0 \mod \mathfrak{m}^2$.

As an example, let $X = \mathbb{A}_1^{\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[t])$, and let $x = \left\langle t^2 + 1 \right\rangle \in X$. Then $\kappa(x) = \mathbb{Q}(i)$. But $\mathbb{Q}(i)$ can't be a subfield of the local ring

$$\mathcal{O}_{X,x} \subseteq \mathbb{Q}(t)$$

But the completion

$$\varprojlim_n \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x^n}$$

does contain a copy of $\kappa(x)$.

1.3 Nonsingular varieties

Let X be an integral scheme of finite type over a field k. Let $d = \dim(X)$.

We say that X is *smooth* over k if $\Omega_{X/k}$ is locally free of rank d. If k is algebraically closed, then this is equivalent to the other conditions:

Theorem 1.15. Suppose X is an integral scheme of dimension d, and of finite type over an algebraically closed field k. Then the following are equivalent.

(i) X is smooth over k,

(ii) for all $x \in |X|$, $\dim_k(\mathrm{T}_{X,x}) = d$,

(iii) for all $x \in |X|$, the local ring $\mathcal{O}_{X,x}$ is *regular*[a] (of dimension d).

[a]A Noetherian local domain R of dimension d is regular if it's maximal ideal $\mathfrak{m}$ can be generated by d elements.

Proof. We know $\dim_k(\mathrm{T}_{X,x}) = \dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2)$.

To show (ii) and (iii) are equivalent, note that

$$\mathfrak{m}_x/\mathfrak{m}_x^2 = \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} k$$

and $\mathfrak{m}_x$ is a finite $\mathcal{O}_X$-module. Nakayama's lemma implies that if $t_1, \dots, t_n \in \mathfrak{m}_x$, then $\mathfrak{m}_x = \langle t_1, \dots, t_n \rangle$ if and only if $t_i + \mathfrak{m}_x^2$ generate $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a k-vector space.

To show (i) implies (ii),

$$\mathrm{T}_{X,x} = \mathrm{Hom}_k(\Omega_{X/k}(x), k)$$

this has dimension d as $\Omega_{X/k}$ is locally free of rank d. For (ii) implies (i), let K be the function field[1] of X. Now as k is algebraically closed, K is a finite separable extension of $K_0 = k(t_1, \dots, t_d)$, for $t_i \in K$ algebraically independent over k. Hence

$$\Omega_{K/k} = \Omega_{K_0/K} \otimes_{K_0} K$$

and so

$$\Omega_{K_0/k} = \bigoplus_i K_0 \mathrm{d}t_i$$

Thus, $\dim_K(\Omega_{K/k}) = d$. But

$$\Omega_{K/k} = \Omega_{X/k,\eta}$$

Lemma 1.16. Let X be as above.

(i) $\mathcal{F}$ a coherent sheaf $\mathcal{O}_X$-module, such that for all closed points $x \in |X|$, $\dim_k(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k)$ is the same as $\dim_k(\mathcal{F}_\eta) = n$, then $\mathcal{F}$ is locally free.

(ii) In general, there exists a non-empty open subscheme $U \subseteq X$, such that for all $x \in |X|$, $\mathcal{F}_x$ is free if and only if $x \in U$.

Remark 1.17. There are two notions of locally free for any A-module M. The first is that $M_\mathfrak{p}$ is free for all $\mathfrak{p} \in \mathrm{Spec}(A)$. The second is that $\widetilde{M}$ is a locally free sheaf on $\mathrm{Spec}(A)$.

These are not equivalent in general. For example, $A = \mathbb{Z}$ and

$$M = \left\{ \frac{m}{n} \;\middle|\; n \text{ squarefree} \right\} \subseteq \mathbb{Q}$$

$M_\mathfrak{p}$ is free, but $\widetilde{M}$ is not locally free.

In the course, we will always use the second one. If A is Noetherian, and M is finite, then the two are equivalent.

Lecture 5

Proof of lemma. For both of these statements, we can assume that $X = \mathrm{Spec}(A)$ is affine, and $\mathcal{F} = \widetilde{M}$, for a finite A-module M. $K = \mathrm{Frac}(A)$, so $n = \dim_K(M \otimes_A K)$.

Let $\mathfrak{m} \in \mathrm{maxSpec}(A)$ be a maximal ideal. By assumption,

$$n = \dim_K(M \otimes (A/\mathfrak{m}))$$

[1]i.e. the local ring at the unique generic point.

Choose $t_1, \dots, t_n \in M$, whose images in $M/\mathfrak{m}M$ form a basis. Now consider the natural map[2] $A^n \to M$ from this basis. By Nakayama, the localisation

$$A^n_{\mathfrak{m}} \to M_{\mathfrak{m}}$$

is surjective. So the map $K^n \to M \otimes_A K$ is surjective, so it is an isomorphism. Hence the above map is an isomorphism.

We need to show that there exists a nonempty open $U \subseteq \operatorname{Spec}(A)$, such that the map

$$\mathcal{O}_U^n \to \widetilde{M}|_U$$

is an isomorphism. We know that the map $A^n \to M$ is injective, and has finite torsion cokernel N say, since it is an isomorphism after tensoring with K. Then the map

$$\mathcal{O}_U^n \to \widetilde{M}|_U$$

is an isomorphism for $U = X \setminus \mathbb{V}(\operatorname{Ann}(N))$. Now note that $\operatorname{Ann}(N)$ is non-zero, so U is non-empty. Moreover, $\operatorname{Ann}(M) \not\subseteq \mathfrak{m}$, so $\mathfrak{m} \in U$.

For (ii), choose $t_i \in M$ such that the map $K^n \to M \otimes_A K$ is an isomorphism. The corresponding map $A^n \to M$ has finite torsion cokernel as above, and so at a maximal ideal $\mathfrak{m}$, we have an isomorphism $A^n_{\mathfrak{m}} \to M_{\mathfrak{m}}$ if and only if $\mathfrak{m} \notin \mathbb{V}(\operatorname{Ann}(\operatorname{coker}(\cdot)))$. ☐

☐

Remark 1.18. The theorem holds more generally, for k perfect. But it fails for general k. Let $X = \operatorname{Spec}(K)$, where $K = k(t^{1/p})$, $k = \mathbb{F}_p(t)$. Here,

$$\Omega_{K/k} = K(t)\,\mathrm{d}\left(t^{1/p}\right)$$

is non-zero, but K is clearly regular (of dimension 0).

Thus, over non-perfect fields, regular and smoothness are not equivalent.

Theorem 1.19. Let k be algebraically closed, X/k an integral scheme of finite type over k, then there exists a maximal non-empty open $U \subseteq X$ which is smooth over k.

Proof. As in the proof of (ii) implies (i) in the previous theorem, we know that

$$\Omega_{X/k,\eta} \cong \kappa(X)^d$$

The second part of the lemma says that there exists a non-empty open U, such that for all $x \in |X|$, $x \in U$ if and only if $\Omega_{X/k,x}$ is free of rank d. But by the theorem, this is the same as

$$\dim_k(T_{X,x}) = d$$

☐

1.4 Digression - some general (nice) properties of schemes

Let X be a scheme. Recall:

- X is *quasicompact* if X is a finite union of open affines.

  Example 1.20 (non-example)

  Let $X = \mathbb{A}_k^\infty = \operatorname{Spec}(k[x_1, x_2, \dots])$, and $U = X \setminus \{0\}$. This is not quasicompact.

- X is *separated* if the diagonal map is closed. In this case, the intersection of open affines is an open affine.

[2] i.e. $(a_1, \dots, a_n) \mapsto a_1 t_1 + \dots + a_n t_n$.

Example 1.21 (non-example)

The bug-eyed plane is not separated.

- (for completeness) X is *quasi-separated* if the intersection of any two open affines is quasicompact.

Example 1.22 (non-example)

Let X be the union of two copies of $\mathbb{A}^\infty_k$, glued on the complement of the origin.

Sometimes, people will say "X is *qcqs*", to say X is quasicompact and quasiseparated.
If $f : X \to Y$ is a morphism,

- it is *quasicompact*, if for all open affines $V \subseteq Y$, $f^{-1}(V)$ is quasicompact,
- it is *locally of finite type*, if for all $x \in X$, there exists an open affine neighbourhoods $x \in U = \operatorname{Spec}(B)$, and $f(U) \subseteq V = \operatorname{Spec}(A)$, where B is a finite type A-algebra.
- it is of *finite type* if it is quasicompact and locally of finite type.
- (for completeness) it is *locally of finite presentation* if locally it is $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$, where B is a finitely presented A-algebra. So
$$B = \frac{A[x_1, \dots, x_n]}{\langle f_1, \dots, f_m \rangle}$$
When A is Noetherian, this is the same as locally of finite type. That is, when Y is locally Noetherian. In this case, f is *of finite presentation* if moreover it is quasicompact and quasiseparated.

2 Flatness and related notions

Let A be a ring. An A-module M is *flat over* A if for every injection $N \to N'$ of A-modules, the corresponding map $M \otimes_A N \to M \otimes_A N'$ is injective.

We say that an A-algebra B is *flat* if it is flat as an A-module.

Proposition 2.1. Some facts:

(i) A is flat over itself,

(ii) any free module is flat, so vector spaces over a field are flat,

(iii) M is flat if and only if for all ideals $I \trianglelefteq A$, the map $M \otimes_A I \to M$ is injective,

(iv) M is flat if and only if for every short exact sequence
$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$
The corresponding sequence
$$0 \longrightarrow M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$
is still exact. That is, $M \otimes_A \cdot$ is an *exact functor*. Note that it is always exact except at $M \otimes_A N_1$. So tensoring is a *right exact functor*.

(v) if M is A-flat, $A \to B$ is a ring homomorphism, then $M \otimes_A B$ is B-flat. This follow from the fact that
$$(M \otimes_A B) \otimes_B N = M \otimes_A N$$

(vi) M is A-flat if and only if $M_\mathfrak{p}$ is $A_\mathfrak{p}$-flat for all $\mathfrak{p} \in \operatorname{Spec}(A)$. This follows from the fact that A-modules are the same as quasicoherent sheaves on $\operatorname{Spec}(A)$, and exactness of sheaves is equivalent to exactness on stalks.

(vii) if A is a PID (or more generally, a Dedekind domain), then M is flat if and only if it is torsion-free. By the above, we can reduce to the case of $M_\mathfrak{p}$ being $A_\mathfrak{p}$-flat. But $A_\mathfrak{p}$ is (a field or) a discrete valuation ring. The result then follows from (iii), since in a DVR, $I = \langle \pi^n \rangle$ for some n.

(viii) Let A be Noetherian, M a finite A-module. Then the following are equivalent.

- M is flat,
- M is projective,
- $M_\mathfrak{p}$ is free over $A_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$,
- $\widetilde{M}$ is locally free. That is, there exists $f_1, \dots, f_n \in A$, such that $\langle f_1, \dots, f_n \rangle = A$, and M_{f_i} is free over A_{f_i} for all i.

So over a Noetherian ring, finite flat is the same as being finite and locally free.

(ix) Since localisation is exact, it preserves flatness.

Lecture 6

Proposition 2.2. Let

$$\cdots \longrightarrow K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \longrightarrow \cdots$$

be a complex of A-modules. Now suppose we have an A-module M, then we have a natural map

$$\mathrm{H}^n(K^\bullet, d^\bullet) \otimes_A M \to \mathrm{H}^n(K^\bullet \otimes_A M, \mathrm{d}^\bullet \otimes \mathrm{id})$$

which is an isomorphism if M is flat.

Proof. The sequence

$$0 \longrightarrow \ker(\mathrm{d}^n) \hookrightarrow K^n \longrightarrow K^{n+1}$$

is exact, so we get

$$\ker(\mathrm{d}^n) \otimes_A M \to \ker(\mathrm{d}^n \otimes \mathrm{id}_M) \subseteq K^n \otimes_A M \tag{$*$}$$

and we have

$$K^{n-1} \xrightarrow{\mathrm{d}^{n-1}} \ker(\mathrm{d}^n) \longrightarrow \mathrm{H}^n(K^\bullet, d^\bullet) \longrightarrow 0$$

which is also exact. So the rows of

$$\begin{array}{ccccccc} K^{n-1} \otimes_A M & \longrightarrow & \ker(\mathrm{d}^n) \otimes_A M & \longrightarrow & \mathrm{H}^n(K) \otimes_A M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K^{n-1} \otimes_A M & \longrightarrow & \ker(\mathrm{d}^n \otimes_A \mathrm{id}_M) & \longrightarrow & \mathrm{H}^n(K \otimes_A M) & \longrightarrow & 0 \end{array} \tag{$**$}$$

are exact. Hence we have an induced map $\mathrm{H}^n(K) \otimes M \to \mathrm{H}^n(K \otimes M)$. If M is flat, then the map $\ker(\mathrm{d}_n) \otimes_A M \to \ker(\mathrm{d}^n \otimes_A \mathrm{id}_M)$ is an isomorphism. Hence by the five lemma, the map $\mathrm{H}^n(K) \otimes_A M \to \mathrm{H}^n(K \otimes_A M)$ is an isomorphism. $\square$

Returning to (algebraic) geometry: Let $f : X \to Y$ be a morphism of schemes. We say that f is *flat* if one of the following (equivalent) conditions are satisfied:

- locally, f is of the form $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$, where B is a flat A-algebra,
- for all $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$.

More generally, a queasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat over* Y if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module.

Example 2.3

Open immersions are flat, since locally, they are isomorphisms. Closed immersions are in general, not flat.

Let X, Y be k-schemes. Then

$$X \times_{\mathrm{Spec}(k)} Y \to Y$$

is flat. Morally, flatness is saying that "the fibres vary continuously".

See more in section in Hartshorne on flat morphisms, especially Example III.9.8.4, of a particular flat family of curves.

Fact: Let $f : X \to Y$ be a flat morphism of integral k-schemes of finite type. Then for all $y \in Y$, and every irreducible component $Z \subseteq f^{-1}(y)$, $\dim(Z)$ is independent of choice of y, Z, and is equal to $\dim(X) - \dim(Y)$.

Example 2.4

Let $Y = \mathbb{P}^2_k$, and let X be the blowup of Y at a point $y \in Y(k)$[a]. Then for all $y' \neq y$, $f^{-1}(y')$ is a point, as we have an isomorphism $X \setminus f^{-1}(y) \cong Y \setminus \{y\}$. But $f^{-1}(y) = \mathbb{P}^1_k$. So f is not flat.

[a] $Y(k) = \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k), Y)$ is the set of k-points of Y.

As an example of blowup for $\mathbb{A}^2$, consider

$$\{(x_0, x_1), (y_0 : y_1) \mid x_0 y_1 = x_1 y_0\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$$

and the projection onto the $\mathbb{A}^2$-factor. Away from the origin, the fibre is a line. At the origin, the fibre is a $\mathbb{P}^1$.

Lecture 7

One way to think about this is that fibre dimension is constant across a flat family.

Example 2.5

Let $Y = \mathrm{Spec}(k[t])$, and take $X = \mathrm{Spec}(k[x, y, t]/\langle xy - t\rangle)$. We have a ring map $k[t] \to k[x, y, t]/\langle xy - t\rangle$, so we have a scheme map $X \to Y$.

When a is non-zero, the fibre is the (smooth irreducible) conic $xy = a$. When $a = 0$, then we have the singular curve $xy = 0$, and the fibre is reducible.

If instead we considered $X = \mathrm{Spec}(k[x, y, t]/\left\langle x^2 - ty\right\rangle)$. When $t = 0$, we have $x^2 = 0$, which is a non-reduced scheme.

We have the very useful result:

Theorem 2.6 ((a special case of) miracle flatness)**.** Let X, Y be integral k-schemes of finite type and smooth over k. Let $f : X \to Y$ be a morphism. Suppose that for all $y \in Y$, and for all irreducible components $Z \subseteq f^{-1}(y)$, $\dim(Z) = \dim(X) - \dim(Y)$. Then f is flat.

This is actually true under much weaker hypotheses on X. See Eisenbud Commutative Algebra for a proof.

Fact: Let $f : X \to Y$ be flat, and locally of finite presentation (for example, if X, Y are Noetherian, then f just has to be flat of finite type). Then f is an open map.

Note that a finiteness condition is necessary, for example, if $X = \mathrm{Spec}(\mathbb{Q}) \hookrightarrow \mathrm{Spec}(Z) = Y$, then this is flat but not open.

3 Sheaf cohomology

In this section, we will assume (unless stated otherwise), all schemes are Noetherian and separated. In particular, they are quasicompact.

3.1 Homological algebra

Recall a *(cochain) complex of abelian groups (or R-modules, or sheaves)* is a sequence of maps $(A^\bullet, \mathrm{d})$

$$A^0 \xrightarrow{\mathrm{d}} A^1 \xrightarrow{\mathrm{d}} \cdots$$

which may or may not be infinite, such that $\mathrm{d}^2 = 0$. Associated to a (cochain) complex is the *cohomology*, which is

$$\mathrm{H}^p(A^\bullet) = \frac{\ker(\mathrm{d} : A^p \to A^{p+1})}{\mathrm{im}(\mathrm{d} : A^{p-1} \to A^p)}$$

We write

$$\mathrm{H}^*(A^\bullet) = \bigoplus_p \mathrm{H}^p(A^\bullet)$$

for the graded R-module. A *morphism of complexes* $f : (A, \mathrm{d}) \to (B, \mathrm{d})$ is a family of maps $f^p : A^p \to B^p$, such that

$$\mathrm{d}f = f\mathrm{d}$$

This induces homomorphisms

$$\mathrm{H}^p(f) : \mathrm{H}^p(A^\bullet) \to \mathrm{H}^p(B^\bullet)$$

Suppose

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

is an exact sequence of complexes, then we have homomorphisms $\partial : \mathrm{H}^p(C^\bullet) \to \mathrm{H}^{p+1}(A^\bullet)$, such thats

$$0 \longrightarrow \mathrm{H}^0(A) \longrightarrow \mathrm{H}^0(B) \longrightarrow \mathrm{H}^0(C) \longrightarrow \mathrm{H}^1(A) \longrightarrow \mathrm{H}^1(B) \longrightarrow \cdots$$

is exact.

Suppose $f, g : A \to B$ are morphisms of chain complexes. They are *homotopic* if there exists maps $h^p : A^p \to B^{p-1}$, such that

$$\mathrm{d}h + h\mathrm{d} = f - g$$

If f and g are homotopic, then they induce the same map on cohomology. In particular, if $A = B$, and $h^p : A^p \to A^{p-1}$ are such that

$$\mathrm{d}h + h\mathrm{d} = \mathrm{id}$$

Then the identity and zero maps on cohomology are equal, so the cohomology is zero. If such an h exists, we say that A is null-homotopic.

