Elliptic PDEs

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Contents

Lecture 1

0 Introduction

We will study second order elliptic PDEs on (a domain in) $\mathbb{R}^n$. For example, arising from variational problems. Ultimately, we are interested in non-linear PDEs. To do this, we will first understand the linear theory.

Setup: Consider a domain $\Omega \subseteq \mathbb{R}^n$, i.e. open, bounded and connected, and a function

$$\begin{aligned} F : \Omega \times \mathbb{R} \times \mathbb{R}^n &\to \mathbb{R} \\ (x, z, p) &\mapsto F(x, z, p) \end{aligned}$$

and consider the functional

$$\mathcal{F}[u] = \int_\Omega F(x, u(x), \partial u(x)) \mathrm{d}x$$

Note we will use ∂, D, ∇ essentially interchangably. Assume F is sufficiently regular. Let $u \in \mathcal{S}$, a suitable vector space of functions $u : \Omega \to \mathbb{R}$. Frequently, $\mathcal{S} = \mathrm{H}^1(\Omega)$, or $\mathcal{S} = C^{1,\alpha}(\Omega)$.

Suppose u minimises $\mathcal{F}$, subject to $u|_{\partial\Omega} = g$, for some given $g : \partial\Omega \to \mathbb{R}$[1]. So for all $\varphi \in \mathcal{S}$,

$$\mathcal{F}[u + t\varphi] \geq \mathcal{F}[u]$$

In particular, this means that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}[u + t\varphi] = 0$$

*Based on lectures by Greg Taujunskas (lectures 1 - 12) and Neshan Wickramasekera (lectures 13 - 24). Last updated February 19, 2024.

[1] Boundary conditions are needed for well-posedness

Or another words,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_\Omega F(x, u + t\varphi, \partial u + t\partial\varphi)\mathrm{d}x = 0$$

Assume enough regularity so that we can exchange the derivative and integral, we get that

$$\int_\Omega (\partial_z F)(x, u, \partial u)\varphi + \partial_i\varphi(\partial p_i F)(x, u, \partial u)\mathrm{d}x = 0 \tag{1}$$

As usual, we will use the summation convention. To ensure that $u+t\varphi$ satisfies the correct boundary conditions, $\varphi|_{\partial\Omega} = 0$. Integrate eq. (1) by parts, we get

$$\int_\Omega \varphi(x)\left(\partial_z F - \partial_i \partial p_i \mathcal{F}\right)(x, u, \partial u)\mathrm{d}x = 0$$

This is true for all $\varphi \in \mathcal{S}$, and so by the fundamental lemma of calculus of variations,

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial p_i}\right) = 0$$

which is the Euler-Lagrange equation for F. We can rewrite this as

$$\frac{\partial F}{\partial z} - \partial_i\partial_j u \frac{\partial^2 F}{\partial p_i \partial p_j} = 0 \tag{2}$$

This is now a second order quasilinear PDE in u. More generally, consider

$$a^{ij}(x, u, \partial u)\partial^2_{ij}u - b(x, u, \partial u) = 0 \tag{3}$$

Definition 0.1
We say that eq. (3) is *elliptic* in Ω if $a^{ij}(x, u, \partial u)$ is a positive definite matrix in Ω.

In the case of eq. (2), this is equivalent to F being convex in the variable p.

Example 0.2 (Dirichlet energy)
When

$$F(x, z, p) = |p|^2$$

One gets

$$\Delta u = 0$$

Extremisers of this are called *harmonic functions*.

Example 0.3 (Minimal surfaces)
When

$$F(x, z, p) = \sqrt{1 + |p|^2}$$

We leave as an exercise to interpret $\mathcal{F}[u]$. In this case, we get the *minimal surface equation*, which is

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \tag{4}$$

Locally, ∇u is a constant, and so eq. (4) looks like $\Delta u = 0$, and so solutions have similar local properties. But the existence theory for Laplace's equation is 'trivial', whereas the existence theory for eq. (4) may fail. That is, global properties are important.

For entire solutions (i.e. solutions defined all of $\mathbb{R}^n$), global behaviour is very different.

Theorem 0.4 (Liouville). If $u : \mathbb{R}^n \to \mathbb{R}$ is C^2, $\Delta u = 0$ and u is bounded, then u is a constant.

Theorem 0.5 (Bernstein). (The only entire solutions to eq. (4) in $\mathbb{R}^n$ are planar (i.e. u is linear)) if and only if $n \leq 7$.

1 Harmonic functions

1.1 Basic properties

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, i.e. open and connected.

Definition 1.1 (hamonic, subharmonic, superharmonic)
A function $u \in C^2(\Omega)$ is *harmonic* if $\Delta u = 0$. It is *subharmonic* if $\Delta u \geq 0$ and *superharmonic* if $\Delta u \leq 0$.

Let $B_\rho(y)$ denote the open ball with centre y and radius ρ. Then

Theorem 1.2 (mean value property). If $u \in C^2(\Omega)$ is subharmonic, and $B_r(y) \subseteq \Omega$, then

$$u(y) \leq \frac{1}{\omega_n r^n} \int_{B_r(y)} u(x) \mathrm{d}x \tag{5}$$

where $\omega_n = |B_1(0)|$. Moreover,

$$u(y) \leq \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(y)} u(x) \mathrm{d}x \tag{6}$$

If u is superharmonic, then the inequalities are reversed. If u is harmonic, then equality holds.

Proof. We have that

$$\begin{aligned} 0 &\leq \int_{B_\rho(y)} \Delta u \mathrm{d}x \\ \text{integrating by parts } &= \int_{\partial B_\rho(y)} \nabla u \cdot w \mathrm{d}x \text{ where } w \text{ is the outwards normal} \\ &= \rho^{n-1} \int_{S^{n-1}} w \cdot \nabla u(y + \rho w) \mathrm{d}w \\ &= \rho^{n-1} \int_{S^{n-1}} \frac{\partial}{\partial \rho} u(y + \rho w) \mathrm{d}w \end{aligned}$$

where we use the fact that $\rho w = x - y$. Exchanging integrals and derivatives,

$$0 \leq \frac{\partial}{\partial \rho} \int_{S^{n-1}} u(y + \rho w) \mathrm{d}w$$

Thus, the map

$$\rho \mapsto \int_{S^{n-1}} u(y + \rho w) \mathrm{d}w$$

is increasing. Thus,

$$\int_{S^{n-1}} u(y + \rho w) \mathrm{d}w \leq \int_{S^{n-1}} y(y + rw) \mathrm{d}w$$

for $0 \leq \rho \leq r$. Taking the limit as $\rho \to 0$, we get eq. (6). Integrating in r to get eq. (5). The superharmonic case is similar. The harmonic case follows from the subharmonic and superharmonic cases. ☐

Remark 1.3. The mean value property characterises harmonic functions. See examples sheet 1.

Lecture 2

Theorem 1.4 (strong maximum principle)**.** Suppose $u \in C^2(\Omega)$ is subharmonic on Ω. Suppose there exists $y_0 \in \Omega$ such that

$$u(y_0) = \sup_\Omega u$$

Then u is constant.

Remark 1.5. If u is superharmonic, then we have a corresponding statement for when $u(y_0) = \inf_\Omega u$. If u is harmonic, then either sup or inf work.

Proof. Let $M = \sup_\Omega u < \infty$, and

$$\Sigma = \{y \in \Omega \mid u(y) = M\}$$

By assumption, Σ is non-empty, as $y_0 \in \Sigma$. Since u is continuous, Σ is closed. Since Ω is connected, it suffices to show that Σ is open, since this implies $\Sigma = \Omega$.

Pick $y \in \Sigma$. By the mean value property, for $\rho > 0$ such that $\overline{B_\rho(y)} \subseteq \Omega$, we have that

$$M = u(y) \leq \frac{1}{\omega_n \rho^n} \int_{B_\rho(y)} u(x)\mathrm{d}x$$

So

$$\int_{B_\rho(y)} (M - u(x))\mathrm{d}x \leq 0$$

But $M - u(x) \geq 0$, and so it must be identically zero, i.e. $u \equiv M$ on $B_\rho(y)$. Hence $B_\rho(y) \subseteq \Sigma$, and Σ is open. $\square$

Here, the strong maximum principle is easy as we have the mean value property. For more general elliptic PDEs, this is not the case. We will prove a weaker statement which does generalise.

Theorem 1.6 (weak maximum principle)**.** Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. If u is subharmonic on Ω, then

$$\sup_\Omega u = \sup_{\partial\Omega} u$$

Proof. This follows from the strong maximum principle. Since Ω is bounded, $\sup_\Omega u$ is attained in $\overline{\Omega}$. So by the maximum principle, these cannot be attained in Ω unless u is constant. $\square$

Remark 1.7. If u is superharmonic, we replace sup with inf, and if u is harmonic then both hold.

The mean value property states that u always an average of itself. In particular, this suggest that u cannot vary too much. Can we use this to relate $\sup u$ and $\inf u$?

Theorem 1.8 (Harnack's inequality)**.** Suppose $u \in C^2(\Omega)$, $u \geq 0$ and $\Delta u = 0$ in Ω. Then if $\Omega' \Subset \Omega$ is any bounded subdomain, we have

$$\sup_{\Omega'} u \leq C \inf_{\Omega'} u$$

where $C = C(n, \Omega, \Omega')$ does not depend on u.

Proof. First, choose $y \in \Omega$ and $\rho > 0$, such that $\overline{B_{4\rho}(y)} \subseteq \Omega$. Choose $x_1, x_2 \in B_\rho(y)$. By the mean value property,

$$u(x_1) = \frac{1}{\omega_n \rho^n} \int_{B_\rho(x_1)} u \leq \frac{1}{\omega_n \rho^n} \int_{B_{2\rho}(y)} u$$

On the other hand,

$$u(x_2) = \frac{1}{\omega_n(3\rho)^n}\int_{B_{3\rho}(y)} u \geq \frac{1}{\omega_n(3\rho)^n}\int_{B_{2\rho(y)}} u$$

Combining these, we see that

$$u(x_1) \leq 3^n u(x_2)$$

for all $x_1, x_2 \in B_\rho(y)$. So Harnack holds locally in balls, with constant independent of u. It is also independent of ρ, y as long as ρ is sufficiently small. Now choose x_1, x_2 in $\overline{\Omega'} \subseteq \Omega$, such that

$$\sup_{\Omega'} u = u(x_1) \quad \text{and} \quad \inf_{\Omega'} u = u(x_2)$$

By path connectedness of Ω, there exists a continuous map $\gamma : [0, 1] \subseteq \overline{\Omega'}$, with $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Choose ρ such that $4\rho < d(\gamma, \partial\Omega)$, and $N = N(\Omega', \Omega)$ such that we can cover γ by N-balls of radius ρ[2].