Example 3.1

Let A be a cochain complex of abelian groups, $n \geq 0$. Let $\mathbb{Z}[-n]$ be the complex which is $\mathbb{Z}$ in degree n, and zero otherwise. Then the set of morphisms $\mathbb{Z}[-n] \to A$ is the same as the set of maps $f : \mathbb{Z} \to A^n$, with $\mathrm{d}f = 0$. But any such map is of the form $f(k) = kx$ for some $x \in \ker(\mathrm{d} : A^p \to A^{p+1})$. Moreover, the homotopy classes of such maps is the above quotiented by $\mathrm{im}(\mathrm{d} : A^{p-1} \to A^p)$, i.e. $\mathrm{H}^n(A)$. So we obtain the identification

$$\mathrm{H}^n(A) = \{\text{homotopy classes of morphisms } \mathbb{Z}[-n] \to A\}$$

In general, we can have that $\mathrm{H}^*(A) = 0$, with A not null-homotopic. For example, consider

$$\mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

This is exact, but not null-homotopic.

If A is a complex of vector spaces over a field k, then we can always "split off" the cohomology as a direct summand of A^n. In particular, if $\mathrm{H}^n(A) = 0$ for all n, then we can write A as

$$B^0 \longrightarrow B^0 \oplus B^1 \longrightarrow B^1 \oplus B^2 \longrightarrow \cdots$$

The obvious map $h^p : B^{p-1} \oplus B^p \to B^{p-2} \oplus B^{p-1}$ satisfies

$$\mathrm{d}h + h\mathrm{d} = \mathrm{id}$$

and so for complexes of k-vector spaces, $\mathrm{H}^* = 0$ is equivalent to being null-homotopic.

As a special case of the long exact sequence, if we have long exact sequences of length 2, so $A \to A'$ and so on.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

Then the long exact sequence is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(\alpha) & \longrightarrow & \ker(\beta) & \longrightarrow & \ker(\gamma) \\ & \swarrow & & & & & \\ \operatorname{coker}(\alpha) & \longrightarrow & \operatorname{coker}(\beta) & \longrightarrow & \operatorname{coker}(\gamma) & \longrightarrow & 0 \end{array}$$

It is very easy to see that this holds with some of the zeroes removed. This is the *Snake lemma*.

Lecture 8

Suppose $(A^\bullet, \mathrm{d}_A), (B^\bullet, \mathrm{d}_B)$ are complexes of R-modules. We define the *tensor product* $A \otimes_R B$ as follows: In degree n, we have

$$(A \oplus_R B)^n = \bigoplus_{p+q=n} A^p \otimes_R B^q$$

For $x \in A^p, y \in B^q$, the differential is

$$\mathrm{d}(x \otimes y) = \mathrm{d}_A x \otimes y + (-1)^p x \otimes \mathrm{d}_B y$$

Note that the sign is required for $\mathrm{d}^2 = 0$.

Theorem 3.2 (naïve Künneth). Suppose $R = k$ is a field, then for all $n \geq 0$,

$$\mathrm{H}^n(A \otimes_k B) = \bigoplus_{p+q=n} \mathrm{H}^p(A) \otimes_k \mathrm{H}^q(B)$$

Proof. Consider $\mathrm{H}^\bullet(A)$ to be a complex, with differential being zero. The cohomology groups are $\mathrm{H}^\bullet(A)$. Now as the A^p are k-vector spaces, we can write

$$A^\bullet \cong \mathrm{H}^\bullet(A) \oplus C^\bullet$$

where $\mathrm{H}^*(C) = 0$. Thus, C is null-homotopic, via a chain homotopy $h^\bullet$. Then

$$A \otimes_k B = H^\bullet(A) \otimes_k B \oplus C^\bullet \otimes_k B^\bullet$$

It's easy to see that $h \otimes \mathrm{id}_B$ shows $C^\bullet \otimes_k B^\bullet$ is null-homotopic. Do the same with B, and we see that

$$\mathrm{H}^n(A \otimes_k B) = \mathrm{H}^n(\mathrm{H}^\bullet(A) \otimes_k \mathrm{H}^\bullet(B)) = \bigoplus_{p+q=n} \mathrm{H}^p(A) \otimes_k \mathrm{H}^q(B)$$

□

3.2 Sheaf cohomology

Let X be a topological space. An exact sequence of sheaves (of abelian groups)

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of abelian groups, then we have an exact sequence

$$0 \longrightarrow \mathcal{F}_1(X) \longrightarrow \mathcal{F}_2(X) \longrightarrow \mathcal{F}_3(X)$$

But the last map is not usually surjective.

There exists families of groups $\mathrm{H}^i(X, \mathcal{F})$ of abelian groups, such that

- $\mathrm{H}^0(X, \mathcal{F}) = \mathcal{F}(X)$,
- there exists a long exact sequence (associated to a short exact sequence of sheaves)

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^0(X, \mathcal{F}_1) & \longrightarrow & \mathrm{H}^0(X, \mathcal{F}_2) & \longrightarrow & \mathrm{H}^0(X, \mathcal{F}_3) \\ & \swarrow & & & & & \\ \mathrm{H}^1(X, \mathcal{F}_1) & \longrightarrow & \mathrm{H}^1(X, \mathcal{F}_2) & \longrightarrow & \mathrm{H}^1(X, \mathcal{F}_3) & \longrightarrow & \cdots \end{array}$$

- If $f : X \to Y$ is a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y, and a map of sheaves $f^*\mathcal{G} \to \mathcal{F}$ (or $G \to f_*\mathcal{F}$), then there are maps

$$f^* : \mathrm{H}^*(Y, \mathcal{G}) \to \mathrm{H}^*(X, \mathcal{F})$$

compatible with the long exact sequence.

Now suppose X is a scheme (which is Noetherian and separated).

- If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then for any open affine cover U_i of X.

$$\mathrm{H}^i(X, \mathcal{F}) = \check{\mathrm{H}}^i(U_i, \mathcal{F})$$

- for general $\mathcal{F}$,

$$\mathrm{H}^0(X, \mathcal{F}) = \check{\mathrm{H}}^0(U_i, \mathcal{F})$$

and,

$$\mathrm{H}^1(X, \mathcal{F}) = \varinjlim_{\text{open covers}} \check{\mathrm{H}}^1(U_i, \mathcal{F})$$

- if $i \geq \dim(X)$, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all sheaves $\mathcal{F}$ on X.
- if X is affine and $\mathcal{F}$ is quasicoherent, then $\mathrm{H}^i(X, \mathcal{F}) = 0$ for all $i \geq 1$.

Theorem 3.3 (finiteness)**.** Suppose $X \to \mathrm{Spec}(A)$ is proper, with A Noetherian. Suppose $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module. Then for all i, $\mathrm{H}^i(X, \mathcal{F})$ is a finite A-module.

There is a proof in Hartshorne for X projective, For a full proof, see Itaka's book on Birational Geometry, or Stacks project.

A particular case is that if $A = k$ is a field, $X \to \mathrm{Spec}(k)$ is proper, and $\mathcal{F}$ is coherent, then $\mathrm{H}^i(X, \mathcal{F})$ are finite dimensional.

Finally, if $X \to \mathrm{Spec}(k)$ is a scheme, $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, and K/k is a field extension, then

$$\mathrm{H}^i(X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K), \mathcal{F}_K) = \mathrm{H}^i(X, \mathcal{F}) \otimes_k K$$

where $\mathcal{F}_K$ is the pullback. We will prove a more general result later.

3.3 Čech cohomology

Now suppose we have a finite open cover $\mathcal{U} = (U_1, \ldots, U_m)$, $\mathcal{F}$ is a sheaf on X, then we define the Čech cohomology for the cohomology of the complex

$$\prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i<j} \mathcal{F}(U_i \cap U_j) \longrightarrow \prod_{i<j<k} \mathcal{F}(U_i \cap U_j \cap U_k)$$

where

$$\mathrm{d} : \prod_{i_0 < \cdots < i_{p-1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_{p-1}}) \to \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p})$$
$$(f_{i_0 \cdots i_{p-1}}) \mapsto (g_{i_0 \cdots i_p})$$

where

$$g_{i_0 \cdots i_p} = \sum_{q=0}^{p} (-1)^q f_{i_0 \cdots \widehat{i_q} \cdots i_p}|_{U_{i_0} \cap \cdots \cap U_{i_p}}$$

If $I = \{i_0, \ldots, i_p\}$, with $i_0 < \cdots < i_p$, we will write $U_I = U_{i_0} \cap \cdots \cap U_{i_p}$. So

$$\check{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{\#I = p+1} \mathcal{F}(U_I)$$

We can also sheafify the Čech complex. For $V \subseteq X$ open, define

$$\check{\mathcal{C}}(\mathcal{U}, X)(V) = \check{C}((V \cap U_i), \mathcal{F})$$

So

$$\check{\mathcal{C}}(\mathcal{U}, X) = \prod_{\#I=p+1} (j_I)_* \mathcal{F}|_{U_I}$$

where $j_I : U_I \to X$ is the inclusion. So $\check{\mathcal{C}}(\mathcal{U}, X)$ is a sheaf, and the differential gives a complex of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

If $\mathcal{F}$ is quasicoherent and $\mathcal{U}$ is an affine cover, then the above is exact, and $\check{\mathcal{C}}$ has no higher cohomology. Lecture 9

Proposition 3.4. Let X be a (Noetherian, separated) scheme, and $\mathcal{U}$ an open affine cover. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Then

(i) we have an exact sequence of quasicoherent sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

(ii) for all $p \geq 0$,

$$\mathrm{H}^i\left(X, \check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})\right) = \begin{cases} \check{C}^p(\mathcal{U}, \mathcal{F}) & i = 0 \\ 0 & i > 0 \end{cases}$$

Lemma 3.5. Let X be a Noetherian separated scheme. Let $j : V \hookrightarrow X$ be an open affine, $\mathcal{G}$ be a quasicoherent $\mathcal{O}_V$-module, then $j_*\mathcal{G}$ is quasicoherent, and for all $i > 0$, $\mathrm{H}^i(X, j_*\mathcal{G}) = 0$.

Proof. Since X is separated, j is an affine map. That is, the preimage of any affine is an affine. Therefore, $j_*\mathcal{G}$ is quasicoherent: If we have $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, and $\mathcal{G} = \widetilde{M}$ for an B-module M. Then $f_*\mathcal{G} = \widetilde{N}$, where N is M viewed as an A-module.

To compute its cohomology, we will compute its Čech cohomology for an open affine cover $\mathcal{U} = (U_i)$ of X. Now

$$\check{C}^p(\mathcal{U}, j_*\mathcal{G}) = \prod_{\#I=p+1} \mathcal{G}(U_I \cap V) = \check{C}^p((U_i \cap V), \mathcal{G})$$

So $\mathrm{H}^i(X, j_*\mathcal{G}) = \mathrm{H}^i(V, \mathcal{G})$ as $(U_i \cap V)$ is an open affine cover of V, and this is zero if $i > 0$. □

Proof of proposition 3.4. By the lemma, $(j_I)_*\mathcal{F}|_{U_I}$ is quasicoherent, so the Čech complex sheaves are quasicoherent. Moreover,

$$\mathrm{H}^i(X, (j_I)_*\mathcal{F}|_{U_I}) = 0$$

for all $I = \{i_0 < \cdots < i_p\}$, with $p \geq 0$. This proves (ii) for $i > 0$. For $i = 0$ it is true by definition.

Thus it remains to show exactness in (i). It suffices to check on an open affine $V \subseteq X$. Then we have

$$0 \longrightarrow \mathcal{F}(V) \longrightarrow \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F})(V) \longrightarrow \cdots$$

But

$$\check{\mathcal{C}}^i(\mathcal{U}, \mathcal{F})(V) = \check{C}^i((U_i \cap V), \mathcal{F}|_V)$$

But we know that

$$C^i((U_i \cap V), \mathcal{F}|_V) = \begin{cases} \mathcal{F}(V) & i = 0 \\ 0 & i > 0 \end{cases}$$

and so the result follows. □

Definition 3.6 (resolution)
Suppose

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G}^1 \longrightarrow \cdots$$

is an exact sequence, we say that the complex $(\mathcal{G}^\bullet)$ is a *resolution* of $\mathcal{F}$.

In particular, (i) in the proposition is called the *Čech resolution* of $\mathcal{F}$.

Definition 3.7 (acyclic)
Let X be a topological space. A sheaf $\mathcal{G}$ on X is *acyclic* if for all $i > 0$, $\mathrm{H}^i(X, \mathcal{G}) = 0$.

Theorem 3.8 (resolution principle). Let X be a topological space, $\mathcal{F}$ be an abelian sheaf on X, and $\mathcal{F} \to \mathcal{G}^\bullet$ a resolution of $\mathcal{F}$ by acyclic sheaves $\mathcal{G}^p$. Then

$$\mathrm{H}^n(X, \mathcal{F}) = \mathrm{H}^n(\Gamma(X, \mathcal{G}^\bullet))$$

Example 3.9
Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module, on a scheme X, and $\mathcal{U}$ is an open affine cover of X. By the proposition,

$$0 \longrightarrow \mathcal{F} \longrightarrow \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

is an acyclic resolution of $\mathcal{F}$. Thus, the resolution principle gives us that

$$\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^n(\Gamma(X, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}))) = \check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F})$$

So sheaf cohomology agrees with Čech cohomology.

Proof. Split up the resolution into short exact sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0 \longrightarrow \mathrm{im}(\mathrm{d}^0) \longrightarrow 0$$

where d^i is the map $\mathcal{G}^i \to \mathcal{G}^{i+1}$, and

$$0 \longrightarrow \mathrm{im}(\mathrm{d}^{i-1}) \longrightarrow \mathcal{G}^i \longrightarrow \mathrm{im}(\mathrm{d}^i) \longrightarrow 0$$

Since the $\mathcal{G}^i$ are all acyclic, the long exact sequence of cohomology gives

$$\begin{array}{l} 0 \longrightarrow \mathrm{H}^0(X, \mathcal{F}) \longrightarrow \mathrm{H}^0(X, \mathcal{G}^0) \longrightarrow \mathrm{H}^0(X, \mathrm{im}(\mathrm{d}^0)) \\ \qquad \longrightarrow \mathrm{H}^1(X, \mathcal{F}) \longrightarrow 0 \end{array} \tag{i}$$

and for all $p > 1$, we have

$$0 \longrightarrow \mathrm{H}^{p-1}(X, \mathrm{im}(\mathrm{d}^0)) \longrightarrow \mathrm{H}^p(X, \mathcal{F}) \longrightarrow 0 \tag{ii}$$

so $\mathrm{H}^p(X, \mathcal{F}) \cong \mathrm{H}^{p-1}(X, \mathrm{im}(\mathrm{d}^0))$. For the second short exact sequence, we get

$$\mathrm{H}^{p-1}(X, \mathrm{im}(\mathrm{d}^i)) \cong \mathrm{H}^p(X, \mathrm{im}(\mathrm{d}^{i-1})) \tag{iii}$$

for $i \geq 1, p > 1$.

As $\mathrm{H}^0(X, \mathrm{im}(\mathrm{d}^0)) = \mathrm{H}^0(X, \ker(\mathrm{d}^1)) = \ker(\mathrm{H}^0(X, \mathcal{G}^1) \to \mathrm{H}^0(X, \mathcal{G}^2)) \subseteq \mathrm{H}^0(X, \mathcal{G}^1)$. So (i) gives the result for H^0 and H^1. So we just need to do $n \geq 2$.

Let $n \geq 2$, and consider the resolution

$$0 \longrightarrow \mathrm{im}(\mathrm{d}^{n-2}) \longrightarrow \mathcal{G}^{n-1} \longrightarrow \mathcal{G}^n \longrightarrow \cdots$$

We can apply the result which we have just shown, to get

$$\mathrm{H}^1(X, \mathrm{im}(\mathrm{d}^{n-2})) = \mathrm{H}^n(\Gamma(X, \mathcal{G}^\bullet))$$

But we can use (ii) and (iii) to get

$$\mathrm{H}^n(X, \Gamma(X, \mathcal{G}^\bullet)) = \mathrm{H}^2(X, \mathrm{im}(\mathrm{d}^{n-3})) = \cdots = \mathrm{H}^{n-1}(X, \mathrm{im}(\mathrm{d}^0)) = \mathrm{H}^n(X, \mathcal{F})$$

as required. □

We can also use other acyclic resolution.

Definition 3.10 (flasque)
Let X be a topological space. A sheaf $\mathcal{F}$ on X is *flasque*[a] (or *flabby*[b], or *flask*[c]) if for all opens $V \subseteq U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

[a] in French
[b] when translated
[c] when translated in America

Fact: Flasque sheaves are acyclic.
For any sheaf $\mathcal{F}$, there exists an injection

$$\mathcal{F} \to \mathrm{G}\mathcal{F} = \prod_{x \in X} (i_x)_* \mathcal{F}_x$$

where $i_x : \{x\} \hookrightarrow X$ is the inclusion map. Then $\mathrm{G}\mathcal{F}$ is flasque, and we have a resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathrm{G}\mathcal{F} \longrightarrow \mathrm{G}(\mathrm{G}\mathcal{F}) \longrightarrow \cdots$$

of $\mathcal{F}$ by flasque sheaves. So every sheaf have a (canonical) flasque sheaves, where G refers to Goderment. Lecture 10

Theorem 3.11. Let $\iota : Y \hookrightarrow X$ be the inclusion of a closed subspace of a topological space X. Let $\mathcal{F}$ be a sheaf on Y. Then

$$\mathrm{H}^*(Y, \mathcal{F}) = \mathrm{H}^*(X, \iota_* \mathcal{F})$$

By abuse of notation, often we will write $\mathrm{H}^*(X, \mathcal{F})$.

Proof. Choose a flasque resolution $\mathcal{F} \to \mathcal{G}^\bullet$ on Y. By the resolution principle,

$$\mathrm{H}^*(Y, \mathcal{F}) = \mathrm{H}^*(\Gamma(Y, G^\bullet))$$

By the detinition, since $\mathcal{G}^p$ is flasque, so is $\iota_* \mathcal{G}^p$, and $\iota_* \mathcal{F} \to \iota_* \mathcal{G}^\bullet$ is also a resolution. For this, note that we can check exactness on stalks, to see that ι_* is exact.