Apply the local result to each ball, and we get

$$u(x_1) \leq (3^n)^N u(x_2) = 3^{nN} u(x_2)$$

□

Theorem 1.9 (derivative estimates)**.** Suppose $u \in C^3(\Omega)$ is harmonic on Ω. Then if $\overline{B_\rho(y)} \subseteq \Omega$, we have that

$$|\mathrm{D}u(y)| \leq \frac{C}{\rho}\sup_{\partial B_\rho(y)} |u|$$

where $C = C(n)$.

Proof. Since $\Delta u = 0$,

$$0 = \mathrm{D}_i(\Delta u) = \Delta(\mathrm{D}_i u)$$

So $\mathrm{D}_i u$ is harmonic. By the mean value property,

$$\begin{aligned}
\mathrm{D}_i u(y) &= \frac{1}{\omega_n\rho^n}\int_{B_\rho(y)} u \\
&= \frac{1}{\omega_n\rho^n}\int_{B_\rho(y)} \nabla\cdot(0, \dots, u, \dots, 0)\mathrm{d}x \\
&= \frac{1}{\omega_n\rho^n}\int_{\partial B_\rho(y)} u(x)\cdot\nu_i(x)\mathrm{d}x
\end{aligned}$$

where $\nu(x)$ is the unit normal at x. But $|\nu_i(x)| \leq 1$, and so

$$|\mathrm{D}_i u(y)| \leq \frac{1}{\omega_n\rho^n}\sup_{\partial B_\rho(y)}|u|\int_{\partial B_\rho(y)}\mathrm{d}x = \frac{n}{\rho}\sup_{\partial B_\rho(y)}|u|$$

□

Remark 1.10. We can apply this result repeatedly, to get that for $\Omega' \subseteq \Omega'' \subseteq \Omega$, and any multi-index α, if $u \in C^{|\alpha|+2}(\Omega)$, with $\Delta u = 0$ in Ω, then

$$\sup_{\Omega''}|\mathrm{D}^\alpha u| \leq C\sup_{\Omega''}|u|$$

for some $C = C(n, \alpha, \Omega, \Omega')$.

That is,

$$\left\|\mathrm{D}^\alpha u\right\|_{L^\infty(\Omega')} \leq C\|u\|_{L^\infty(\Omega')}$$

By the mean value property, for some $y \in \overline{\Omega''} \subseteq \Omega$,

$$\sup_{\Omega''} u = |u(y)| = \left|c\int_{B_\rho(y)} u(x)\mathrm{d}x\right| \leq C\int_\Omega |u|$$

[2]This follows from the fact that Ω' is relatively compact

So

$$\left\|\mathrm{D}^{\alpha}(u)\right\|_{L^{\infty}(\Omega')} \leq C\|u\|_{L^1(\Omega)}$$

Theorem 1.11 (uniqueness of solutions for the Dirichlet problem)**.** Suppose Ω is bounded, and $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with

$$\begin{cases} \Delta u_1 = \Delta u_2 & \text{in } \Omega \\ u_1 = u_2 & \text{on } \partial\Omega \end{cases}$$

Then $u_1 = u_2$ in $\overline{\Omega}$.

Proof. Set $w = u_2 - u_2$. Then w is harmonic function in Ω, and $w = 0$ on $\partial\Omega$. By the weak maximum principle, we get that $w = 0$ in Ω. □

Remark 1.12. Of course, we can integrate by parts to get the same result, but the weak maximum principle will apply for non divergence form equations.

Lecture 3

Theorem 1.13 (Liouville)**.** If $u \in C^{\infty}(\mathbb{R}^n)$ is harmonic, and grows sublinearly at infinity. Then u is constant.

Remark 1.14. "growing sublinearly" means that

$$|u(x)| \leq C\left(1 + |x|^{\alpha}\right)$$

where $0 < \alpha < 1$.

Proof. From derivative estimates, we know that for all $x \in \mathbb{R}^n$, we have that

$$|\mathrm{D}u(y)| \leq \frac{C}{\rho} \sup_{B_\rho(y)} |u|$$

Plugging in the growth assumption, we get that

$$|\mathrm{D}u(y)| \leq \frac{C}{\rho}(1 + (\rho + |y|)^{\alpha})$$

Taking $\rho \to \infty$, we get that $\mathrm{D}u(y) = 0$. But y was arbitrary, and so we are done. □

1.2 Existence theory for harmonic functions

We will consider the classical problem of solving the Dirichlet problem for the Laplacian on bounded domains $\Omega \subseteq \mathbb{R}^n$, and $\varphi : \overline{\Omega} \to \mathbb{R}$ continuous. We wish to find $u \in C^{\infty}(\Omega) \cap C^0(\overline{\Omega})$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

We will assume for simplicity that $\partial\Omega$ is smooth, and $\varphi \in C^{\infty}(\Omega)$.