So

$$\mathrm{H}^*(X, \iota_* \mathcal{F}) = \mathrm{H}^*(\Gamma(X, \iota_* G^\bullet)) = \mathrm{H}^*(\Gamma(Y, G^\bullet))$$

□

Theorem 3.12 (Mayer-Vietoris). Suppose $X = U \cup V$, and $\mathcal{F}$ is an abelian sheaf on X. There is a long exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{H}^0(X,\mathcal{F}) \longrightarrow \mathrm{H}^0(U,\mathcal{F}) \oplus \mathrm{H}^0(V,\mathcal{F}) \longrightarrow \mathrm{H}^0(U\cap V,\mathcal{F}) \\ \mathrm{H}^1(X,\mathcal{F}) \longrightarrow \mathrm{H}^1(U,\mathcal{F}) \oplus \mathrm{H}^1(V,\mathcal{F}) \longrightarrow \mathrm{H}^1(U\cap V,\mathcal{F}) \longrightarrow \cdots \end{array}$$

Proof. Choose a flasque resolution $\mathcal{F} \to \mathcal{G}^\bullet$. Then we have an exact sequence of complexes

$$0 \longrightarrow \Gamma(X,\mathcal{G}^\bullet) \longrightarrow \Gamma(U,\mathcal{G}^\bullet) \oplus \Gamma(V,\mathcal{G}^\bullet) \longrightarrow \Gamma(U\cap V,\mathcal{G}^\bullet) \longrightarrow 0$$

This is exact on the right as the $\mathcal{G}^p$ are flat. Taking the long exact sequence on cohomology gives the result, since $\mathrm{H}^*(\cdot,\mathcal{F}) = \mathrm{H}^*(\Gamma(\cdot,\mathcal{G}^\bullet))$. □

Another application:

Theorem 3.13 (Künneth)**.** Let X, Y be Noetherian separated schemes over a field k, $\mathcal{F},\mathcal{G}$ quasicoherent sheaves on X and Y respectively. Then

$$\mathrm{H}^n(X\times_k Y, \mathrm{pr}_1^*\mathcal{F} \underset{\mathcal{O}_{X\times_k Y}}{\otimes} \mathrm{pr}_2^*\mathcal{G}) = \bigoplus_{p+q=n} \mathrm{H}^p(X,\mathcal{F}) \otimes_k \mathrm{H}^q(Y,\mathcal{G})$$

In particular,

$$\mathrm{H}^n(X\times_k Y, \mathcal{O}_{X\times_k Y}) = \bigoplus_{p+q=n} \mathrm{H}^p(X,\mathcal{O}_X) \otimes_k \mathrm{H}^q(Y,\mathcal{O}_Y)$$

Notation 3.14. We will denote

$$\mathcal{F}\boxtimes\mathcal{G} = \mathcal{F} \underset{X,Y}{\boxtimes} \mathcal{G} = \mathrm{pr}_1^*\mathcal{F} \underset{\mathcal{O}_{X\times_k Y}}{\otimes} \mathrm{pr}_2^*\mathcal{G}$$

Lemma 3.15.

(i) Let A, B be k-algebras, M an A-module, N a B-module, then on $\mathrm{Spec}(A)\times_k\mathrm{Spec}(B) = \mathrm{Spec}(A\otimes_k B)$,

$$\widetilde{M}\boxtimes\widetilde{N} = \widetilde{M\otimes_k N}$$

(ii) if $\mathcal{F}\to\mathcal{K}^\bullet$, $\mathcal{G}\to\mathcal{L}^\bullet$ are resolutions of quasicoherent sheaves on k-schemes X, Y, then

$$\mathcal{F}\boxtimes\mathcal{G} \to \mathcal{K}^\bullet\boxtimes\mathcal{L}^\bullet$$

is a resolution of quasicoherent sheaves on $X\times Y$, where

$$(\mathcal{K}^\bullet\boxtimes\mathcal{L}^\bullet)^n = \bigoplus_{p+q=n} \mathcal{K}^p\boxtimes\mathcal{L}^q$$

Proof. For (i), note

$$\mathrm{pr}_1^*(\widetilde{M}) = M\otimes_A \widetilde{(A\otimes_k B)}$$

and so

$$\begin{aligned} \widetilde{M}\boxtimes\widetilde{N} &= (M\otimes_A \widetilde{(A\otimes_k B)}) \otimes (N\otimes_B \widetilde{(A\otimes_K B)}) \\ &= \widetilde{M\otimes_k N} \end{aligned}$$

For (ii), we can check exactness on an affine cover, so we reduce to the case where X, Y are affine. So $\mathcal{F} = \widetilde{M}, \mathcal{G} = \widetilde{N}, \mathcal{K}^\bullet = \widetilde{K^\bullet}, \mathcal{L}^\bullet = \widetilde{L^\bullet}$. By (i), we reduce to a statement about modules, showing that if $M\to K^\bullet, N\to L^\bullet$ are resolutions, then

$$M\otimes_k N \to K^\bullet\otimes_k L^\bullet$$

is a resolution. That is,

$$\mathrm{H}^n(K^\bullet \otimes_k L^\bullet) = \begin{cases} M \otimes_k N & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since

$$\mathrm{H}^n(K^\bullet) = \begin{cases} M & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

This follows from the naïve Künneth formula. □

Proof of theorem 3.13. Take open affine covers $\mathcal{U}, \mathcal{V}$ of X and Y respectively. So we have Čech resolutions,

$$\mathcal{F} \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \quad \text{and} \quad \mathcal{G} \to \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$$

By the lemma, we get a resolution

$$\mathcal{F} \boxtimes \mathcal{G} \to \mathcal{K}^\bullet$$

where

$$\mathcal{K}^\bullet = \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \boxtimes \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{G})$$

Now note

$$\mathcal{K}^n = \bigoplus_{p+q=n} \left(\prod_{\#I=p+1, \#J=q+1} \mathcal{K}_{I,J} \right)$$

where

$$\mathcal{K}_{I,J} = (\iota_I \times \iota_J)_*(\mathcal{F} \boxtimes \mathcal{G})$$

where $\iota_I : U_I \to X$, $\iota_J : V_J \to Y$ are the inclusion maps. This is acyclic, as for the usual Čech complex. Thus,

$$\mathrm{H}^*(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) = \mathrm{H}^*(\Gamma(X \times Y, \mathcal{K}^\bullet))$$

Now

$$\Gamma(X \times Y, \mathcal{K}_{I,J}) = \Gamma\left(U_I \times U_J, (\mathcal{F} \boxtimes \mathcal{G})|_{U_I \times U_J}\right) = \mathcal{F}(U_I) \otimes \mathcal{G}(U_I)$$

by the lemma. Thus,

$$\Gamma(X \times Y, \mathcal{K}^\bullet) = \check{\mathcal{C}}(\mathcal{U}, \mathcal{F}) \otimes \check{\mathcal{C}}(\mathcal{V}, \mathcal{G})$$

So by naïve Künneth, the cohomology of this in degree n is

$$\bigoplus_{p+q=n} \mathrm{H}^p(X, \mathcal{F}) \otimes_k \mathrm{H}^q(Y, \mathcal{G})$$

□

Suppose $Y = X$. Then for $p, q \geq 0$, $\mathcal{F}, \mathcal{G}$ quasicoherent $\mathcal{O}_X$-modules, $\Delta : X \hookrightarrow X \times_k X$ is the diagonal map. Then $\Delta^*(\mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} \otimes \mathcal{G}$, and so we get

$$\mathrm{H}^p(X, \mathcal{F}) \otimes_k \mathrm{H}^q(X, \mathcal{G}) \xrightarrow{K} \mathrm{H}^{p+q}(X \times X, \mathcal{F} \otimes \mathcal{G}) \xrightarrow{\Delta^*} \mathrm{H}^{p+q}(X, \mathcal{F} \otimes \mathcal{G})$$

The map K is from the Künneth formula.

Definition 3.16
The composition $\Delta^* \circ K$ is called the *cup product*.

When $\mathcal{F} = \mathcal{G} = \mathcal{O}_X$, the cup product makes $\mathrm{H}^*(X, \mathcal{O}_X)$ into a graded k-algebra.
Fact: The multiplication in $\mathrm{H}^*(X, \mathcal{O}_X)$ is graded-commutative. That is,

$$x \smile y = (-1)^{\deg(x)\deg(y)} y \smile x$$

Lecture 11

3.4 Cohomology of sheaves on $\mathbb{P}^N$

Definition 3.17 (support)
Let X be any Noetherian scheme, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then the *support of $\mathcal{F}$* is

$$\mathrm{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$$

This is a closed subset of X. In fact, it suppose to check for $X = \mathrm{Spec}(R)$ and $\mathcal{F} = \widetilde{M}$. Then

$$\mathrm{supp}(\mathcal{F}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$$

But $M_{\mathfrak{p}} = 0$ if and only if there exists $r \in R \setminus \mathfrak{p}$, with $rM = 0$. This is because M is finite. Equivalently, $\mathrm{Ann}(M) \not\subseteq \mathfrak{p}$. So

$$\mathrm{supp}(\mathcal{F}) = \mathbb{V}(\mathrm{Ann}(M))$$

More precisely, there exists a closed subscheme $i : Z \hookrightarrow X$, with underlying point set $\mathrm{supp}(\mathcal{F})$, such that $\mathcal{F} = i_*\mathcal{G}$ for some coherent $\mathcal{O}_Z$-module $\mathcal{G}$. When $X = \mathrm{Spec}(R)$ is affine, $Z = \mathrm{Spec}(R/\mathrm{Ann}(M))$.

If X is proper over a field k, then $\mathrm{H}^*(X, \mathcal{F})$ is finite dimensional for coherent $\mathcal{F}$. In this case, we can define the *Euler characteristic*

$$\chi(X, \mathcal{F}) = \sum_i (-1)^i \dim_l \left(\mathrm{H}^i(X, \mathcal{F})\right)$$

By the long exact sequence of cohomology, if we have a short exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of $\mathcal{O}_X$-modules, then

$$\chi(X, \mathcal{F}_2) = \chi(X, \mathcal{F}_1) + \chi(X, \mathcal{F}_3)$$

Recall that on

$$X = \mathbb{P}^N_k = \mathrm{Proj}\,(k[T_0, \dots, T_N])$$

we have an invertible $\mathcal{O}_X$-module (i.e. a line bundle) $\mathcal{O}_{\mathbb{P}^n}(n)$, with sections being quotients f/g, with $f, g \in k[T_0, \dots, T_N]$ homogeneous, and $\deg(f) = n + \deg(g)$.

In Part III Algebraic Geometry, we computed

$$\mathrm{H}^i(\mathbb{P}^N, \mathcal{O}(n)) = \begin{cases} k[T_0, \dots, T_N]_n & i = 0, n > 0 \\ k[T_0, \dots, T_N]^\vee_{-N-n-1} & i = n, n \leq -N-1 \end{cases}$$

In the first case, the dimension is $\binom{N+n}{n}$, and in the second case $\binom{-n-1}{N}$. In this case,

$$\chi(\mathbb{P}^N, \mathcal{O}(n)) = P(n)$$

where

$$P(t) = \binom{t+N}{N} = \frac{(t+N)\cdots(t+1)}{N!}$$

This generalises to any coherent $\mathcal{F}$ on $\mathbb{P}^N$.

Theorem 3.18. There exists a polynomial $P(\mathcal{F}, t) \in \mathbb{Q}[t]$, such that

$$\chi(\mathbb{P}^N_k, \mathcal{F}(n)) = P(\mathcal{F}, n)$$

for all $n \in \mathbb{Z}$, and $\deg(P(\mathcal{F}, t)) = \dim(\mathrm{supp}(\mathcal{F}))$. Here, $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}(n)$.

Suppose $i : X \hookrightarrow \mathbb{P}^N_k$ is a closed immersion. Then $\mathcal{O}_X(n) = i^*(\mathcal{O}_{\mathbb{P}^1}(n))$. More generally, we can replace k by any (Noetherian) ring. For any k-scheme X, an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is *very ample* if

$$\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}^N}(1)$$

If we have a projective embedding $i : X \hookrightarrow \mathbb{P}^N$ as above, $\mathcal{O}_X(1)$ is a very ample line bundle on X, then we can define for coherent $\mathcal{F}$ on X,

$$P(X, \mathcal{F}, t) = P(i_*\mathcal{F}, t)$$

and $P(X,t) = P(X, \mathcal{O}_X, t)$. These are called *HIlbert polynomials*.

Since

$$\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^i(\mathbb{P}^N, i_*\mathcal{F})$$

We must have that

$$P(X, \mathcal{F}, n) = \chi(X, \mathcal{F}(n))$$

Moreover, as χ is additive in short exact sequences, so is the Hilbert polynomial.

A related notion is an *ample* line bundle, which is one which satisfies one of the following conditions.

Theorem 3.19. Let X/k be proper, $\mathcal{L}$ a line bundle on X. Then the following are equivalent:

(i) for some $r \geq 1$, $\mathcal{L}^{\otimes r}$ is very ample,

(ii) for some $r_0 \geq 1$ and all $r \geq r_0$, $\mathcal{L}^{\otimes r}$ is very ample,

(iii) for every coherent $\mathcal{O}_X$-module $\mathcal{F}$, there exists n_0 such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. That is, there exists a surjection

$$\mathcal{O}_X^m \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

(iv) for every coherent sheaf $\mathcal{F}$, there exists n_0 such that for all $n \geq n_0, i > 0$,

$$\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$$

For a curve, $\mathcal{L}$ is ample if and only if $\mathcal{L} \cong \mathcal{O}(D)$ for some $\deg(D) \geq 0$.

Proposition 3.20. Let $X \subseteq \mathbb{P}_k^N$ be an integral closed subscheme, of dimension d. Let $\eta \in X$ be its generic point. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then

$$P(X, \mathcal{F}, t) = \dim_{\kappa(\eta)}\left(\mathcal{F}_\eta\right) P(X, t) + \text{terms of degree less than } d$$

Proof. Let $e_1, \ldots, e_r$ be a $\kappa(\eta)$ basis for $\mathcal{F}_\eta$. Then there exists n_0 such that $e_1, \ldots, e_r$ extend to sections of $\mathcal{F}(n_0)$. Thus, we have an exact sequence

$$0 \longrightarrow \mathcal{O}^r \xrightarrow{(e_i)} \mathcal{F}(n_0) \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{G}_\eta = 0$, and so $\dim(\mathrm{supp}(\mathcal{G})) < d$. But then

$$P(X, \mathcal{F}, tn_0) = \mathcal{P}(X, \mathcal{F}(n_0), t) = rP(X,t) + P(X, \mathcal{G}, t)$$

where $P(X, \mathcal{G}, t)$ has degree less than d. □

Idea of proof of the existence of Hilbert polynomials. First of all, we can assume k is algebraically closed.

We can use exact sequences and induction on dimension of $\mathrm{supp}(\mathcal{F})$, to reduce to the case where $\mathrm{supp}(\mathcal{F})$ is irreducible, of dimension d, and we can find a hyperplane $H = \{f = 0\}$, such that the map $\mathcal{F} \to \mathcal{F}(1)$, given by multiplication by f is injective. Then we have an exact sequence

$$0 \longrightarrow \mathcal{F}(n) \xrightarrow{f\cdot} \mathcal{F}(n+1) \longrightarrow \mathcal{G}(n+1) \longrightarrow 0$$

and $\mathrm{supp}(\mathcal{G}) = \mathrm{supp}(\mathcal{F}) \cap H$, which has dimension $d-1$. Then

$$\chi(\mathcal{F}(n+1)) - \chi(\mathcal{F}(n)) = \chi(\mathcal{G}(n))$$

which is a polynomial in n, of degree $d-1$. □

Lecture 12

3.5 Cohomology and base change

How does cohomology vary in a family? That is, if we have a morphism $f : X \to S$ and a sheaf $\mathcal{F}$ on X, we would like to compare $\mathrm{H}^*(X_s, \mathcal{F}|_{X_s})$ and $\mathrm{H}^*(X_t, \mathcal{F}|_{X_t})$ for $s, t \in S$.

Suppose that $S = \mathrm{Spec}(A)$ is Noetherian and separated, and we have a ring homomorphism $A \to B$. Then we have the fibre product

$$\begin{array}{ccc} X_B = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(B) & \xrightarrow[g]{} & \mathrm{Spec}(A) \end{array}$$

Suppose $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module. In this case, $\mathrm{H}^*(X, \mathcal{F})$ is an A-module, and by functoriality, we have a map

$$\mathrm{H}^p(X, \mathcal{F}) \to \mathrm{H}^p(X_B, \mathcal{F}_B)$$

where $\mathcal{F}_B = (g')^*\mathcal{F}$. In particular, we have

$$\begin{array}{ccc} \mathrm{H}^p(X, \mathcal{F}) & \longrightarrow & \mathrm{H}^p(X_B, \mathcal{F}_B) \\ \downarrow & \nearrow{\scriptstyle \beta_B} & \\ \mathrm{H}^p(X, \mathcal{F}) \otimes_A B & & \end{array}$$

We would like to know when is β_B an isomorphism. For example, if $B = \kappa(s)$, where $s \in \mathrm{Spec}(A)$, then we would like to know when $\mathrm{H}^p(X_s, \mathcal{F}|_{X_s})$ is isomorphic to $\mathrm{H}^p(X, \mathcal{F}) \otimes_A \kappa(s)$.

We will approach this using Čech cohomology. Say $X = \bigcup_i U_i$ is an open affine cover. Then

$$X_B = \bigcup U_i \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) = \bigcup U_{i,B}$$

is also an open affine cover. Now

$$\mathcal{F}_B(U_{I,B}) = \mathcal{F}(U_I) \otimes_A B$$

Hence

$$\check{C}^i((U_{I,B}), \mathcal{F}_B) = \check{C}^i((U_i), \mathcal{F}) \otimes_A B$$

Say $K^\bullet = \check{C}^\bullet((U_i), \mathcal{F})$. So we have

$$\begin{array}{ccc} \mathrm{H}^p(X, \mathcal{F}) \otimes_A B & \xrightarrow{\cong} & \mathrm{H}^p(K^\bullet) \otimes_A B \\ \downarrow{\scriptstyle \beta_B} & & \downarrow \\ \mathrm{H}^i(X_B, \mathcal{F}_B) & \xrightarrow[\cong]{} & \mathrm{H}^p(K^\bullet \otimes_A B) \end{array}$$

Theorem 3.21. Suppose B is A-flat. Then β_B is an isomorphism.

Proof. Since in this case, we know that $\mathrm{H}^*(K^\bullet \otimes_A B) \cong \mathrm{H}^*(K^\bullet) \otimes_A B$. □

Example 3.22
Suppose $k \subseteq K$ are fields, with $X/\mathrm{Spec}(k)$. Then

$$\mathrm{H}^*(X_K, \mathcal{F}_K) = \mathrm{H}^*(X, \mathcal{F}) \otimes_k K$$

For the general case, we will need to put some hypotheses on X and on $\mathcal{F}$. The problem is that the modules $K^\bullet$ are typically large. For example, they are rarely finite. Assume from now on, $f : X \to \mathrm{Spec}(A)$ is proper, $\mathcal{F}$ is coherent and flat over $\mathrm{Spec}(A)$. In this case. $\mathrm{H}^p(X, \mathcal{F})$ is a finite A-module, and vanishes for p sufficiently large.

Theorem 3.23. Suppose $\mathrm{H}^p(X, \mathcal{F})$ is zero for all $p > n$. Then there exists a complex

$$L^0 \longrightarrow L^1 \longrightarrow \cdots \longrightarrow L^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

of finite flat A-modules, and for all A-algebras B, isomorphisms

$$\mathrm{H}^*(L^\bullet \otimes_A B) \to \mathrm{H}^*(X_B, \mathcal{F}_B)$$

which are functorial in B.

Recall finite flat is equivalent to finite and locally free, since A is Noetherian. In fact, the same is true for any A-module M,

$$\mathrm{H}^*(L^\bullet \otimes_A M) \cong \mathrm{H}^*(X, \mathcal{F} \otimes_A M)$$

Here, $\mathcal{F} \otimes_A M = \mathcal{F} \otimes f^*\widetilde{M}$.