We have (at least) three methods to solve this problem.

1. Hilbert space methods (c.f. Analysis of PDEs). Use the Riesz representation theorem to obtain a solution $u \in \mathrm{H}^1(\Omega)$, and deal with regularity afterwards. This relies on the equation being linear.

2. Direct method of calculus of variations. We can rephrase $\Delta u = 0$ as a variational problem. That is, the Euler-Lagrange equation of

$$\int |\mathrm{D}u|^2$$

and prove existence using the functional.

3. Perron's method. Use the fact that solvability in balls impies solvability in more general domains. This method is based on maximum principles.

> **Remark 1.15.** In all cases, we obtain a rougher solution first, and improve regulaity later.

We will focus on the second method. Define

$$\mathscr{S} = \left\{ w \in \mathrm{H}^1(\Omega) \mid w - \varphi \in \mathrm{H}^1_0(\Omega) \right\}$$

That is, H^1 functions which agree with φ on the boundary. Clearly $\varphi \in \mathscr{S}$, and so $\mathscr{S}$ is non-empty. Set

$$\mathcal{E}[w] = \int_\Omega |\mathrm{D}w|^2$$

and define

$$\beta = \inf_{w \in \mathscr{S}} \mathcal{E}[w]$$

By the definition of the infimum, there exists a seqeunce $(w_j) \subseteq \mathscr{S}$, such that

$$\mathcal{E}[w_j] \to \beta$$

We want to extract a convergent subsequence and show that its limit is a solution. Clearly for j large, we have that

$$\int_\Omega \left|\mathrm{D}w_j\right|^2 \leq \beta + 1$$

Since $w_j - \varphi \in \mathrm{H}^1_0(\Omega)$, by the Poincaré inequality,

$$\int_\Omega \left|w_j - \varphi\right|^2 \leq C \int_\Omega \left|\mathrm{D}(w_j - \varphi)\right|^2$$

In particular, this implies that

$$\left\|w_j\right\|^2_{L^2(\Omega)} \leq C(\Omega, \varphi, \beta) < \infty$$

Indeed,

$$\left\|w_j\right\|^2_{L^2(\Omega)} \leq C\left\|\mathrm{D}w_j - \mathrm{D}\varphi\right\|^2_{L^2(\Omega)} \leq C(\Omega, \varphi, \beta)$$

Expanding the left hand side, we get

$$\left\|w_j\right\|^2_{L^2(\Omega)} - 2\left\langle w_j, \varphi\right\rangle_{L^2(\Omega)} \leq C(\Omega, \varphi, \beta)$$

In particular, by Young's inequality,

$$\left\|w_j\right\|^2_{L^2(\Omega)} \leq C(\varphi, \Omega, \beta) + \varepsilon\left\|w_j\right\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\left\|\varphi\right\|^2_{L^2(\Omega)}$$

Take $0 < \varepsilon < 1$, and rearrange.

So we have that $\left\|w_j\right\|^2_{\mathrm{H}^1(\Omega)} \leq C$, so by Banach-Alaoglu,

$$w_{j_k} \rightharpoonup w \text{ in } \mathrm{H}^1(\Omega)$$

By Rellich-Kondrachov[3],

$$w_{j_k} \to w \text{ in } L^2(\Omega)$$

[3]Recall

> **Theorem 1.16** (Rellich-Kondrachov). Let Ω be bounded, $1 \leq p < n$, then
>
> $$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$
>
> and
>
> $$W^{1,p}(\Omega) \Subset L^q(\Omega)$$
>
> for $1 \leq q < p^*$. Here, p^* is the conjugate
>
> $$p^* = \frac{np}{n-p}$$

In particular, when $p = 2$, $p^* > 2$ if $n > 2$.

for some $w \in \mathrm{H}^1(\Omega)$. Hence for all $v \in \mathrm{H}^1(\Omega)$, we have that

$$\int_\Omega \mathrm{D}w_j \cdot \mathrm{D}v \to \int_\Omega \mathrm{D}w \cdot \mathrm{D}v$$

Also, clearly we have that $w_{j_k} - \varphi \rightharpoonup w - \varphi$ in $\mathrm{H}^1(\Omega)$, as φ is smooth. But $w_{j_k} - \varphi \in \mathrm{H}^1_0(\Omega)$, but $\mathrm{H}^1_0(\Omega) \subseteq \mathrm{H}^1(\Omega)$ is norm closed, so it is weakly closed[4]. Hence $w - \varphi \in \mathrm{H}^1_0(\Omega)$, so $w \in \mathscr{S}$. Finally, since $\mathcal{E}[\cdot]$ is *sequentially weakly lower semicontinuous*[5] in $\mathrm{H}^1(\Omega)$, we have that

$$\mathcal{E}[w] \leq \liminf \mathcal{E}[w_{j_k}] = \beta$$

Hence $\mathcal{E}[w] = \beta$. We have found a global minimum w. That is, for all $v \in \mathrm{H}^1_0(\Omega)$, $w + tv \in \mathscr{S}$,

$$f(t) = \mathcal{E}[w + tv] \geq \mathcal{E}[w]$$

Thus, $f'(0) = 0$. But

$$\begin{aligned} f'(0) &= \mathrm{D}\mathcal{E}[w](v) \\ &= \lim_{t \to 0} \frac{\mathcal{E}[w + tv] - \mathcal{E}[w]}{t} \\ &= 2 \int_\Omega \mathrm{D}w \cdot \mathrm{D}v \end{aligned}$$

In particular,

$$\int_\Omega \mathrm{D}w \cdot \mathrm{D}v = 0$$

is the weak formulation of $\Delta w = 0$. We will next upgrade the regularity, and show that weak solutions are in fact smooth.

1.3 Interior regularity

We wish to improve the regularity of the weak solution. What we have shown is that there exists a weak solution $u \in L^1(\Omega)$, such that

$$\int_\Omega u \Delta v = 0$$

for all $v \in C^\infty_c(\Omega)$.

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[4] This follows from Hahn-Banach. In fact, it follows for any convex subset of a Banach space.

> **Lemma 1.17.** Let X be a Banach space, for $C \subseteq X$ convex, C is norm-closed if and only if C is weakly closed.

Proof. Weak closed implies norm closed is clear. For the converse, we can show that $X \setminus C$ is weakly open. Let $x_0 \in X \setminus C$, by the Hahn-Banach separation, there exists $\phi \in X'$ such that

$$\varphi|_C = 0 \quad \text{and} \quad \phi(x_0) \neq 0$$

Then

$$\left\{ x \in X \;\middle|\; |\phi(x)| > \frac{1}{2}|\phi(x_0)| \right\} \subseteq X \setminus C$$

is a weakly open neighbourhood of x_0. □

[5] That is, if $u_j \rightharpoonup u$ in $\mathrm{H}^1(\Omega)$, then

$$\mathcal{E}[u] \leq \liminf \mathcal{E}[u_j]$$

To see this, note that

$$\int_\Omega \mathrm{D}u_j \cdot \mathrm{D}v \to \int_\Omega \mathrm{D}u \cdot \mathrm{D}v$$

Setting $u = v$, we see that

$$\int_\Omega \mathrm{D}u_j \cdot \mathrm{D}u \to \int_\Omega |\mathrm{D}u|^2$$

Thus,

$$\begin{aligned} \mathcal{E}[u] &= \lim \int_\Omega \mathrm{D}u_j \cdot \mathrm{D}u \\ &= \liminf \int_\Omega \mathrm{D}u_j \cdot \mathrm{D}u \\ &\leq \liminf \mathcal{E}[u_j]^{1/2} \mathcal{E}[u]^{1/2} \end{aligned}$$

Theorem 1.18 (Weyl's lemma)**.** Weakly harmonic functions are smooth. That is, for $\Omega \subseteq \mathbb{R}^n$ a domain, $u \in L^1_{\text{loc.}}(\Omega)$, if we have u is a weak solution to Laplace's equation, then u is C^∞ and $\Delta u = 0$ in Ω.

Proof. Mollify u. Take $\varphi \in C^\infty(\mathbb{R}^n)$, such that

- $0 \leq \varphi$,
- $\varphi(x) = 0$ for $|x| \geq 1$,
- $\int_{\mathbb{R}^n} \varphi = 1$,
- φ is radially symmetric[6].

For $\sigma > 0$, set

$$\varphi_\sigma(x) = \frac{1}{\sigma^n}\varphi\left(\frac{x}{\sigma}\right)$$

Then $\varphi_\sigma \in C_c^\infty(B_\sigma(0))$ is nonnegative and has integral 1.

Define

$$u_\sigma(x) = (\varphi_\sigma * u)(x)$$

This is well defined for

$$x \in \Omega_\sigma = \{x \in \Omega \mid d(x, \partial\Omega) \geq \sigma\}$$

Then u_σ is smooth, $u_\sigma \to u$ in $L^1_{\text{loc.}}(\Omega)$. Moreover, $\Delta u_\sigma = 0$. To see this,

$$\frac{\partial}{\partial x^i} u_\sigma = \int_\Omega u(y)\frac{\partial}{\partial x^i}\varphi_\sigma(x-y)\mathrm{d}y = -\int_\Omega u(y)\frac{\partial}{\partial y^i}\varphi_\sigma(x-y)\mathrm{d}y$$

and so

$$\Delta_x u_\sigma(x) = \int_\Omega u(y)\Delta_y\varphi_\sigma(x-y)\mathrm{d}y = 0$$

as u is weakly harmonic.