Since $\mathrm{H}^*(X_B, \mathcal{F}_B) = \mathrm{H}^*(K^\bullet \otimes_A B)$, where $K^\bullet$ is the Čech complex, by our assumptions it suffices to prove the purely algebraic statement:

Theorem 3.24. Let A be a Noetherian ring, $K^\bullet$ a complex of A-modules, such that $\mathrm{H}^p(K^\bullet)$ is finite for all p, and zero for $p > n$. Then there exists a complex

$$L^0 \longrightarrow L^1 \longrightarrow \cdots \longrightarrow L^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

of finite A-modules and a morphisms of complexes $L^\bullet$ to $K^\bullet$, inducing an isomorphism $\mathrm{H}^i(L^\bullet) \cong \mathrm{H}^i(K^\bullet)$, and $L^1, \ldots, L^n$ are all free.

If in addition all K^p are flat, then L^0 is locally free, and for every A-module M, we have an isomorphism

$$\mathrm{H}^*(L^\bullet \otimes_A M) \cong \mathrm{H}^*(K^\bullet \otimes_A M)$$

Remark 3.25. For the application to the previous theorem, note that if $\mathcal{F}$ is A-flat, then the

$$K^p = \prod_{\#I = p+1} \mathcal{F}(U_I)$$

is also A-flat.

Proof. We start from the top. Pick a finite free module L^n, and a surjection $L^n \twoheadrightarrow \mathrm{H}^n(K)$. Since L^n is free, we can lift this to a map $L^n \to \ker(\mathrm{d} : K^n \to K^{n+1})$. So we have

$$\begin{array}{ccccccccccc}
 & & & & & & & & L^n & \longrightarrow & 0 & & \\
 & & & & & & & & \downarrow{\scriptstyle f^n} & & \downarrow & & \\
K^0 & \longrightarrow & K^1 & \longrightarrow & \cdots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \cdots
\end{array}$$

Now choose a finite free module P, and a surjection $P \twoheadrightarrow \ker(L^n \to \mathrm{H}^n(K))$. So we have

$$\begin{array}{ccccccccccccc}
 & & & & & & P & \longrightarrow & L^n & \longrightarrow & 0 & & \\
 & & & & & & \vdots & & \downarrow{\scriptstyle f^n} & & \downarrow & & \\
K^0 & \longrightarrow & K^1 & \longrightarrow & \cdots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \cdots
\end{array}$$

Since P is free, we have a lift $P \to K^{n-1}$ making the square commute. To see this, the composition $P \to \mathrm{H}^n(K)$ is zero, and so P maps to $\mathrm{im}(\mathrm{d} : K^{n-1} \to K^n)$, hence we have a lift.

Next, choose a finite free Q and a surjection $Q \twoheadrightarrow \mathrm{H}^{n-1}(K)$, which lifts to a map $Q \to \ker(\mathrm{d} : K^{n-1} \to K^n)$. Now let $L^{n-1} = Q \oplus P$. The map $L^{n-1} \to L^n$ is given by $Q \to 0$. Now we have $\mathrm{H}^n(L) \cong \mathrm{H}^n(K)$, and $\mathrm{H}^{n-1}(L) \twoheadrightarrow \mathrm{H}^{n-1}(K)$. We can continue this, and after n steps, we get

$$\begin{array}{ccccccccccccc}
L^0 & \longrightarrow & L^1 & \longrightarrow & \cdots & \longrightarrow & L^{n-1} & \longrightarrow & L^n & \longrightarrow & 0 & & \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow & & \\
K^0 & \longrightarrow & K^1 & \longrightarrow & \cdots & \longrightarrow & K^{n-1} & \longrightarrow & K^n & \longrightarrow & K^{n+1} & \longrightarrow & \cdots
\end{array}$$

where for $0 < p \leq n$, $\mathrm{H}^p(K^\bullet) \cong \mathrm{H}^p(L^\bullet)$, and $\mathrm{H}^0(L^\bullet) \twoheadrightarrow \mathrm{H}^0(K^\bullet)$. Replacing L^0 bt $L^0/\ker(\mathrm{H}^0(f))$ gives the isomorphism at degree 0. Let $C^\bullet$ be the following complex (usually called the *mapping cone* of f).

Lecture 13

$$C^p = L^p \oplus K^{p-1}$$

and

$$\mathrm{d}(x, y) = (\mathrm{d}x, -\mathrm{d}y + f(x))$$

We have an exact sequence of complexes

$$0 \longrightarrow K[-1] = (K^{\bullet-1}, -\mathrm{d}_K) \longrightarrow (C^\bullet, \mathrm{d}_C) \longrightarrow (L^\bullet, \mathrm{d}_L) \longrightarrow 0$$

Therefore, we have a long exact sequence of cohomology

$$\mathrm{H}^{p-1}(K^\bullet) \longrightarrow \mathrm{H}^p(C^\bullet) \longrightarrow \mathrm{H}^p(L^\bullet) \xrightarrow{\delta} \mathrm{H}^p(K)$$

and $\delta = \mathrm{H}^p(f)$. In particular, this is an isomorphism for all p, and so $C^\bullet$ is acyclic, i.e $\mathrm{H}^*(C) = 0$.

Since $K^p = 0$ for sufficiently large p, for some N,

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^N \longrightarrow 0$$

is exact. But $C^1, \ldots, C^N$ are all flat, so C^0 is also flat (see examples sheet 2).

Finally, let M be any A-module. Write

$$0 \longrightarrow Q \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P free. Assume for all such M, and $q > p$, the map

$$\mathrm{H}^q(L \otimes M) \to \mathrm{H}^q(K \otimes M)$$

is an isomorphism. For example, this is true with $p = n$. As $K^\bullet, L^\bullet$ are flat, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L \otimes Q & \longrightarrow & L \otimes P & \longrightarrow & L \otimes M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K \otimes Q & \longrightarrow & K \otimes P & \longrightarrow & K \otimes M & \longrightarrow & 0 \end{array}$$

which has exact rows. Hence we get a commutative diagram with long exact rows

$$\begin{array}{ccccccccc} \mathrm{H}^p(L \otimes Q) & \longrightarrow & \mathrm{H}^p(L \otimes P) & \longrightarrow & \mathrm{H}^p(L \otimes M) & \longrightarrow & \mathrm{H}^{p+1}(L \otimes Q) & \longrightarrow & \mathrm{H}^{p+1}(L \otimes P) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow & & \downarrow \gamma & & \downarrow \varepsilon \\ \mathrm{H}^p(K \otimes Q) & \longrightarrow & \mathrm{H}^p(K \otimes P) & \longrightarrow & \mathrm{H}^p(K \otimes M) & \longrightarrow & \mathrm{H}^{p+1}(K \otimes Q) & \longrightarrow & \mathrm{H}^{p+1}(K \otimes P) \end{array}$$

By induction hypothesis, γ, ε are isomorphisms. Since P is free, β is an isomorphism. By the five lemma, the middle map is surjective. The same argument for α shows that α is surjective, and so the middle map is injective. □

Lemma 3.26 (five lemma)**.** Suppose we have

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

where the rows are exact, b, d are isomorphisms. Then:

- if e is a monomorphism, then c is an epimorphism,
- if a is an epimorphism, then c is monomorphism.

Proof. Omitted, just a diagram chase. □

Recall the assumptions we have made, that is, $X \to \mathrm{Spec}(A)$ proper, $\mathcal{F}$ on X is coherent and flat over $\mathrm{Spec}(A)$.

$\mathrm{H}^0(X_B, \mathcal{F}_B) = \ker(\mathrm{d} : L^0 \otimes B \to L^1 \otimes B)$, and for any finite flat (or equivalently, locally free) A-module M,

$$M \otimes_A B = \mathrm{Hom}_A(M^\vee, B) = \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), B)$$

and so

$$\mathrm{H}^0(X_B, \mathcal{F}_B) = \ker(\mathrm{Hom}_A((L^0)^\vee, B) \to \mathrm{Hom}_A((L^1)^\vee, B))$$

Let

$$Q = \mathrm{coker}((L^1)^\vee \to (L^0)^\vee)$$

and so

Corollary 3.27. There exists a finite A-module Q, such that for all B,

$$\mathrm{H}^p(X_B, \mathcal{F}_B) = \mathrm{Hom}_A(Q, B)$$

and this is functorial in B.

Corollary 3.28 (semicontinuity for H^0)**.** For every $r \geq 0$, the subset

$$Z_r = \left\{ s \in \mathrm{Spec}(A) \;\middle|\; \dim_{\kappa(s)} \left(\mathrm{H}^0(X_s, \mathcal{F}(s)) \geq r \right) \right\} \subseteq \mathrm{Spec}(A)$$

is closed. Here, $\mathcal{F}(s) = \mathcal{F} \otimes_A \kappa(s)$, viewed as a coherent $\mathcal{O}_{X_s}$-module.

Remark 3.29. Note Mumford calls $\mathcal{F}(s)$ $\mathcal{F}_s$, but this can cause confusion with stalks.

Remark 3.30. This is also true for all H^p. The statement implies that if $s \in \overline{\{t\}}$, then $\dim(\mathrm{H}^0(X_s)) \geq \dim(\mathrm{H}^0(X_r))$.

Proof. By localising on $\mathrm{Spec}(A)$, we can assume that L^0, L^1 are free, isomorphic to A^m, A^n respectively. Then $(\mathrm{d}^0)^\mathsf{T}$ is represented by an $m \times n$ matrix over A. So

$$\begin{aligned} Z_i &= \left\{ s \mid \mathrm{rank}\left((\mathrm{d}^0)^\mathsf{T} \otimes \mathrm{id}_{\kappa(s)} \right) \leq m - r \right\} \\ &= \{ s \mid \text{all } m - r + 1 \text{ minors of } C \text{ vanish in } \kappa(s) \} \end{aligned}$$

□

Corollary 3.31. Assume $\mathrm{Spec}(A)$ is connected. Then $\chi(X_s, \mathcal{F}(s))$ is independent of $s \in \mathrm{Spec}(A)$.

If $X \subseteq \mathbb{P}^N_k$ is projective, then the Hilbert polynomial $P(X_s, \mathcal{F}_s, t)$ is independent of s.

Proof. Localising, we can assume L^0 is free. But then

$$\begin{aligned} \chi(X_s, \mathcal{F}(s)) &= \sum_p (-1)^p \dim_{\kappa(s)} (\mathrm{H}^p(L \otimes \kappa(s))) \\ &= \sum_p (-1)^p \dim_{\kappa(s)} (L^p \otimes \kappa(s)) \end{aligned}$$

But

$$\dim_{\kappa(s)}(L^p \otimes \kappa(x)) = \mathrm{rank}(L)$$

which is independent of s. □

Note here, we use the fact from homological algebra that if we have a chain complex consisting of k-vector spaces, we can compute the Euler characteristic using cohomology or as the alternating sum of the dimensions. Lecture 14

Now lets look at the top dimensional cohomology, since $L^p = 0$ for $p > n$, we must have that $\mathrm{H}^p(X_B, \mathcal{F}_B) = 0$ for $p > n$. Moreover,

$$\mathrm{H}^n(X_B, \mathcal{F}_B) = \mathrm{coker}(L^{n-1} \otimes_A B \to L^n \otimes_A B) = \mathrm{coker}(L^{n-1} \to L^n) \otimes_A B = \mathrm{H}^n(X, \mathcal{F}) \otimes_A B$$

Here, we use the fact that the tensor product is right exact. Hence the base change map is an isomorphism on the top degree.

Fibres: Assume A is reduced. Then the map

$$A \to \prod_{\mathfrak{p} \in \mathrm{Spec}(A)} \frac{A}{\mathfrak{p}}$$

is injective. As $\mathrm{H}^n(X, \mathcal{F})$ is a finite A-module, we get that

$$\begin{aligned} \mathrm{H}^n(X, \mathcal{F}) - 0 &\iff \forall s \in \mathrm{Spec}(A), \mathrm{H}^n(X, \mathcal{F}) \otimes_A \kappa(s) = 0 \\ &\iff \forall s \in \mathrm{Spec}(A), \mathrm{H}^n(X_s, \mathcal{F}(s)) = 0 \end{aligned}$$

If these hold, then we can replace n with $n-1$, and by descending induction,

Corollary 3.32. Suppose A is reduced, and let $p \geq 0$. Then the following are equivalent:

(i) for all $i \geq p$, $\mathrm{H}^i(X, \mathcal{F}) = 0$,

(ii) for all $i \geq p, s \in \mathrm{Spec}(A)$, $\mathrm{H}^i(X_s, \mathcal{F}(s)) = 0$

In particular, setting $p = 0$, $\mathrm{H}^*(X, \mathcal{F}) = 0$ if and only if $\mathrm{H}^*(X_s, \mathcal{F}(s)) = 0$ for all $s \in \mathrm{Spec}(A)$.

Higher direct images: Suppose $f : X \to Y$ is a morphism of (separated Noetherian) schemes, $\mathcal{F}$ a quasicoherent sheaf on X. Let $V = \mathrm{Spec}(A) \subseteq Y$ be an open affine. Let

$$\mathcal{G}_p(V) = \mathrm{H}^p(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

This is an A-module. Now if we have $V' = \mathrm{Spec}(A') \subseteq V$ affine, then $\mathcal{G}_p(V') = \mathcal{G}_p(V) \otimes_A A'$, by flat base change.

in particular, by question 2 on examples sheet 2, there exists a unique quasicoherent sheaf on Y, extending $\mathcal{G}_p$. We write $R^p f_* \mathcal{F}$ for this sheaf, which is called the *higher direct image sheaf*.

Remark 3.33. For any abelian sheaf $\mathcal{F}$ on X, there is are sheaves $R^p f_* \mathcal{F}$ on Y, which are the sheafifications of $V \mapsto \mathrm{H}^p(f^{-1}(V), \mathcal{F})$.

We can also define this as $\mathrm{H}^*(f_* K^\bullet)$, for a suitable resolution of $\mathcal{F}$, see the relevant section of Hartshorne.

We can rephrase the earlier results in terms of higher direct images. For example,

Theorem 3.34 (restatement of theorem 3.3)**.** Suppose $f : X \to Y$ is proper, $\mathcal{F}$ is q coherent $\mathcal{O}_X$-module, then $R^p f_* \mathcal{F}$ is coherent.

and we can restate corollary 3.32 as

Corollary 3.35. Suppose Y is reduced. Then for all $p \geq 0$, the following are equivalent:

(i) for all $i \geq p$, $R^i f_* \mathcal{F} = 0$,

(ii) for all $i \geq p, y \in Y$, $\mathrm{H}^i(X_y, \mathcal{F}(y)) = 0$

4 Group schemes over a field

Fix a field k. Write $\mathbf{Sch}/k$ for the category of k-schemes, i.e. schemes with a morphism $a_X : X \to \mathrm{Spec}(K)$. The morphisms are morphisms of schemes such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(k) & \end{array}$$

commutes. We write $\mathbf{Aff}/k$ for the category of affine k-schemes. Equivalently, the opposite category of the category of k-algebras.

For $X, S \in \mathbf{Sch}/k$, write $X(S) = \mathrm{Mor}_k(S, X)$, and if R is a k-algebra, we write $X(R) = X(\mathrm{Spec}(R))$ for the R-points.

If $X = \mathbb{V}(I) \subseteq \mathbb{A}^n_k$, then

$$X(R) = \{(x_i) \in R^n \mid f(x_i) = 0 \text{ for all } f \in I\}$$

We will write $X \times Y = X \times_k Y = X \times_{\mathrm{Spec}(k)} Y$. Finally, in this course,

Definition 4.1 (variety)
A *variety* is a separated k-scheme X of finite type which is geometrically integral[a].

[a] That is, $X \times_k \mathrm{Spec}(\overline{k})$ is integral.

Example 4.2
$X = \mathrm{Spec}(\mathbb{Q}(\sqrt{2})) = \mathbb{V}(t^2 - 2) \subseteq \mathbb{A}^1_{\mathbb{Q}}$ is not a variety. This is not a variety as $X \times \mathrm{Spec}(\overline{\mathbb{Q}})$ is two points.

Theorem 4.3. Let X/k be a proper variety[a]. Then $\mathrm{H}^0(X, \mathcal{O}_X) = k$.

[a] That is, the map $X \to \mathrm{Spec}(k)$ is proper.

Proof. Since

$$\mathrm{H}^0(X, \mathcal{O}_X) \otimes_k \overline{k} = \mathrm{H}^0(X_{\overline{k}}, \mathcal{O}_{X_{\overline{k}}})$$

we may assume that $k = \overline{k}$ is algebraically closed. Now $\mathrm{H}^0(X, \mathcal{O}_X)$ is a finite integral k-algebra, which must be just k. □

Remark 4.4. Proper varieties are also called *complete* varieties.

Definition 4.5
A *k-group scheme* is a k-scheme G, with a morphism $m : G \times G \to G$, such that for every k-algebra R, m makes $G(R)$ into a group.

Note the set G is rarely a group.

Example 4.6
The additive group $\mathbb{G}_a = \mathrm{Spec}(k[t])$. The group operation is given by

$$\begin{aligned} k[t] &\to k[u, v] \\ t &\mapsto u + v \end{aligned}$$

and for any k-algebra R, $\mathbb{G}_a(R) = (R, +)$.

Example 4.7

The multiplicative group $\mathbb{G}_m = \operatorname{Spec}(k[t, t^{-1}])$. The group operation is given by

$$\begin{aligned} k[t] &\to k[u, v, u^{-1}, v^{-1}] \\ t &\mapsto uv \end{aligned}$$

For any k-algebra R, $\mathbb{G}_m(R) = (R^*, \cdot)$.

Example 4.8

The general linear group is

$$\mathrm{GL}_n = \operatorname{Spec}\left(k\left[(t_{ij}), \frac{1}{\det(t_{ij})}\right]\right)$$

with multiplication

$$t_{ij} \mapsto \sum_{i=1}^{n} t_{ir} t'_{rj} \in k\left[(t_{ij}), (t_{ij})', \frac{1}{\det(t_{ij}) \det(t'_{ij})}\right]$$

Example 4.9

Let Γ be a group, and

$$G = \bigsqcup_{\gamma \in \Gamma} \operatorname{Spec}(k)$$

with the discrete topology. There's an obvious way to make this into a group scheme using the group structure on Γ.

Lecture 15

Recall from earlier that $\mathbb{G}_a = \operatorname{Spec}(k[t])$, and this does not form a group. For example, what do we do with the generic point? Even if we just consider closed points, in general, they don't form a group.

The S-valued points, $X(S) = \operatorname{Mor}(S, X)$ (as S varies), plays the role of the usual points.

Let S be a k-scheme. Then $X(S) = \{f : S \to X\}$. Now if we have $g : S' \to S$, then we have a map of sets

$$(\cdot \circ g) : X(S) \to X(S')$$

and this is compatible with composition $S'' \to S' \to S$. In particular, for each X, we have a functor, called the *functor of points*,

$$h_X : \mathbf{Sch}/k \to \mathbf{Sets}^{\mathrm{op}}$$

or quivalently, a contravariant functor $\mathbf{Sch}/k \to \mathbf{Sets}$.