By the a priori derivative estimates for harmonic functions, for $\Omega' \Subset \Omega$,

$$\sup_{\Omega'} |\mathrm{D}^\alpha u_\sigma| \leq C\int_{\Omega'_{\sigma_1}} |u_\sigma|$$

for some $\sigma_1(\Omega')$ small, where

$$\Omega'_\sigma = \Omega' \cup \{x \in \Omega \mid d(x, \partial\Omega') < \sigma\}$$

Since $u_\sigma \to u$ in $L^1_{\text{loc.}}(\Omega)$, for σ small enough,

$$C\int_{\Omega'_{\sigma_1}} |u_\sigma| \leq C\int_{\Omega'_{\sigma'}} (|u|+1)$$

Hence

$$\sup_{\Omega'} |\mathrm{D}^\alpha u_\sigma| \leq C\int_{\Omega'_{\sigma_1}} (|u|+1)$$

So $\mathrm{D}^\alpha u_\sigma$ is uniformly bounded in $L^\infty(\Omega')$. Hence (as bounded derivatives imply equicontinuity) by Arzela-Ascoli, there exists a subsequence (σ_j) such that $\sigma_j \to 0$, and there exists $\widetilde{u} \in C^\infty(\Omega)$ such that $u_{\sigma_j} \to \widetilde{u}$ in $C^k(\Omega')$ for all k. Hence

$$\Delta\widetilde{u} = \lim_{j\to\infty} \Delta u_{\sigma_j} = 0$$

in Ω, as Ω' was arbitrary. By properties of mollifiers, $u_\sigma \to u$ a.e. in Ω, and so $\widetilde{u} = u$ a.e. □

Remark 1.19. We do not say anything about boundary regularity. It is possible to get (at least) $u \in C^0(\overline{\Omega})$.

Let's now improve our C^∞ existence result to C^0.

[6] This is not a standard assumption for mollifiers, but we can assume this.

Theorem 1.20 (existence and uniqueness for the Dirichlet problem with C^0 data)**.** Suppose Ω is bounded with $\partial\Omega$ sufficiently regular, then for any $\varphi \in C^0(\partial\Omega)$, there exists a unique $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ solving

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Remark 1.21. We might have

$$\int_\Omega |Du|^2 = \infty$$

for this solution.

Proof. Choose a sequence $(\varphi_n) \subseteq C^\infty(\mathbb{R}^n)$, such that $\varphi_n \to \varphi$ on $\partial\Omega$, in L^∞. Then we know that there exists $u_n \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$, such that

$$\begin{cases} \Delta u_n = 0 & \text{in } \Omega \\ u_n = \varphi_n & \text{on } \partial\Omega \end{cases}$$

Then for all $n, m \in \mathbb{N}$, $\Delta(u_n - u_m) = 0$ in Ω, and $u_n - u_m = \varphi_n - \varphi_m$ on $\partial\Omega$. By the weak maximum principle,

$$\sup_{\overline{\Omega}} |u_n - u_m| \le \sup_{\partial\Omega} |u_n - u_m| = \left\|\varphi_n - \varphi_m\right\|_{L^\infty} \to 0$$

as $n, m \to \infty$. So (u_n) is Cauchy in $C^0(\overline{\Omega})$, which is a Banach space, so by completeness, there exists $u \in C^0(\overline{\Omega})$ such that $u_n \to u$ uniformly on $\overline{\Omega}$. In particular, $u = \varphi$ on $\partial\Omega$.

By the derivative estimates, (u_n) converges in $C^k(\Omega')$, for any $\Omega' \Subset \Omega$, and so u is smooth in the interior. Thus, $\Delta u = 0$ in the interior. $\square$

Remark 1.22. A sufficient condition for regularity is that $\partial\Omega$ is C^2. More generally, it is enough to have the *exterior sphere condition*, which says that for all $z \in \partial\Omega$, there exists $B_\rho(y)$, such that $\overline{B_\rho(y)} \cap \partial\Omega = \{z\}$. There exists bounded domains in which this fails, and the conclusion of the theorem fails in that case as well. For example, when the boundary has a cusp.

2 General second order elliptic operators

From now on, write

$$Lu = a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu$$

We will work on $\Omega \subseteq \mathbb{R}^n$ open, $u \in C^2(\Omega)$, $a^{ij}, b^i, c : \Omega \to \mathbb{R}$ and consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

for given f and φ. If we can write L in divergence form

$$Lu = \partial_i(a^{ij}\partial_j u) + \widehat{b}^i\partial_i u + cu$$

then we can use Hilbert space methods, as in Analysis of PDEs. But if a^{ij} is only C^0 say, we will need Schauder theory.

The idea is to deform L into Δ using a series of rescalings. In particular, this does not involve Sobolev spaces. Since $u \in C^2(\Omega)$, we can assume a^{ij} is symmetric.

Definition 2.1

We say that L is *elliptic* in Ω if the matrix (a^{ij}) is positive definite in Ω. So that

$$0 \le \lambda(x)|\xi|^2 \le a^{ij}(x)\xi^i\xi^j \le \Lambda(x)\left|\xi^2\right|$$

for all $\xi \in \mathbb{R}^n$ non-zero. In particular, λ is the minimum eigenvalue, and Λ is the maximum.

L is *strictly elliptic* if there exists λ_0 such that $0 < \lambda_0 \le \lambda(x)$ for all x. L is *uniformly elliptic* if it is elliptic, and $\Lambda(x)/\lambda(x)$ is uniformly bounded.

Remark 2.2. In general, uniformly elliptic does not imply strictly elliptic.

Example 2.3
The minimal surface equation
$$\nabla \cdot \left(\frac{Du}{\sqrt{1+|Du|^2}} \right) = 0$$
has
$$a^{ij} = \left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2} \right) \frac{1}{\sqrt{1+|Du|^2}}$$
This is elliptic but not uniformly elliptic.

Lecture 4

We are interested in general second order elliptic operators, with $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$. In particular, we cannot write them in divergence form, as a^{ij} are not C^1.

2.1 Basic properties

Theorem 2.4 (weak maximum principle). Suppose that L is elliptic and that
$$\sup_{\Omega} \left| \frac{b^i}{\lambda} \right| < \infty \tag{(*)}$$
for some i. Moreover, suppose Ω is bounded, open, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, such that $Lu \geq 0$[a].
Then

- if $c = 0$, then
$$\sup_{\Omega} u = \sup_{\partial\Omega} u$$
- if $c \leq 0$, then
$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$$
where $u^+ = \max(u, 0)$.

[a] We say that u is a *subsolution*.

Remark 2.5. The assumption that $c \leq 0$ is crucial. For example, when $n = 1$, let $\Omega = (0, \pi)$, and $u'' + u = 0$. Then $u(x) = \sin(x)$ is a solution. But then $\sup_{\Omega} u = 1$, and $\sup_{\partial\Omega} u^+ = 0$.
For $n = 2$, $\Omega = (0, \pi)^2$, and $\Delta u + 2u = 0$. Then
$$u(x, y) = \sin(x)\sin(y)$$
has the same properties as the above.

Proof. First suppose $c = 0$. If $Lu > 0$ in Ω, then in fact the strong maximum principle holds. Indeed, if $x_0 \in \Omega$ is a local maximum, then
$$\partial_i u(x_0) = 0 \quad \text{and} \quad \partial_i \partial_j u(x_0) \preccurlyeq 0$$
Since $a^{ij}(x_0) \succcurlyeq 0$, we have that
$$a^{ij} \partial_i \partial_j u(x_0) = \operatorname{tr}\big(A \nabla^2 u(x_0)\big) \leq 0$$
Hence $0 < Lu(x_0) = a^{ij}\partial_i\partial_j u(x_0) + b^i \partial_i u(x_0) \leq 0$. Contradiction. More generally, if $Lu \geq 0$ in Ω, consider
$$v(x) = e^{\gamma x_1}$$

for some $\gamma > 0$ to be chosen. Here, we assume without loss of generality that $(*)$ holds for $i = 1$. Then

$$\partial_1 v = \gamma e^{\gamma x_1} \quad \text{and} \quad \partial_i v = 0 \text{ for } i \neq 1$$

and

$$\partial_1 \partial_1 v \gamma^2 e^{\gamma x_1}$$

and all other second derivatives are zero. So

$$\begin{aligned} Lv &= e^{\gamma x_1}(a^{11}\gamma^2 + b^1\gamma) \\ &\geq e^{\gamma x_1}(\lambda\gamma^2 + b^1\gamma) \\ &= \gamma e^{\gamma x_1}\left(\gamma^2 + \frac{b^1}{\lambda}\gamma\right) \end{aligned}$$

This is positive for γ large enough. Hence

$$L(u + \varepsilon v) > 0$$

for all $\varepsilon > 0$. By the above,

$$\begin{aligned} u(x) &\leq \sup_{\Omega}(u + \varepsilon v) \quad \text{as } v \geq 0 \\ &\leq \sup_{\partial\Omega}(u + \varepsilon v) \\ &\leq \sup_{\partial\Omega} u + \varepsilon \sup_{\partial\Omega} v \end{aligned}$$

Taking $\varepsilon \to 0$, we get that $u(x) \leq \sup_{\partial\Omega} u$. Since this is true for all $x \in \Omega$, then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u$$

The reverse inequality is trivial.

Now suppose $c \leq 0$. Define

$$L_0 u = a^{ij}\partial_i\partial_j u + b^i \partial_i u$$

and consider

$$\Omega^+ = \{x \in \Omega \mid u(x) > 0\}$$

Since $cu \leq 0$ on Ω^+,

$$L_0 u = Lu - cu \geq 0$$

on Ω^+. Note if $\Omega^+ = \varnothing$, then $\sup_{\Omega} u \leq 0$, and $u^+ = 0$, and the conclusion is trivial. Thus, without loss of generality assume $\Omega^+ \neq \varnothing$. Then there exists $x_0 \in \partial\Omega \cap \partial\Omega^+$, with $u(x_0) \geq 0$. If not, then

$$\partial\Omega^+ \cap \partial\Omega = \varnothing$$

and so $\partial\Omega^+ \subseteq \Omega$, and so it $\partial\Omega^+ \subseteq \Omega \setminus \Omega^+$. So $u|_{\partial\Omega^+} \leq 0$. But this contradicts the first part for L_0 on Ω^+.

Hence

$$\begin{aligned} \sup_{\Omega} u &= \sup_{\Omega^+} u \\ &= \sup_{\partial\Omega^+} u \\ &\leq \sup_{\partial\Omega} u \\ &\leq \sup_{\partial\Omega} u^+ \end{aligned}$$

□

Corollary 2.6. Let Ω be bounded open, $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, L is elliptic, with the same bound on Ω, and $c \leq 0$ in Ω. Then

1. if $Lu \leq 0$ in Ω, then

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-$$

where $u^- = \min(u, 0)$.