Lemma 4.10 (Yoneda). We have a natural bijection

$$\operatorname{Mor}(X, Y) \leftrightarrow \{(f_S : X(S) \to Y(S))_{S \in \mathbf{Sch}/k} \mid \forall g : S' \to S, x \in X(S), f_S(x) \circ g = f_{S'}(x \circ g)\}$$

Proof. Given $f : X \to Y$ a morphism, $x \in X(S)$, define $f_S = f \circ x \in Y(S)$. On the other hand, given a family (f_S), define $f = f_X(\mathrm{id}_X) \in Y(X) = \operatorname{Mor}(X, Y)$. □

Lemma 4.11 (Yoneda+). The same bijection holds when we restrict to affine S. That is, X is determined by the functor $R \mapsto X(R)$ on k-algebras.

Proof. We need to reconstruct $X(S)$ from the $X(R)$. Let

$$S = \bigcup_i U_i$$

where U_i are affine. Then

$$X(S) = \mathrm{Mor}(S, X) = \{(f_i \in \mathrm{Mor}(U_i, X)) \mid \text{for all } V \subseteq U_i \cap U_j \text{affine, } f_i|_V = f_j|_V\}$$

So the restriction to affines determine $X(S)$. □

Proposition 4.12. Suppose we have a k-group scheme G. Then

(i) for all $S \in \mathbf{Sch}/k$, $G(S)$ is a group,

(ii) for all $S' \to S$ morphism of k-schemes, the corresponding map $G(S) \to G(S')$ is a group homomorphism

Proof. First we prove (ii) for affines. Suppose $S' = \mathrm{Spec}(R') \to \mathrm{Spec}(R) = S$. Then $G(R) \to G(R')$ is a group homomorphism if and only if the right hand square in

$$\begin{array}{ccccc} (G \times G)(R) & \xrightarrow{\cong} & G(R) \times G(R) & \xrightarrow{m_R} & G(R) \\ \downarrow & & \downarrow & & \downarrow \\ (G \times G)(R') & \xrightarrow[\cong]{} & G(R') \times G(R') & \xrightarrow[m_{R'}]{} & G(R') \end{array}$$

commutes. But the outer rectangle commutes, as it is the map induced by $G \times G \to G$, and the left square commutes by definition of the fibre product. Hence the right hand square commutes.

For (i), let $S = \bigcup_i U_i$, U_i affine. Then

$$G(S) = \{(g_i)_{i \in I} \mid g_i \in G(U_i), \text{ forall } V \subseteq U_i \cap U_j \text{ affine }, g_i|_V = g_j|_V\}$$

But we just saw that the restriction maps $G(U_i) \to G(V)$ are homomorphisms, so $G(S)$ is a subgroup of $\prod G(U_i)$.

Same argument as before implies (ii) for general morphisms $S' \to S$. □

Corollary 4.13. There exists $e \in G(k)$, $i : G \to G$ such that for all S, e maps to the identity element of $G(S)$, and $i_S : G_S \to G_S$ is the inverse.

Proof. The first part is because $G(k) \to G(S)$ is a homomorphism, induced by the map $S \to \mathrm{Spec}(k)$. The second part is just Yoneda. Explicitly, $i \in G(G)$ is the inverse of $\mathrm{id}_G \in G(G)$.

Here, id_G is the identity map $G \to G$. □

Remark 4.14. An equivalent definition of a group scheme consists of $(G, m : G \times G \to G, e \in G(k), i : G \to G)$, such that various diagrams commute, such as

$$\begin{array}{ccccc} (G \times G) \times G & \xrightarrow{m \times \mathrm{id}} & G \times G & & \\ & & & \searrow^{m} & \\ \cong \Big| & & & & G \\ & & & \nearrow & \\ G \times (G \times G) & \xrightarrow[m \times \mathrm{id}]{} & G \times G & & \end{array}$$

for associativity, and so on. That is, it is a group object in $\mathbf{Sch}/k$.

Definition 4.15

A *homomorphism of group schemes* is a morphisms $f : G \to G'$, such that for all R, $f_R : G(R) \to G(R')$ is a homomorphism.

By Yoneda, $f_S : G(S) \to G(S')$ is a homomorphism for all $S \in \mathbf{Sch}/k$. We leave as an exercise to show that f is a homomorphism if and only if a certain diagram commutes.

Definition 4.16

A *closed subgroup scheme* is a closed subscheme $i : H \to G$, such that for all R, $H(R) \subseteq G(R)$ is a subgroup.

We need to check that H is a group scheme. For all S, $H(S) \subseteq G(S)$ is a subgroup (same argument as before), and we have a diagram

$$\begin{CD} (H \times H)(S) @>{i \times i}>> (G \times G)(S) \\ @VVV @VV{m_S}V \\ H(S) @>>> G(S) \end{CD}$$

So we need to show that we have the dotted arrow making the diagram commute. Taking $S = H \times H$, and $\mathrm{id}_{H \times H} \in (H \times H)(H \times H)$, this maps to some $H \times H \to H$ making H into a group scheme.

Example 4.17 (trivial)

$e : \mathrm{Spec}(k) \hookrightarrow G$ is a closed subgroup scheme.

Example 4.18

Let $f : G \to G'$ be a homomorphism of group schemes. Define $\ker(f)$ to be the fibre product

$$\begin{CD} \ker(f) @>>> G \\ @VVV @VV{f}V \\ \mathrm{Spec}(k) @>>{e'}> G' \end{CD}$$

By definition of the fibre product, $\ker(f)(S) = \ker(f_S : G(S) \to G'(S))$. e' is a closed point, and so $\ker(f)$ is a closed subscheme. Hence it is a subgroup scheme.

Example 4.19

Let $G = \mathrm{GL}_n$. Then $\det : \mathrm{GL}_n \to \mathrm{GL}_1 = \mathbb{G}_m$, where $\det_R : \mathrm{GL}_n(R) \to \mathbb{R}^\times$ is the usual determinant, is a homomorphism of group schemes. Then we have

$$\mathrm{SL}_n := \ker(\det)$$

Remark 4.20.
1. Existence of quotient schemes is much harder.
2. Essentially everything works as is, if $\mathrm{Spec}(k)$ is replaced by an arbitrary base scheme T. So we replace $\mathbf{Sch}/k$ by $\mathbf{Sch}/T$ and so on. Note

$$\mathbf{Aff}/T = \{S \to T \mid S \text{ is affine}\}$$

Note this is not the same as $S \to T$ being an affine morphism.

Lecture 16

5 Abelian varieties

Definition 5.1 (left translation)
Suppose G/k is a group scheme, and $x \in G(k)$. The *(left) translation* morphism $T_x : G \to G$ is the unique morphism such tha for all k-algebras R, $y \in G(R)$,

$$T_x(y) = xy = m(x, y)$$

That is, T_x is the composition

$$G \times \mathrm{Spec}(k) \times G \xrightarrow{x \times \mathrm{id}_G} G \times G \xrightarrow{m} G$$

Similarly, we can define right translation. Clearly

- $T_e = \mathrm{id}$,
- $T_{xy} = T_x T_y$

So $T_x : G \to G$ is an isomorphism of k-schemes. We don't just need to consider k-points.

Definition 5.2
Let S be a k-scheme, $x \in G(S)$, then define $T_x : G \times S \to G \times S$ to be the morphism

$$G \times S \xrightarrow{\mathrm{id} \times (x, \mathrm{id}_S)} G \times (G \times S) = (G \times G) \times S \xrightarrow{m^t \times \mathrm{id}_S} G \times S$$

where $m^t(g, h) = m(h, g)$.

When $S = \mathrm{Spec}(k)$, this is the same as the above. In particular, taking $S = G, x = \mathrm{id}_G \in G(G)$ to be the "tautological point", we get the universal translation

$$G \times G \to G \times G$$

given by $(g, h) \mapsto (hg, h)$. This is universal since if $x \in G(S)$ for any S, then we have the commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{(m^t, \mathrm{pr}_2)} & G \times G \\ \uparrow{\scriptstyle \mathrm{id}_G \times x} & & \uparrow{\scriptstyle \mathrm{id}_G \times x} \\ G \times S & \xrightarrow{T_x} & G \times S \end{array}$$

In fact, this is a cartesian diagram.

Definition 5.3 (group variety, abelian variety)
A *group variety over k* is a k-group scheme which is also a variety. An *abelian variety* is a group variety which is proper over k.

Remark 5.4. An *algebraic group* is sometimes a group variety, or sometimes a group scheme of finite type.

Example 5.5
$\mathbb{G}_a, \mathbb{G}_m, \mathrm{GL}_n, \mathrm{SL}_n$ are group varieties. The simplest non-trivial abelian variety is an elliptic curve E/k (a non-singular plane cubic with a point $e \in E(k)$). The group law on E is the chord/tangent construction for a plane cubic. This makes E into an abelian variety of dimension 1.

Theorem 5.6 ((classical form of) Mumford's rigidity lemma)**.** Let X, Y, Z be varieties over k, where X is proper and $X(k) \neq \varnothing$. Suppose $f : X \times Y \to Z$ is a morphism. If for some $y_0 \in Y$, $z_0 \in Z$,

$$X \times \{y_0\} \subseteq f^{-1}(z_0)$$

then there exists $g : Y \to Z$ such that $f = g \circ \mathrm{pr}_2$. In particular, for all $y \in Y$, $f(X \times \{y\})$ is a single point.

In words, suppose X is proper, and we are given a family of maps $X \to Z$ indexed by Y. If one of the maps is constant, then they all are.

Note that here

$$X \times \{y_0\} = \mathrm{pr}_2^{-1}(y_0)$$

is the fibre.

Remark 5.7 (properness)**.** Suppose $X = Y = Z = \mathbb{A}^1_k$, and $f : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is given by $f(x, y) = xy$. For $y = 0$, this is the zero morphism $\mathbb{A}^1 \to \mathbb{A}^1$. But if $y \neq 0$ it is an isomorphism.

Corollary 5.8. Suppose X is an abelian variety, G any group variety. If $f : X \to G$ is a morphism and $g = f(e_X)$. Then $T_{g^{-1}} \circ f$ is a homomorphism.

Proof. Replacing f with $T_{g^{-1}} \circ f$, we may assume $f(e_X) = e_G$. Consider the morphism $h : X \times X \to G$, given on (R-valued) points by

$$h(x, y) = f(x)f(y)f(xy)^{-1}$$

Then

$$h(X \times \{e_X\}) = \{e_G\} = h(\{e_X\} \times X)$$

Applying rigidity, the first equality implies that h factors through pr_2. The second equality implies h factors through pr_1. Hence $h(x, y) = e_G$ for all $x, y \in X(R)$. Hence f is a homomorphism. □

Corollary 5.9. Abelian varieties are commutative.

Proof. Consider the inverse map $\iota : X \to X$. As $\iota(e) = e$, ι is a homomorphism. This means $X(R)$ is commutative for all R, since a group is abelian if and only if the inverse map is a homomorphism. □

Remark 5.10. The "abelian" in abelian varieties is the same Abel as for abelian groups, but not for the same reason. Similarly, abelian functions are a generalisation of elliptic functions.

Instead of the rigidity lemma, we will prove a stronger statement:

Theorem 5.11. Let $p : X \to Y$ be a proper morphism of schemes, where Y is integral, and admits a section $s : Y \to X$. That is, $p \circ s = \mathrm{id}_Y$. Assume $p_*\mathcal{O}_X = \mathcal{O}_Y$.

Let $f : X \to Z$ be a morphism where Z is a separated scheme. Suppose there exists $y \in Y, z \in Z$, such that $X_y = p^{-1}(y) \subseteq f^{-1}(z)$. So f collapses X_y to a point. Then $f = g \circ p$, for some unique $g : Y \to Z$.

Remark 5.12. To deduce theorem 5.6, we have $p = \mathrm{pr}_2 : X \times Y \to Y$. Integral and separated follow from Y being a variety. s is given by any element of $X(k)$. $(\mathrm{pr}_2)_*\mathcal{O}_{X\times Y} = \mathcal{O}_Y$, since $\mathrm{H}^0(X, \mathcal{O}_X) = k$ for a proper variety.

Proof. If g exists, then $f \circ s = g \circ (p \circ s) = g$, and so g is unique, and it exists if and only if $f = f \circ (s \circ p)$. Suppose also there exists an open dense $Y' \subseteq Y$, over which g exists. Then $f|_{f^{-1}(Y')} = (f \circ s \circ p)|_{f^{-1}(Y')}$. But as Y is reduced and Z is separated, $f = f \circ s \circ p$ everywhere, see Hartshorne.

We know f maps $X_y \subseteq X$ to $\{z\} \subseteq Y$, and so let W be an open affine neighbourhood of z. Then $f^{-1}(W)$ is an open neighbourhood of X_y. Since p is proper, it is closed, and any such neighbourhood of $f^{-1}(W)$ contains $p^{-1}(Y')$ for some sufficiently small open affine neighbourhood Y' of y.

To see this, let $T = p(X \setminus f^{-1}(W))$. This is closed and does not contain y. Hence any open affine neighbourhood of y in $Y \setminus T$ works. So let $X' = p^{-1}(Y')$, and let $p' : X' \to Y'$ be the restriction. Then f maps X' to $W = \mathrm{Spec}(B) \subseteq Z$. Hence this is induced by a ring homomorphism $B \to \Gamma(X', \mathcal{O}_{X'}) = \Gamma(Y', \mathcal{O}_{Y'})$, since $f_*\mathcal{O}_X = \mathcal{O}_Y$. So $f|_{V'}$ factors as $X' \to Y' \to W$. Since Y is integral, Y' is dense in Y. □

Lecture 17

Proposition 5.13. Let k be a field, G a group variety over k, then G is smooth over k.

Proof. First of all, if X/k is a scheme of finite type, let $\overline{X} = X \times_k \mathrm{Spec}(\overline{k})$. Then X/k is smooth if and only if $\overline{X}/\overline{k}$ is smooth.

To see this, smooth is equivalent to Ω being locally free (and of the correct rank). Let $p_X : \overline{X} \to X$ be the projection map. Then

$$\Omega_{\overline{X}/\overline{k}} = p_X^* \Omega_{X/k}$$

Thus, if $\Omega_{X/k}$ is locally free, then so is $\Omega_{\overline{X}/\overline{k}}$. For the other direction, it reduces to $X = \mathrm{Spec}(A)$ and $\Omega_{X/k} = \widetilde{M}$, where $M = \Omega_{A/k}$ which is a finite A-module. We know that $M \otimes_k \overline{k}$ is a free $A \otimes \overline{k}$-module. For this, we need to take a sufficiently small open affine. Then $M \subseteq M \otimes_k \overline{k}$ is a direct summand as an A-module,, since $k \subseteq \overline{k}$ has a vector space complement. Hence $M \otimes_k \overline{k}$ is a free A-module, so M is projective and thus locally free.

So if G is a group variety, we may assume $k = \overline{k}$. Then G is integral, and so there exists a non-empty open $U \subseteq G$ which is smooth. If $x \in G(k)$ and $y \in U(k)$, then $T_{xy^{-1}}(U) \subseteq G$ is a smooth open containing x. So we can cover G by smooth opens, so G is smooth. □

Remark 5.14. If $\mathrm{char}(k) = 0$, then Cartier's theorem says that every finite type group scheme over k is smooth. This is false for $\mathrm{char}(k) = p > 0$.

Our aim is now to:

- prove that abelian varieties are projective,
- show $X[n] = \{x \in X(\overline{k}) \mid nx = e\}$ is finite (and determine the structure),
- relate $X(\overline{k})$ and $\mathrm{Pic}(X \times \mathrm{Spec}(\overline{k}))$, where the Picard group is the group of isomorphism classes of line bundles.

The last one is a generalisation of the Abel-Jacobi theorem, where if E is an elliptic curve, then we have a homomorphism

$$\begin{aligned} E(\overline{k}) &\to \mathrm{Cl}^0(E \times \mathrm{Spec}(\overline{k})) \\ p &\mapsto [p] - [e] \end{aligned}$$

6 Seesaw, cube and square

Theorem 6.1 (seesaw). Let X, Y be varieties over k, with X proper. Let $\mathcal{L}$ be a line bundle[a] on $X \times Y$. For $y \in Y$, write

$$i_y : X \times \mathrm{Spec}(\kappa(y)) \to X \times Y$$

for the inclusion of a fibre $\mathrm{pr}_2^{-1}(y)$. Suppose for all closed points $y \in Y$, $\mathcal{L}(y) = i_y^*\mathcal{L}$ is trivial[b]. Then there exists a line bundle $\mathcal{M}$ on Y, such that $\mathcal{L} \cong \mathrm{pr}_2^*\mathcal{M}$. Morover, $\mathcal{M} \cong (\mathrm{pr}_2)_*\mathcal{L}$, and so it is unique up to isomorphism.

[a] i.e. an invertible $\mathcal{O}_{X\times Y}$-module, or locally free of rank 1

[b] i.e. isomorphic to $\mathcal{O}_{X\times y}$, where $X \times y = X \times \mathrm{Spec}(\kappa(y))$.

Proof. First, we will show that every $y \in Y$ has an open affine neighbourhood V, such that $\mathcal{L}|_{X\times V} \cong \mathcal{O}_{X\times V}$, and so $\mathcal{L}|_{X\times V} \cong \mathrm{pr}_2^* \mathcal{O}_V$. Suppose $Y = \mathrm{Spec}(A)$ is affine. Then $X \times Y \to Y$ is flat, and so there exists a finite A-module Q, such that for all A-algebras B,

$$\mathrm{H}^0(X \times \mathrm{Spec}(B), \mathcal{L}_B) = \mathrm{Hom}_A(Q, B)$$

In particular,

$$\mathrm{H}^0(X \times y, i_y^* \mathcal{L}) = \mathrm{Hom}_A(Q, \kappa(y)) = (Q \otimes_A \kappa(y))^\vee$$

As $\mathrm{H}^0(X \times y, i_y^* \mathcal{L}) \cong \mathrm{H}^0(X \times y, \mathcal{O}_{X\times y})$, which has dimension 1 for all closed points $y \in Y$. Hence

$$\dim_{A/\mathfrak{m}}(Q/\mathfrak{m}Q) = 1$$

for all $\mathfrak{m} \in \mathrm{maxSpec}(A)$, and so Q is locally free of rank 1.

Localising further, every y_0 has an open neighbourhood for which Q is free of rank 1, say equal to $A \cdot q$. Then $\mathrm{H}^0(X \times \mathrm{Spec}(A), \mathcal{L}) = \mathrm{Hom}(Q, A) = A \cdot q^\vee$. For each closed $y \in \mathrm{Spec}(A)$, $q^\vee$ maps to a generator of $\mathrm{H}^0(X \times y, \iota_y \mathcal{L})$, which is everywhere non-vanishing on $X \times y$, since $i_y^* \mathcal{L} \cong \mathcal{O}$. Hence $q^\vee$ defines an isomorphism

$$\mathcal{O}_{X\times \mathrm{Spec}(A)} \cong \mathcal{L}$$

and as

$$(\mathrm{pr}_2)_* \mathcal{O}_{X\times \mathrm{Spec}(A)} = \mathcal{O}_{\mathrm{Spec}(A)}$$

we have that

$$\mathrm{pr}_2^* (\mathrm{pr}_2)_* \mathcal{L} \cong \mathcal{L}$$

and so in this case, $\mathcal{M} = \mathcal{O}_{\mathrm{Spec}(A)}$.

For general Y, the preceding argument shows that if $\mathcal{M} = (\mathrm{pr}_2)_* \mathcal{L}$, then $\mathcal{M}$ is locally free of rank 1, and the map $\mathrm{pr}_2^* \mathcal{M} \to \mathcal{L}$ (through adjunction) is an isomorphism.

If $\mathcal{L} \cong \mathrm{pr}_2^* \mathcal{M}'$ say, then we have an isomorphism $\mathcal{M}' \cong (\mathrm{pr}_2)_* \mathcal{L}$ (as locally on Y, both $\mathcal{L}$ and $\mathcal{M}'$ are free), and so $\mathcal{M} \cong \mathcal{M}'$. ☐

> **Remark 6.2.** If we have the fibre product $X \times Y$, with $Y \to \mathrm{Spec}(k)$ flat, and so by flat base change,
>
> $$(\mathrm{pr}_2)_* \mathcal{O}_{X\times Y} = \mathcal{O}_Y$$
>
> as $\mathrm{H}^0(X, \mathcal{O}_X) = k$.

If moreover, $x \in X(k)$ and $\mathcal{L}|_{x\times Y}$ is trivial (as $\mathcal{L}_{x\times Y} \cong M$), and $\mathcal{L}$ is trivial. However, it is not sufficient to just assume that $\mathcal{L}_{x\times Y}, \mathcal{L}_{X\times y}$ for fixed (x, y). We need it to be true for *all* (x, y).

On the other hand, if we have $X \times Y \times Z$, then the analogous statement would be true. That is, if

$$\mathcal{L}|_{x\times Y\times Z}, \mathcal{L}|_{X\times y\times Z}, \mathcal{L}|_{X\times Y\times z}$$

are trivial, then so is $\mathcal{L}$.

Lecture 18

> **Theorem 6.3** (cube)**.** Let X, Y, Z be k-varieties, with X, Y proper. Let $x \in X, y \in Y, z \in Z$, and $\mathcal{L}$ be a line bundle on $X \times Y \times Z$. Suppose that each of
>
> $$\mathcal{L}|_{x\times Y\times Z}, \mathcal{L}|_{X\times y\times Z}, \mathcal{L}|_{X\times Y\times z}$$
>
> is trivial. Then $\mathcal{L}$ is trivial.

Proof. See section 8. ☐

> **Remark 6.4.** (i) In fact, this is true for more general k-schemes Z. On the other hand, the assumption of X, Y being proper is essential.
>
> (ii) For all line bundles $\mathcal{L}$ on $X \times Y \times Z$, there exists an isomorphism
>
> $$\mathcal{L} \cong \mathrm{pr}_{X\times Y}^* \mathcal{L}_1 \otimes \mathrm{pr}_{X\times Z}^* \mathcal{L}_2 \otimes \mathrm{pr}_{Y\times Z}^* \mathcal{L}_3$$

for line bundles $\mathcal{L}_i$ on the appropriate spaces. On the other hand, this is not true for the product of two varieties.

Corollary 6.5. Let X be an abelian variety over k, and $\mathcal{L}$ a line bundle on X. Let Y be any k-scheme, $f, g, h : Y \to X$ morphisms. Then

$$\mathcal{M} = \mathcal{M}_{f,g,h} = (f+g+h)^*\mathcal{L} \otimes (f+g)^*\mathcal{L}^\vee \otimes (f+h)^*\mathcal{L}^\vee \otimes (g+h)^*\mathcal{L}^\vee \otimes f^*\mathcal{L} \otimes g^*\mathcal{L} \otimes h^*\mathcal{L}$$

is trivial. Here, $f+g : Y \to X$ is the composition

$$Y \xrightarrow{f\times g} X \times X \xrightarrow{m} X$$

Proof. First consider the case where $Y = X \times X \times X$, $f = \mathrm{pr}_1^3, g = \mathrm{pr}_2^3, h = \mathrm{pr}_3^3$ are the projections. Here, pr_i^n is the projection $X^n \to X$ on the i-th factor. Let $x = y = z = e$. By the theorem of the cube, we need to show that $\mathcal{M}|_{X\times X\times e} \cong \mathcal{O}_{X\times X}$. By symmetry it'll hold for the other cases. Now, $\mathcal{M}|_{X\times X\times e} = q^*\mathcal{M}$, where

$$q : X \times X \to X \times X \times X$$
$$q(x, y) = (x, y, e)$$

Now

$$\mathrm{pr}_1^3 \circ q = \mathrm{pr}_1^2$$
$$\mathrm{pr}_2^3 \circ q = \mathrm{pr}_2^2$$
$$\mathrm{pr}_3^3 \circ q = \text{constant morphism } e$$

Hence

$$(\mathrm{pr}_1^3 + \mathrm{pr}_2^3 + \mathrm{pr}_3^3) \circ q = (\mathrm{pr}_1^3 + \mathrm{pr}_2^3) \circ q = m$$
$$(\mathrm{pr}_1^3 + \mathrm{pr}_3^3) \circ q = \mathrm{pr}_1^2$$
$$(\mathrm{pr}_2^3 + \mathrm{pr}_3^3) \circ q = \mathrm{pr}_2^2$$

With this,

$$q^*\mathcal{M} = m^*\mathcal{L} \otimes m^*\mathcal{L}^\vee \otimes (\mathrm{pr}_1^2)^*\mathcal{L}^\vee \otimes (\mathrm{pr}_2^2)^*\mathcal{L}^\vee \otimes (\mathrm{pr}_1^2)^*\mathcal{L} \otimes (\mathrm{pr}_2^2)^*\mathcal{L} \otimes \mathcal{O}_{X\times X} \cong \mathcal{O}_{X\times X}$$

and so $\mathcal{M} \cong \mathcal{O}_{X\times X\times X}$. For the general case, we have that

$$\mathcal{M}_{f,g,h} = (f \times g \times h)^*\mathcal{M}_{\mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_3}$$

□

Theorem 6.6 (square). Let X be an abelian variety, $\mathcal{L}$ a line bundle on X, $x, y \in X(k)$. Then $T_{x+y}^*\mathcal{L} = T_x^*\mathcal{L} \otimes T_y^*\mathcal{L} \otimes \mathcal{L}^\vee$.

Proof. Apply the corollary with $Y = X, f = x, g = y, h = \mathrm{id}_X$. Then $f+h = T_x, g+h = T_y$ and $f+g+h = T_{x+y}$. $f+g$ is the constant morphism $x+y$. If $p : X \times X$ is constant, then $p^*\mathcal{L} \cong \mathcal{O}_X$. Then we get

$$T_{x+y}^*\mathcal{L} \otimes T_x^*\mathcal{L}^\vee \otimes T_y^*\mathcal{L}^\vee \otimes L \cong \mathcal{O}_X$$

□

Corollary 6.7. Let X be an abelian variety, $\mathcal{L}$ an abelian variety on X. For $n \in \mathbb{Z}$, let $[n] : X \to X$ be the multiplication by n morphism. Then

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n(n+1)/2} \otimes (i^*\mathcal{L})^{\otimes n(n-1)/2}$$

Here, $i = [-1]$ is the inversion map.

In particular, if $\mathcal{L}$ is symmetric, i.e. if $\mathcal{L}$ is isomorphic to $i^*\mathcal{L}$, then $[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n^2}$. On the other hand, if $i^*\mathcal{L} \cong \mathcal{L}^\vee$, then $[n]^*\mathcal{L} \cong \mathcal{L}^{\otimes n}$.

Here, for n negative,

$$\mathcal{L}^{\otimes n} = (\mathcal{L}^\vee)^{\otimes |n|}$$

Proof. It's clear when $n = 0, 1$. For $n \geq 2$, apply corollary 6.5 with $f = [n-1], g = \mathrm{id}_X = [1], h = i = [-1]$. Then

$$\begin{aligned} f + g + h &= [n-1] \\ f + g &= [n] \\ f + h &= [n-2] \\ g + h &= [0] \end{aligned}$$

Hence we get that

$$[n-1]^*\mathcal{L} \cong [n]^*\mathcal{L} \otimes [n-2]^*\mathcal{L} \otimes [0]^*\mathcal{L} \otimes [n-1]^*\mathcal{L}^\vee \otimes [1]^*\mathcal{L}^\vee \otimes [-1]^*\mathcal{L}^\vee$$

In particular,

$$[n]^*\mathcal{L} \cong [n-1]^*\mathcal{L}^{\otimes 2} \otimes [n-2]^*\mathcal{L}^\vee \otimes \mathcal{L} \otimes i^*\mathcal{L}$$

Inserting the result for $n-1$ and $n-2$, we get the result for n. This shows the result for $n \geq 0$. For $n < 0$, note that

$$[-n]^*\mathcal{L} = [n]^*(i^*\mathcal{L})$$

□

7 Picard group of abelian varieties

Recall for any scheme X, $\mathrm{Pic}(X)$ is the group of isomorphism classes of line bundles on X. The group law is tensor product of line bundles. We will study $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X_{\overline{k}})$ for an abelian variety X/k.

Proposition 7.1. Suppose X is a k-variety, which is proper. Then the natural map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\overline{k}})$ induced by the map $X_{\overline{k}} \to X$ is injective.

So we can check identities between line bundles after passing to $\overline{k}$.

Proof. It is a homomorphism, and so it suffices to show that if $\mathcal{L}_{\overline{k}} \cong \mathcal{O}_{X_{\overline{k}}}$, then $\mathcal{L}$ is trivial. But on the examples sheet, we see that $\mathcal{L} \cong \mathcal{O}_X$ if and only if $\mathrm{H}^0(X, \mathcal{L})$ and $\mathrm{H}^0(X, \mathcal{L}^\vee)$ are non-zero. But as $\mathrm{H}^0(X_{\overline{k}}, \mathcal{L}_{\overline{k}}) = \mathrm{H}^0(X, \mathcal{L}) \otimes_k \overline{k}$, we get the result. □

Lecture 19

From now on, assume X is an abelian variety over k. To study $\mathrm{Pic}(X)$, we will use:

Proposition 7.2. Let $\mathcal{L}$ be a line bundle on X. For $x \in X(\overline{k})$, define

$$\varphi_{\mathcal{L}}(x) = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \in \mathrm{Pic}(X_{\overline{k}})$$

More precisely, it's the isomorphism class of $T_x^*\mathcal{L} \otimes \mathcal{L}^\vee$ in the Picard group. Then

$$\varphi_{\mathcal{L}} : X(\overline{k}) \to \mathrm{Pic}(X_{\overline{k}})$$

is a homomorphism.

Proof.

$$\varphi_{\mathcal{L}}(x+y) = T_{x+y}^*\mathcal{L} \otimes \mathcal{L}^\vee$$
$$\varphi_{\mathcal{L}}(x) + \varphi_{\mathcal{L}}(y) = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes T_y^* \otimes \mathcal{L}^\vee$$

These two are equal by the theorem of the square. □

Note we will write the group operation in $\text{Pic}(X)$ either as $+$ or as $\otimes$. Thus, $\varphi_{\mathcal{L}}$ combines the group law on X and on $\text{Pic}(X_{\overline{k}})$.

Definition 7.3 (Néron-Severi group)
Define

$$\text{Pic}^0(X_{\overline{k}}) = \{\mathcal{L} \mid \varphi_{\mathcal{L}} = 0\}$$

and

$$K(\mathcal{L}) = \ker(\varphi_{\mathcal{L}})$$

Define the *Néron-Severi group* of X to be

$$\text{NS}(X_{\overline{k}}) = \frac{\text{Pic}(X_{\overline{k}})}{\text{Pic}^0(X_{\overline{k}})}$$

Note that $\text{NS}(X_{\overline{k}})$ is a subgroup of $\text{Hom}(X(\overline{k}), \text{Pic}(X_{\overline{k}}))$. In the end, $\text{Pic}^0(X_{\overline{k}})$ will turn out to be the $\overline{k}$-points of another abelian variety $\widehat{X}$, the *dual abelian variety*, and $\text{NS}(X_{\overline{k}})$ is a finitely generated free abelian group.

First of all, we will give an alternative characterisation of Pic^0.

Proposition 7.4. Let $\Lambda(\mathcal{L}) = m^*\mathcal{L} \otimes \text{pr}_1^*\mathcal{L}^\vee \otimes \text{pr}_2^*\mathcal{L}^\vee \in \text{Pic}(X \times X)$. Then $\mathcal{L} \in \text{Pic}^0(X)$ if and only of $\Lambda(\mathcal{L})$ is trivial. That is,

$$m^*\mathcal{L} \cong \text{pr}_1^*\mathcal{L} \otimes \text{pr}_2^*\mathcal{L}$$

$\Lambda(\mathcal{L})$ is called the *Mumford line bundle*.

Proof. $\varphi_{\mathcal{L}}$ depends only on the isomorphism class of $\mathcal{L}_{\overline{k}} \in \text{Pic}(X_{\overline{k}})$, and so we can assume $k = \overline{k}$. Let $x \in X(k)$. Then

$$m \circ (\text{id} \times x) = T_x$$
$$\text{pr}_1 \circ (\text{id} \times x) = \text{id}_X$$
$$\text{pr}_2 \circ (\text{id} \times x) = x$$

Hence

$$\Lambda(\mathcal{L})|_{X \times x} = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes \mathcal{O}_X = \varphi_L(x)$$

Moreover, $\Lambda(\mathcal{L})|_{e \times X} = \varphi_{\mathcal{L}}(e) = \mathcal{O}_X$ is trivial. Thus, by seesaw, $\Lambda(\mathcal{L})$ is trivial if and only if for all $x \in X(k)$, $\mathcal{L}|_{X \times x} \cong \mathcal{O}_X$. That is, $\varphi_{\mathcal{L}} = 0$. □

Proposition 7.5.

(i) for all line bundles $\mathcal{L}$, $\text{im}(\varphi_{\mathcal{L}}) \subseteq \text{Pic}^0(X_{\overline{k}})$,

(ii) if $\mathcal{L} \in \text{Pic}^0(X_{\overline{k}})$, then $i^*\mathcal{L} \cong \mathcal{L}^\vee$,

Proof. Without loss of generality, assume $k = \overline{k}$. For (i), let $x \in X(k)$ and $\mathcal{M} = \varphi_{\mathcal{L}}(x) = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee$. For $y \in X(k)$,

$$\varphi_{\mathcal{M}}(y) = T_y^*(T_x^*\mathcal{L} \otimes \mathcal{L}^\vee) \otimes (T_x^*\mathcal{L} \otimes \mathcal{L}^\vee)^\vee \cong \mathcal{O}_X$$

by the theorem of the square. Hence $\varphi_{\mathcal{M}} = 0$, and so $\mathcal{M} \in \text{Pic}^0(X_{\overline{k}})$.

For (ii), by the previous proposition,

$$m^*\mathcal{L} \cong \text{pr}_1^*\mathcal{L} \otimes \text{pr}_2^*\mathcal{L}$$

and consider $d : X \to X \times X$, $d(x) = (x, -x)$. Now

$$d^*(\mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{L}) = \mathcal{L} \otimes i^* \mathcal{L}$$

and

$$d^* m^* \mathcal{L} = (m \circ d)^* \mathcal{L} = \mathcal{O}_X$$

as $m \circ d$ is the constant map. Thus, $\mathcal{L} \otimes i^* \mathcal{L} \cong \mathcal{O}_X$,a nd so $i^* \mathcal{L} \cong \mathcal{L}^\vee$. ☐

Theorem 7.6. Let $\mathcal{L} \in \mathrm{Pic}^0(X)$ be non-trivial. Then $\mathrm{H}^r(X, \mathcal{L}) = 0$ for all $r \geq 0$.

Proof. To start off, we will show $\mathrm{H}^0(X, \mathcal{L})$ is zero. If not, let $s \in \mathrm{H}^0(X, \mathcal{L})$ be a non-zero section. This has an effective divisor $D = \mathrm{div}(s)$, and $\mathcal{L} \cong \mathcal{O}_X(D)$. Now $\mathcal{L} \in \mathrm{Pic}^0(X)$, and so $\mathcal{L}^\vee \cong i^* \mathcal{L} = \mathcal{O}_X(i^* D)$. and so $\mathrm{H}^0(X, \mathcal{L}^\vee) \cong \mathrm{H}^0(X, \mathcal{O}_X(i^* D)) \neq 0$. But then from the examples sheet, if both $\mathcal{L}$ and $\mathcal{L}^\vee$ have a non-zero global section, then $\mathcal{L}$ is trivial.

So assume for all $0 \leq i < n$, $\mathrm{H}^i(X, \mathcal{L}) = 0$. Consider

$$\begin{array}{ccc} X & \xrightarrow{\quad \mathrm{id} \quad} & X \\ & {\scriptstyle \mathrm{id} \times e} \searrow \qquad \nearrow {\scriptstyle m} & \\ & X \times X & \end{array}$$

So on $\mathrm{H}^n(X, \mathcal{L})$, the identity map factors as

$$\mathrm{H}^n(X, \mathcal{L}) \xrightarrow{m^*} \mathrm{H}^n(X \times X, m^* \mathcal{L}) \xrightarrow{(\mathrm{id} \times e)^*} \mathrm{H}^n(X, \mathcal{L})$$

Now $m^* \mathcal{L} \cong \mathrm{pr}_1^* \mathcal{L} \otimes \mathrm{pr}_2^* \mathcal{L} = \mathcal{L} \boxtimes \mathcal{L}$. By Künneth,

$$\mathrm{H}^n(X \times X, \mathcal{L} \boxtimes \mathcal{L}) \cong \bigoplus_{p+q=n} \mathrm{H}^p(X, \mathcal{L}) \otimes_k \mathrm{H}^q(X, \mathcal{L})$$

The right hand side vanishes by the induction hypothesis. But the identity map factors through the zero map, and so $\mathrm{H}^n(X, \mathcal{L}) = 0$. ☐

Lecture 20

Proposition 7.7. Suppose $\mathcal{L} \in \mathrm{Pic}(X)$. Then there exists a closed subgroup scheme $Z \subseteq X$ such that $K(\mathcal{L}) = Z(\bar{k})$, and $\Lambda(\mathcal{L})|_{X \times Z}$ is trivial.

Proof. Recall

$$\Lambda(\mathcal{L}) = m^* \mathcal{L} \otimes \mathrm{pr}_1^* \mathcal{L}^\vee \otimes \mathrm{pr}_2^* \mathcal{L}^\vee \in \mathrm{Pic}(X \times X)$$

and that for all $x \in X(\bar{k})$,

$$\Lambda(\mathcal{L})|_{X \times x} = T_x^* \mathcal{L} \otimes \mathcal{L}^\vee = \varphi_{\mathcal{L}}(x)$$

and

$$\Lambda(\mathcal{L})|_{e \times X} = \mathcal{O}_{X_{\bar{k}}}$$

Hence

$$K(\mathcal{L}) = \{x \in X(\bar{k}) \mid \Lambda(\mathcal{L})|_{X \times x} \cong \mathcal{O}_{X_{\bar{k}}}\}$$

Thus, if

$$Z = \{x \in X \mid \Lambda(\mathcal{L})|_{X \times x} \cong \mathcal{O}_{X \times \mathrm{Spec}(\kappa(x))}\}$$

Then by sheet 3 question 2, Z is closed, and $Z(\bar{k})$ is $K(\mathcal{L})$.

Give Z the reduced subscheme structure. Then the seesaw theorem implies that

$$\Lambda(\mathcal{L})|_{X \times Z} = \mathrm{pr}_2^* \mathcal{M}$$

for some $\mathcal{M} \in \mathrm{Pic}(Z)$. Now

$$\Lambda(\mathcal{L})|_{e \times X} \cong \mathcal{O}_X$$

and so restricting to $e \times Z$, we get $\mathcal{M} \cong \mathcal{O}_Z$. So we just need to check that Z is a subgroup scheme.

First, Z is non-empty, and so consider the automorphism of $X \times X$ given by

$$\begin{aligned} f : X \times X &\to X \times X \\ (x, y) &\mapsto (x + y, y) \end{aligned}$$

We just need to check that it takes $Z \times Z$ isomorphically to itself. Since Z is reduced, and X is a variety, it is enough to check

$$f(Z \times Z)(\overline{k}) = (Z \times Z)(\overline{k})$$

But this is true since $Z(\overline{k}) = K(\mathcal{L})$ is a subgroup of $X(\overline{k})$. □

Remark 7.8. (i) In fact, it's not hard to prove that there exists a unique closed subgroup scheme $\underline{K}(\mathcal{L})$, which need not be reduced, such that for any closed subscheme $S \subseteq X$, $S \subseteq \underline{K}(\mathcal{L})$ if and only if

$$\Lambda(\mathcal{L})|_{X \times S} \cong \mathcal{O}_{X \times S}$$

(ii) If $K(\mathcal{L})$ is infinite, then $\dim(Z) > 0$. Thus, taking the irreducible component containing e, there exists an abelian subvariety $Y \subseteq X$ such that $Y(\overline{k}) \subseteq K(\mathcal{L})$. This is immediate if $k = \overline{k}$, and in general, we just need to check that Y is geometrically integral.

7.1 Ampleness criterion

Suppose $\mathcal{L} = \mathcal{O}_X(D)$ for some $D \geq 0$ effective divisor. We can take Weil or Cartier divisors, since they are equivalent in this case. Say

$$D = \sum_i n_i D_i$$

where $n_i \geq 0$ and D_i are integral closed subschemes of codimension 1 of X. In this case,

$$K(\mathcal{L}) \supseteq H(D) = \{x \in X(\overline{k}) \mid T_x(D) = D\}$$

where $T_x(D) = D$ is equality of divisors on $X_{\overline{k}}$. $H(D)$ is a subgroup of $X(\overline{k})$, and it is also the $\overline{k}$-points of a closed subscheme of $X_{\overline{k}}$.

Assume $k = \overline{k}$. So if $x \in X(k)$, and $Y \subseteq X$ any closed subset, $T_x Y = Y$ if and only if for all $y \in Y(k)$, $x + y \in Y$. Equivalently,

$$x \in \bigcap_{y \in Y(k)} \{x \mid (x, y) \in m^{-1}(Y)\} = \bigcap_{y \in Y(k)} \mathrm{pr}_1(X \times y \cap m^{-1}(Y))$$

and the right hand side is a closed subset.

Theorem 7.9. Let $\mathcal{L} = \mathcal{O}_X(D)$. Then the following are equivalent:

(i) $\mathcal{L}$ is ample,

(ii) $K(\mathcal{L})$ is finite,

(iii) $H(D)$ is finite.

Remark 7.10. Since $H(D) \subseteq K(\mathcal{L}) \subseteq X(\overline{k})$, they only depend on $\mathcal{L}_{\overline{k}}$. But if $\mathcal{L}_{\overline{k}}$ is ample, then so is $\mathcal{L}$. For this, note that

$$\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^n) \otimes_k \overline{k} = \mathrm{H}^i(X_{\overline{k}}, \mathcal{F}_{\overline{k}} \otimes \mathcal{L}^n_{\overline{k}})$$

for $\mathcal{F}$ coherent on X. See theorem 3.19(iii).

In particular, we can assume $k = \overline{k}$ to prove the theorem.

Proof. (ii) implies (iii) is obvious, as $H(D) \subseteq K(\mathcal{L})$. We'll now prove (i) implies (ii). Suppose $\mathcal{L}$ is ample, but $K(\mathcal{L})$ is infinite. Then there exists an abelian subvariety Y[3], such that $K(\mathcal{L})$ contains $Y(\overline{k})$ and $\dim(Y) > 0$. But $\mathcal{L}$ restricited to Y (or any closed subscheme) is also ample, by any of the criterion in theorem 3.19.

[3] i.e. a subvariety which is also a closed subgroup scheme

Replacing X by Y, we may assume that $K(\mathcal{L}) = X(k)$. That is, $\mathcal{L} \in \mathrm{Pic}^0(X)$. But then $i^*\mathcal{L} \cong \mathcal{L}^\vee$, and i is an automorphism, and so $\mathcal{L}^\vee$ is ample. Note that if $\mathcal{L}_1, \mathcal{L}_2$ are ample, then for all coherent $\mathcal{F}$, then there exists n_0 such that for $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}_1^n$ and $\mathcal{L}_2^n$ are generated by global sections, and so $F \otimes (\mathcal{L}_1 \otimes \mathcal{L}_2)^n$ is also generated by global sections, and so $\mathcal{L}_1 \otimes \mathcal{L}_2$ is ample.

But in this case, we have that $\mathcal{L} \otimes \mathcal{L}^\vee = \mathcal{O}_X$ is ample. But as $\dim(X) > 0$, there exists a non-empty proper closed subscheme W, and the ideal sheaf $\mathcal{I}_W$ has

$$0 = \mathrm{H}^0(X, \mathcal{I}_W) \subseteq \mathrm{H}^0(X, \mathcal{O}_X) = k$$

So $\dim(X) = 0$. Contradiction.

Lecture 21

It remains to show that if $H(D)$ is finite, then $\mathcal{L}$ is ample. It suffices to show that $\mathcal{L}^2$ is ample. For any $x \in X(k)$, by the theorem of the square,

$$\mathcal{L}^2 \cong T_x^*\mathcal{L} \otimes T_{-x}^*\mathcal{L} \cong \mathcal{O}_X(T_x^*D + T_{-x}^*D)$$

Let $s_x \in \mathrm{H}^0(X, \mathcal{L}^2)$ be non-zero, with $\mathrm{div}(s_x) = T_x^*D + T_{-x}^*D$. Note here $T_x^*D = T_x^{-1}D = T_{-x}D$. If $y \in X(k)$, then $x_x(y) = 0$ if and only if $y \in T_x^*D + T_{-x}^*D$. In turn, this is true if and only if $x \in E_y = T_y^*D + T_{-y}^*D$.

Thus, for all $x \notin (X \setminus E_y)(k)$, $s_x(y) \neq 0$. Therefore, sections of $\mathcal{L}^2$ give a morphism $f : X \to \mathbb{P}_k^N$, where $N = \dim(\mathrm{H}^0(X, \mathcal{L}^2)) - 1$, and $f^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{L}^2$.

Note here if

$$\mathrm{H}^0(X, \mathcal{L}^2) = \mathrm{span}_k\{f_0, \ldots, f_N\}$$

Then $X = \bigcup_i U_i$, where $U_i = \{f_i \neq 0\}$. Define

$$f : U_i \to \mathbb{A}_k^N = \{t_i \neq 0\} \subseteq \mathbb{P}_k^N$$
$$f = \left(\frac{f_0}{f_i}, \ldots, \frac{f_{i-1}}{f_i}, \frac{f_{i+1}}{f_i}, \ldots, \frac{f_n}{f_i}\right)$$

Here, $f_j/f_i \in \mathcal{O}_X(U_i)$ as $\mathcal{L}$ is invertible.

Claim 7.11. f has finite fibres.

Proof of claim. Suppose $y, y' \in X(k)$ are such that $f(y) = f(y')$. Then for all $s \in \mathrm{H}^0(X, \mathcal{L}^2)$, either both $s(y), s(y')$ are zero, or they are both non-zero. In particular, $x \in E_y$ if and only if $x \in E_{y'}$. Thus, as subsets of X, $E_y = E_{y'}$.

So if f does not have finite fibres, then there exists $p \in \mathbb{P}^N(k)$, such that $f^{-1}(p)$ contains a closed subscheme Y of positive dimension, and for all $y, y' \in Y$, $E_y = E_{y'}$. If D' is any component of D, then T_y^*D' is equal to $T_{y'}^*D' \in E_{y'}$, since they are equal for $y = y'$, and as Y is connected, they have to be equal for all y'. Thus, $T_{y'-y}^*D'$ for all components D' of D, and so $y' - y \in H(D)$, which is a finite set. Contradiction.

□

Remark 7.12. Note that it might happen that $f^{-1}(p)$ is not connected, and $y' - y \in H(D)$ only holds for y, y' in the same connected component.

It remains to show that $\mathcal{L}^2$ is ample, using finiteness which we have just proven.

Lemma 7.13. If $f : X \to Y$ is a morphism of proper k-varieties, with finite fibres and $\mathcal{L}$ is an ample line bundle on Y, then $f^*\mathcal{L}$ is an ample line bundle on X.

Sketch proof. f being proper with finite fibres implies that it is finite. This is a consequence of Zariski's main theorem. However, this is easy when f is projective.

Now for any coherent sheaf $\mathcal{F}$ on X, $i > 0$

$$\mathrm{H}^i(X, \mathcal{F} \otimes (f^*\mathcal{L})^n) = \mathrm{H}^i(Y, f_*\mathcal{F} \otimes \mathcal{L}^n) = 0$$

for n sufficiently large. □

□

Corollary 7.14. Abelian varieties are projective.

Proof. It is enough to find an effective divisor such that $H(D)$ is finite. Let U be a non-empty affine open, and let D be the complement, with the reduced subscheme structure. Then (see examples sheet 4), D is a divisor.

Now we will show that $H(D)$ is finite. We can assume k is algebraically closed. Then $H(D)$ is the set of k-points of a closed subgroup scheme $Z \subseteq X$, Here, Z is the closure of $H(D)$. Now for any $x \in H(D)$, $T_x^* D = D$, and so $T_x^* U = U$. Hence if $x_0 \in U$, then $x + x_0 \in U$ for all $x \in H(D)$. So U contains a translate of $H(D)$, and so it contains a translate of Z, since $H(D) = Z(k)$. Now Z is proper, and U is affine, and so $\dim(Z) = 0$. Hence $H(D)$ is finite. □

Corollary 7.15. For all $n \geq 1$, the set

$$\ker([n])(\overline{k}) = \{x \in X(\overline{k}) \mid nx = e\}$$

is finite. The map $[n] : X \to X$ is surjective. Moroever, $X(\overline{k})$ is divisible.

Proof. Suppose $\ker([n])(\overline{k})$ is finite. Then for all $x, x' \in X(\overline{k})$, $[n](x) = [n](x')$ if and only if $x' - x \in \ker([n])$, i.e. for all $y \in X(\overline{k})$, $[n]^{-1}(y)$ is finite.

The morphism $[n] : X \to X$ having finite fibres over each $y \in X(\overline{k})$ has to be dominant, and as X is proper, it has to be surjective. In particular, $[n]^{-1}(y)$ is non-empty for all $y \in X(\overline{k})$, hence $X(\overline{k})$ is divisible.

Thus, it remains to show that $\ker([n])(\overline{k})$ is finite. Assume k is algebraically closed. Suppose if $\ker([n])(\overline{k})$ is infinite. Then it contains a subvariety $V(k)$ of positive dimension. The composition

$$V \hookrightarrow X \xrightarrow{[n]} X$$

is the constant map $v \to \operatorname{Spec}(k) \to X$ at e. Let $\mathcal{L}$ be an ample line bundle on X. If we replace it by $\mathcal{L} \otimes i^*\mathcal{L}$, we may assume that $\mathcal{L} \cong i^*\mathcal{L}$. In this case, $[n]^*\mathcal{L} \cong \mathcal{L}^{n^2}$, which is also ample on X, and so it is ample on V. But $[n]^*\mathcal{L}|_V \cong \mathcal{O}_V$. So V admits a trivial ample line bundle, and so $\dim(V) = 0$. Contradiction. □

Lecture 22

As $[n]$ has finite fibres, and is proper (in fact projective, as X is projective), $[n]$ is a finite morphism (for projective morphisms, this is a rather easy fact, it is enough to find a hypersurface $H \subseteq \mathbb{P}^n$ which (locally) doesn't meet X for $f : X \hookrightarrow \mathbb{P}^N_Y$. Then locally on Y, $X \subseteq \mathbb{P}^N_Y \setminus Y$, which is affine over Y. Then it follows that X is finite over Y).

Now as X is a smooth variety, so $[n] : X \to X$ finite surjective, is flat. Hence $[n]_*\mathcal{O}_X$ is a locally free $\mathcal{O}_X$-module of some rank r, called the degree of $[n]$. This is equal to the degree of the extension of function fields $[n]^* : \kappa(X) \to \kappa(X)$, by passing to the generic point of X.

Theorem 7.16.

$$\deg([n] : X \to X) = n^{2g}$$

where $g = \dim(X)$. In paticular, $\ker([n])(\overline{k})$ has at most n^{2g} points. This holds as

$$n^{2g} = \dim_k(\Gamma(\ker([n]), \mathcal{O}))$$

Proof. Recall if $X \subseteq \mathbb{P}^N$ is projective, and $\mathcal{F}$ is a coherent sheaf on X, then

$$P_X(\mathcal{F}, t) \in \mathbb{Q}[t]$$

is such that

$$P_X(\mathcal{F}, n) = \chi(X, \mathcal{F}(n)) = \chi(X, \mathcal{F} \otimes \mathcal{O}_X(n))$$

with

$$\deg(P_X(\mathcal{F}, \cdot)) = \dim(\operatorname{supp}(\mathcal{F}))$$

Recall from proposition 3.20 that if $X \subseteq \mathbb{P}^N_k$ is integral, $\dim(X) = d$ and with generic point η, then

$$P_X(\mathcal{F}, t) = \dim_{\kappa(\eta)} \left(\mathcal{F}_\eta\right) P_X(t) + \text{terms of degree less than } d$$

Let $\mathcal{L}$ be an ample line bundle on X, and assume $i^*\mathcal{L} \cong \mathcal{L}$. We can do this since we can replace $\mathcal{L}$ with $\mathcal{L} \otimes i^*\mathcal{L}$ if required. This then determines an embedding $X \hookrightarrow \mathbb{P}^N_k$, for which $\mathcal{L} \cong \mathcal{O}_X(1)$. Let $\mathcal{F} = [n]_*\mathcal{O}_X$. Then in this case,

$$P_X([n]_*\mathcal{O}_X, t) = \deg([n])P_X(t) + \text{terms of degree less than } g$$

and $\deg(P_X(t)) = g$. As $[n]$ is finite, for any $m \in \mathbb{Z}$, and any open affine $U \subseteq X$, we have that

$$([n]_*\mathcal{O}_X \otimes \mathcal{L}^m)(U) = ([n]^*\mathcal{L}^m)([n]^{-1}U)$$

and so

$$\mathrm{H}^0(X, [n]_*\mathcal{O}_X \otimes \mathcal{L}^m) = \mathrm{H}^0(X, [n]^*\mathcal{L}^m = \mathrm{H}^0(X, \mathcal{L}^{n^2 m}))$$

Next, recall for m sufficiently large, $P_X(\mathcal{F}, m) = \dim(\mathrm{H}^0(X, \mathcal{F}(m)))$ as the other cohomology groups vanish. For m sufficiently large,

$$\begin{aligned}
\deg([n])P_X(m) &= P_X([n]_*\mathcal{O}_X, m) + \text{terms of degree less than } g \\
&= \dim(\mathrm{H}^0(X, [n]_*\mathcal{O}_X \otimes \mathcal{L}^m)) + \text{terms of degree less than } g \\
&= \dim(\mathrm{H}^0(X, \mathcal{L}^{n^2 m})) + \text{terms of degree less than } g \\
&= P_X(n^2 m) + \text{terms of degree less than } g
\end{aligned}$$

Since $\deg(P_X(t)) = g$,

$$P_X(n^2 t) = n^{2g}P_X(t) + \text{terms of degree less than } g$$

Thus, $\deg([n]) = n^{2g}$. □

Theorem 7.17. If $\mathrm{char}(k) \nmid n$, then $\ker([n])(\overline{k}) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. If $\mathrm{char}(k) = p > 0$, then

$$\ker([p^j])(\overline{k}) \cong (\mathbb{Z}/p_j\mathbb{Z})^r$$

for all $j \geq 1$. Here, $0 \leq r \leq h$ is independent of j. r is called the *p-rank of X*.

To complete the study of $\varphi_\mathcal{L}$, we have the following result:

Theorem 7.18. Suppose $\mathcal{L}$ is ample. Then $\varphi_\mathcal{L} : X(\overline{k}) \to \mathrm{Pic}^0(X_{\overline{k}}) \subseteq \mathrm{Pic}(X_{\overline{k}})$ is surjective. That is,

$$\mathrm{Pic}^0(X_{\overline{k}}) = \frac{X(\overline{k})}{K(\mathcal{L})}$$

where $K(\mathcal{L})$ is finite.

Proof. We may assume $k = \overline{k}$. Let $\mathcal{M} \in \mathrm{Pic}^0(X)$ and suppose $\mathcal{M} \notin \mathrm{im}(\varphi_\mathcal{L})$. Let $\mathcal{F} = \Lambda(\mathcal{L}) \otimes \mathrm{pr}_1^*\mathcal{M}^\vee$. This is a line bundle on $X \times X$. If $x \in X$, then

$$\mathcal{F}|_{X \times x} = T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes \mathcal{M}^\vee \in \mathrm{Pic}(X_{\kappa(x)})$$

By assumption, for all $x \in X(k)$, $\mathcal{F}|_{X \times x} \ncong \mathcal{O}_X$, since $\mathcal{M} \ncong \varphi_\mathcal{L}(x)$. Now by sheet 3, question 2 (ii), $\mathcal{F}|_{X \times x}$ is non-trivial for all $x \in X$. As

$$\mathcal{F}|_{X \times x} = \varphi_\mathcal{L}(x) \otimes \mathcal{M}^\vee \in \mathrm{Pic}^0$$

By theorem 7.6,

$$\mathrm{H}^i(X_{\kappa(x)}, \mathcal{F}|_{X \times x}) = 0$$

for all i. Thus, for any open affine $U \subseteq X$, $\mathrm{H}^i(X \times U, \mathcal{F}|_{X \times U}) = 0$. Therefore, by sheet 2 question 5,

$$\mathrm{H}^i(X \times X, \mathcal{F}) = 0$$

for all i. Equivalently, $R^i\,\mathrm{pr}_2^*\mathcal{F} = 0$. On the other hand, $\mathcal{F}|_{x \times X} = \Lambda(\mathcal{L})|_{x \times X}$ as $\mathrm{pr}_1^*\mathcal{M}|_{x \times X} \cong \mathcal{O}_X = T_x^*\mathcal{L} \cong \mathcal{L}^\vee$. So if $X \notin K(\mathcal{L})$,

$$\mathcal{F}|_{x \times X} \ncong \mathcal{O}_X$$

and so as above,

$$\mathrm{H}^i(x \times X, \mathcal{F}|_{x\times X})$$

is zero for all i. Again, for all open affines $U \subseteq X \setminus K(\mathcal{L})$,

$$\mathrm{H}^i(U \times X, \mathcal{F}|_{U\times X})$$

is zero for all i. As $K(\mathcal{L})$ is finite and X is projective, there exists an open affine V containing $K(\mathcal{L})$. Same exercise shows that

$$0 = \mathrm{H}^i(X \times X, \mathcal{F}) = \mathrm{H}^i(V \times X, \mathcal{F}|_{V\times X})$$

But now for all $x \in V$,

$$\mathrm{H}^i(x \times X, \mathcal{F}|_{x\times X}) = 0$$

Taking $x = e \in K(\mathcal{L}) \subseteq V$, $\mathcal{F}|_{e\times X} = \mathcal{O}_X$. This has non-zero H^0. Contradiction. □

Lecture 23

Proof of theorem 7.17. We may assume $k = \overline{k}$. Since $X(k)$ is divisible, and $\ker([n])(k)$ is finite, it follows that for all primes p, $\ker([p^j])(k) \cong (\mathbb{Z}/p^j\mathbb{Z})^r$, with r independent of j, by considering

$$0 \longrightarrow \ker([p^{j-1}])(k) \longrightarrow \ker([p^j])(k) \xrightarrow{p^{j-1}} \ker([p])(k) \longrightarrow 0$$

Let $G = \ker(n) \subseteq X$. This is a finite group scheme over k. We know that

$$\dim_k(\Gamma(G, \mathcal{O}_G)) = n^{2g} = \deg([n])$$

If n is invertible in k, then we have a map (examples sheet 4 question 1)

$$[n]_* : \mathrm{T}_{G,e} \to \mathrm{T}_{G,e}$$

on the tangent space, which is multiplication by n. Then this is an isomorphism. On the other hand, $[n] : G \to G$ factors through $\mathrm{Spec}(k)$,

$$\begin{array}{ccc} G & \xrightarrow{[n]} & G \\ & {\scriptstyle a_G}\searrow \quad \nearrow{\scriptstyle e} & \\ & \mathrm{Spec}(k) & \end{array}$$

and so we must have that $\mathrm{T}_{G,e}$ is zero. Hence $\mathcal{O}_{G,e} = k$. But then

$$G = \bigsqcup_{x\in G(k)} \mathrm{Spec}(\mathcal{O}_{G,x}) = \bigsqcup_{x\in G(k)} \mathrm{Spec}(k)$$

since translation induces an isomorphism $\mathcal{O}_{G,x} \cong \mathcal{O}_{G,e}$. So G is a constant group scheme, of order n^{2g}. Therefore, G is isomorphci to the constant group scheme $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

Now suppose $\mathrm{char}(k) = p > 0$. It is enough to compute $G(k)$ for $G = \ker([p]) \subseteq X$. From examples sheet 4 question 3, we have a Frobenius homomorphism $F = F_{X/k} : X \to X' = X$, which on the structure sheaf is $t \mapsto t^p$. To make the Frobenius a k-morphism, we have

$$\begin{array}{ccccc} X & \longrightarrow & X' & = & X \\ \downarrow{\scriptstyle a_X} & \swarrow{\scriptstyle a_{X'}} & & & \downarrow{\scriptstyle a_X} \\ \mathrm{Spec}(k) & & \xrightarrow[t\mapsto t^p]{} & & \mathrm{Spec}(k) \end{array}$$

Moreover, $\ker(F)$ is killed by $[p]$, i.e. $\ker(F) \subseteq \ker([p])$. Also,

$$\ker(F) \cong \mathrm{Spec}\left(\frac{k[t_1, \ldots, t_g]}{\langle t_1^p, \ldots, t_g^p \rangle}\right)$$

Let $\mathfrak{m}_e = \langle f_1, \ldots, t_g \rangle$. Then $\ker(F)$ is a one-point scheme, and

$$\dim(\Gamma(\ker(F), \mathcal{O})) = p^g$$

Thus,

$$\dim(\mathcal{O}_{G,e}) \geq p^g$$

and as $p^{2g} = \dim(\Gamma(G, \mathcal{O})) = |G(k)| \dim(\mathcal{O}_{G,e})$, $|G(k)| = p^r$ for some $0 \leq r \leq g$. □

Remark 7.19. If $g = 1$, then X is an elliptic curve and either $r = 1$, so X is ordinary, or $r = 0$ and X is supersingular. In general, for $0 \le r \le g$, there exists X of dimension g, with p-rank r. Take $X = E_1^r \times E_0^{g-r}$, where E_1 is ordinary and E_0 is supersingular.

Remark 7.20. If k is algebraically closed, $\text{char}(k) = p > 0$. Then

- If $r = g$, then
$$\ker([p^j]) = (\mathbb{Z}/p^j\mathbb{Z})^g \times (\mu_{p^j})^g$$
where $\mu_{p^j} = \ker([p^j] : \mathbb{G}_m \to \mathbb{G}_m)$,
- Otherwise,
$$\ker([p^j]) = (\mathbb{Z}/p^j\mathbb{Z})^r \times G^0$$
where G^0 is a single point. In general, G^0 is not isomorphic to $(\mu_p^j)^{2g-r}$.

Moreover, there exists a classification and duality theory for finite group schemes over a perfect field, and G^0 fits into this classification.

Recall that for X/k, if $\mathcal{L}$ is an ample line bundle on X, then we have a homomorphism

$$\varphi_{\mathcal{L}} : X(\overline{k}) \twoheadrightarrow \text{Pic}^0(X_{\overline{k}}) \subseteq \text{Pic}(X_{\overline{k}})$$

with finite kernel.

Theorem 7.21. There exists a dual abelian variety $\widehat{X}$ over k, with $\dim(\widehat{X}) = \dim(X)$, and we have an isomorphism
$$\Psi : \widehat{X}(\overline{k}) \cong \text{Pic}^0(X_{\overline{k}})$$
and for any ample $\mathcal{L}$, there exists a unique surjective homomorphism $\lambda_{\mathcal{L}} : X \to \widehat{X}$ such that

$$X(\overline{k}) \xrightarrow{\lambda_{\mathcal{L}}} \widehat{X}(\overline{k}) \xrightarrow{\Psi} \text{Pic}^0(X_{\overline{k}}), \qquad \Psi \circ \lambda_{\mathcal{L}} = \varphi_{\mathcal{L}}$$

When $g = 1$, X is an elliptic curve E, and we have Abel's theorem, which says that

$$E(\overline{k}) \cong \text{Cl}^0(E_{\overline{k}})$$

where $\text{Cl}^0(E_{\overline{k}})$ are degree zero divisors, sending x to $\langle x \rangle - \langle e \rangle$. Thus, if $\mathcal{L} = \mathcal{O}(e)$, then

$$\varphi_{\mathcal{L}}(-x) = T_{-x}^*\mathcal{L} \otimes \mathcal{L}^\vee = \mathcal{O}(\langle x \rangle - \langle e \rangle)$$

and so we have that $\widehat{E} = E$. Thus, we have a *canonical* line bundle $\mathcal{O}(e)$ on E.

Suppose X/k is any proper variety. Then we would like a group scheme $\underline{\text{Pic}}_{X/k}$ which classify line bundles on X. These are called *Picard schemes*. We certainly want

$$\underline{\text{Pic}}_{X/k}(\overline{k}) = \text{Pic}(X_{\overline{k}})$$

First guess for any k-scheme S,
$$\underline{\text{Pic}}_{X/k}(S) = \text{Pic}(X \times S)$$
However, this cannot be the case. There cannot be a group scheme G with $G(S) = \text{Pic}(X \times S)$, for two reasons:

1. If $S = \bigcup_i U_i$, then
$$G(S) = \{(f_i : U_i \to G) \mid f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}\}$$
But then
$$\text{Pic}(X \times S) = \ker\left(\prod_i \text{Pic}(X \times U_i) \to \prod_{i,j} \text{Pic}(X \times (U_i \cap U_j))\right)$$
where we send $(\mathcal{L}_i)$ to $(\mathcal{L}_i \otimes \mathcal{L}_j^\vee)_{U_i \cap U_j}$. This fails even for $X = \text{Spec}(k)$ and $S = \mathbb{P}^1_k$. Here, we have an "extraneous" $\text{Pic}(S)$.

2. We could instead hope that

$$G(S) = \frac{\operatorname{Pic}(X \times S)}{\operatorname{pr}_2^* \operatorname{Pic}(S)}$$

For any k-variety G, if k'/k is a finite Galois extension, then

$$G(k) = G(k')^{\operatorname{Gal}(k'/k)}$$

Let X be the conic $x_0^2 + x_1^2 + x_2^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. Then $X(\mathbb{R}) = \varnothing$ and $X_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$. But $\operatorname{Pic}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{Z}$. In this case, we have

$$\begin{array}{ccc} \operatorname{Pic}(X_{\mathbb{C}}) & \xrightarrow{\cong} & \mathbb{Z} \\ \uparrow & & \uparrow \\ \operatorname{Pic}(X) & \xrightarrow{\cong} & 2\mathbb{Z} \end{array}$$

Thus, $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ acts trivially on $\operatorname{Pic}(X_{\mathbb{C}})$, and so $\operatorname{Pic}(X) \neq \operatorname{Pic}(X_{\mathbb{C}})^{\operatorname{Gal}(\mathbb{C}/\mathbb{R})}$.

We have two solutions, which give the "correct" $\underline{\operatorname{Pic}}_{X/k}$.

1. sheafification (for étale topology)
2. assume $e \in X(k) \neq \varnothing$. Define

$$\operatorname{Pic}_e(X \times S) = \{\mathcal{L} \in \operatorname{Pic}(X \times S) \mid \mathcal{L}|_{e \times S} \cong \mathcal{O}_S\}$$

A basic result of Grothendieck: If X/k is a projective variety, $e \in X(k)$, then there exists a group scheme $\underline{\operatorname{Pic}}_{X/k}$ locally of finite type over k, such that $\underline{\operatorname{Pic}}_{X/k}(S) = \operatorname{Pic}_e(X \times S)$ and this is natural in S. The proof uses Hilbert schemes.

Lecture 24

8 Proof of theorem 6.3

Step 1: We will prove the statement for $Z = \operatorname{Spec}(A)$, where A is a finite local k-algebra. with residue field k. So $A = k \oplus \mathfrak{m}$ with $\mathfrak{m}$ nilpotent, and so $Z = \{\text{pt}\}$.

Induct on $\dim_k(A)$. If $\dim_k(A) = 1$, then $A = k$, and so $\mathcal{L} = \mathcal{L}|_{X \times Y \times z}$ is trivial. Otherwise, we can find an ideal $I \subseteq \mathfrak{m}$ with dimension 1. To see this, any minimal non-zero ideal of A is necessarily killed by $\mathfrak{m}$, and is a k-vector space. Thus, by minimality it has dimension 1. Say $I = \operatorname{span}_k\{t\}$. Let $Z_1 = \operatorname{Spec}(A/I)$.

If V/k is a proper variety, then for any K-algebra B, $\mathrm{H}^0(V_B, \mathcal{O}_{V_B}) = B$, since $\mathrm{H}^0(V, \mathcal{O}_V) = k$ and we can use flat base change.

Lemma 8.1 (tangent space to Pic)**.** Let V be a proper k-variety. Then there exists an exact sequence, functorial in V,

$$0 \longrightarrow \mathrm{H}^1(V, \mathcal{O}_V) \longrightarrow \operatorname{Pic}(V \times Z) \longrightarrow \operatorname{Pic}(V \times Z_1)$$

where the last map is induced by inclusion.

When $Z = \operatorname{Spec}(k[\varepsilon]/\varepsilon^2)$, then $Z_1 = k$, and $\ker(\operatorname{Pic}(V \times Z) \to \operatorname{Pic}(V))$ is the "tangent space".

For example, this tells us that the dimension of $\underline{\operatorname{Pic}}_{V/k}$ is at most the dimension of $\mathrm{H}^1(V, \mathcal{O}_V)$.

Proof. First note that we have an exact sequence of abelian groups

$$0 \longrightarrow I \xrightarrow{a \mapsto 1+a} A^\times \longrightarrow (A/I)^\times \longrightarrow 1$$

Note that $(1 + a)(1 + b) = 1 + (a + b)$ since $I^2 = 0$. On the other hand,

$$\ker(A^\times \to (A/I)^\times) = 1 + I$$

and the cokernel is trivial, since if $a \in A$ is a unit in A/I, then a is a unit, since $(1+I) \subseteq A^\times$. Therefore, we have an exact sequence of sheaves on V, given by

$$0 \longrightarrow I\mathcal{O}_{V\times Z} \longrightarrow \mathcal{O}^\times_{V\times Z} \longrightarrow \mathcal{O}^\times_{V\times Z_1} \longrightarrow 1$$

Note that as topological spaces, $V, V\times Z, V\times Z_1$ are all the same (or canonically isomorphic). But now multiplication by t defines an isomorphism $\mathcal{O}_V \cong I\mathcal{O}_{V\times Z}$, and so by the long exact sequence of cohomology, we obtain

$$\begin{array}{c} 0 \longrightarrow \mathrm{H}^0(V,\mathcal{O}_V) \longrightarrow \mathrm{H}^0\left(V\times Z,\mathcal{O}^\times_{V\times Z}\right) \longrightarrow \mathrm{H}^0\left(V\times Z_1,\mathcal{O}^\times_{V\times Z_1}\right) \\ \mathrm{H}^1(V,\mathcal{O}_V) \longrightarrow \mathrm{H}^1\left(V\times Z,\mathcal{O}^\times_{V\times Z}\right) \longrightarrow \mathrm{H}^1\left(V\times Z_1,\mathcal{O}^\times_{V\times Z_1}\right) \end{array}$$

On the first row, we get

$$0 \qquad k \qquad A^\times \qquad (A/I)^\times$$

and we saw the last map is surjective. Thus, the map $\mathrm{H}^1(V < \mathcal{O}_V) \to \mathrm{H}^1(V\times Z, \mathcal{O}^\times_{V\times Z})$ is injective. Moroever, for any scheme X,

$$\mathrm{Pic}(X) = \mathrm{H}^1(X,\mathcal{O}^\times_X)$$

□

To see that $\mathrm{Pic}(X) = \mathrm{H}^1(X,\mathcal{O}^\times_X)$, note that if $\mathcal{L}$ is a line bundle on X, then there exists an open cover

$$X = \bigcup_i U_i$$

of X, such that for all i, we have an isomorphism

$$\mathcal{O}_{U_i} \cong \mathcal{L}|_{U_i}$$

say sending 1 to $s_i \in \Gamma(U_i,\mathcal{L})$. On $U_i \cap U_j$, there exists a unique $a_{ij} \in \mathcal{O}_X(U_i\cap U_j)^\times$ such that $s_i = a_{ij}s_j$. Clearly the (a_{ij}) satisfy the cocycle conditions, and it defines a Čech 1-cocycle. Thus, we gave an element of $\check{\mathrm{H}}^1((U_i),\mathcal{O}^\times_X)$. Passing to the limit of all covers, we get an element of $\mathrm{H}^1(X,\mathcal{O}^\times_X)$.

Lots of checking to show that this only depends on the isomorphism class, and that it is an isomorphism of groups and so on...

Returning to the proof. by induction, assume $\mathcal{L}|_{X\times Y\times Z_1}$ is trivial. Apply the lemma, we get

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{H}^1(X\times Y,\mathcal{O}_{X\times Y}) & \longrightarrow & \mathrm{Pic}(X\times Y\times Z) & \xrightarrow{c} & \mathrm{Pic}(X\times Y\times Z_1) \\ & \downarrow a & & \downarrow b & & \downarrow \\ 0 \longrightarrow & \mathrm{H}^1(X,\mathcal{O}_X)\oplus \mathrm{H}^1(Y,\mathcal{O}_Y) & \longrightarrow & \mathrm{Pic}(X\times Z)\oplus\mathrm{Pic}(Y\times Z) & \longrightarrow & \mathrm{Pic}(X\times Z_1)\oplus\mathrm{Pic}(Y\times Z_1) \end{array}$$

The rows are exact, and the vertical maps are induced by restruction y^*, x^*. By Künneth, a is an isomorphism. Now $\mathcal{L} \in \mathrm{Pic}(X\times Y\times Z)$, with $b(\mathcal{L}) = 0$ by hypothesis, and $c(\mathcal{L}) = 0$ by the induction hypothesis. From the diagram, we must then have that $\mathcal{L} = 0 \in \mathrm{Pic}(X\times Y\times Z)$.

Step 2: Consider $Z = \mathrm{Spec}(A)$ where A is a local Noetherian k-algebra, with $A/\mathfrak{m} = k$. Let

$$Z_n = \mathrm{Spec}\left(\frac{A}{\mathfrak{m}^n}\right)$$

by the previous step, $\mathcal{L}|_{X\times Y\times Z_n}$ is trivial for all n. As before, we have finity cyclic A-modules Q, Q', such that for all A-algebras B, $\mathrm{H}^0(\mathcal{L}_B) = \mathrm{Hom}_A(Q,B)$ and $\mathrm{H}^0(\mathcal{L}^\vee_B) = \mathrm{Hom}_A(Q',B)$. For all $n \geq 1$,

$$Q \otimes \left(\frac{A}{\mathfrak{m}^n}\right) \cong \frac{A}{\mathfrak{m}^n}$$

since $\mathcal{L}|_{X\times Y\times Z_n}$ is trivial. So

$$\mathrm{Ann}(Q) \subseteq \bigcap_n \mathfrak{m}^n = 0$$

$$\mathrm{Ann}(Q') \subseteq \bigcap_n \mathfrak{m}^n = 0$$

So Q, Q' are isomorphic to A. Thus, $\mathcal{L}$ is trivial.

Step 3: Now assume that Z is a k-variety. Then $\mathcal{L}|_{X\times Y\times \mathcal{O}_{Z,s}}$ is trivial, by the previous part. Let

$$F = \{z' \in Z \mid \mathcal{L}|_{X\times Y\times z'} \text{ is trivial}\}$$

Then F contains the generic point of z (which is the generic point of $\mathcal{O}_{Z,z}$), and as F is closed, $F = Z$. Thus, by theorem 6.1,

$$\mathcal{L} \cong \mathrm{pr}_3^* \mathcal{M}$$

for a line bundle $\mathcal{M}$ on Z. But then

$$\mathcal{O}_Z \cong \mathcal{L}_{x\times y\times Z} \cong M$$

and thus $\mathcal{L}$ is trivial.