2. if $Lu = 0$, then

$$\sup_{\partial\Omega} |u| = \sup_{\Omega} |u|$$

Proof. Exercise. ☐

Corollary 2.7. Let L as above, and suppose we have $u, v, w \in C^2(\Omega) \cap C^0(\overline{\Omega})$, such that

$$Lu \geq 0 \quad Lv = 0 \quad \text{and} \quad Lw \leq 0$$

Then

1. if $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.
2. if $v \leq w$ on $\partial\Omega$, then $v \leq w$ in $\overline{\Omega}$.

Proof. Exercise. ☐

We want to build towards a strong maximum principle. For this, we need

Theorem 2.8 (Hopf boundary point lemma)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open, and take $y \in \partial\Omega$. Suppose $\partial\Omega$ satisfies the *interior sphere condition*[a] at y. Let L be uniformly elliptic in Ω, with

$$\sup_{\Omega} \frac{|b|}{\lambda} + \sup_{\Omega} |c|\lambda < \infty$$

Suppose $u \in C^2(\Omega) \cap C^0(\{y\} \cup \Omega)$, such that $u(y) \geq u(x)$ for all $x \in \Omega$, and $Lu \geq 0$ in Ω. Finally, assume one of the following holds:

(i) $c = 0$ in Ω,

(ii) $c \leq 0$ in Ω and $u(y) \geq 0$,

(iii) $u(y) = 0$

Then

$$\frac{\partial u}{\partial \nu} > 0$$

if it exists, where ν is the outwards pointing normal at y to $\partial B_R(z)$, coming from the interior sphere condition.

[a] That is, there exists $R > 0$, $z \in \Omega$, such that $B_R(z) \subseteq \Omega$, $y \in \partial B_R(z)$.

Remark 2.9. The weak maximum principle implies that

$$\frac{\partial u}{\partial \nu} \geq 0$$

and so the content of the theorem is the strict inequality.

Proof. Let

$$A = B_R(z) \setminus B_r(z)$$

for some $0 < r < R$. We will first solve cases (i) and (ii). On A, consider

$$v(x) = e^{-\alpha|x-z|^2} - e^{-\alpha R^2}$$

First note that on A, $v > 0$.

$$\partial_i v(x) = -2\alpha(x_i - z_i)e^{-\alpha|x-z|^2}$$

and

$$\partial_i\partial_j v(x) = -2\alpha\delta_{ij}e^{-\alpha|x-z|^2} + 4\alpha^2(x_i - z_i)(x_j - z_j)e^{-\alpha|x-z|^2}s$$

So on A,

$$Lv = e^{-\alpha|x-z|^2}(a^{ij}4\alpha^2(x_i - z_i)(x_j - z_j) - 2\alpha a^{ii} - 2\alpha b^i(x_i - z_i) + c) - ce^{-\alpha R^2}$$

By ellipticity and the sign of c,

$$\begin{aligned} Lv &\geq 2^{-\alpha|x-z|^2}(4\alpha^2\lambda(x)|x-z|^2 - 2\alpha n\Lambda(x) - 2\alpha|b||x-z| - |c|) \\ &\geq e^{-\alpha|x-z|^2}\lambda(x)\left(\alpha^2 R^2 - 2\alpha n \sup_\Omega \frac{\Lambda}{\lambda} - \alpha R \sup_\Omega \frac{|b|}{\lambda} - \sup_\Omega \frac{|c|}{\lambda}\right) \end{aligned}$$

where r is chosen such that $|x-z|^2 \geq (R/2)^2$. In particular, this is positive for α large enough. Fix such an α. Lecture 5

Set $w(x) = u(x) - u(y) + \varepsilon v(x)$, for some small ε to be determined. Now

$$Lw = Lu + \varepsilon Lv - cu(y) \geq 0$$

in A, by above (and the assumptions). Also,

$$v|_{\partial B_R(x)} = 0$$

and because $u(x) \leq u(y)$ on $\overline{\Omega}$, so

$$w|_{\partial B_R(z)} \leq 0$$

Also, $u(x) < u(y)$ on $\partial_{B_r(z)}$, so we can choose ε small enough such that

$$w|_{\partial B_r(z)} < 0$$

So $w|_{\partial A} \leq 0$. Apply the weak maximum principle to w in A, we get $w \leq 0$, and so

$$u(x) - u(y) + \varepsilon v(x) \leq 0$$

in A. Choose $t < 0$, and we have that

$$\frac{u(y+t\nu) - u(y)}{t} \geq -\varepsilon \frac{v(y+t\nu) - v(y)}{t}$$

Note $v(y) = 0$. Sending $t \to 0$,

$$\begin{aligned} \frac{\partial u}{\partial \nu}(y) &\geq -\varepsilon \frac{\partial v}{\partial \nu}(y) \\ &= -\varepsilon \partial_i v(y) \left(\frac{y_i - z_i}{R}\right) \\ &= 2\alpha\varepsilon R e^{-\alpha R^2} \\ &> 0 \end{aligned}$$

For case (iii), consider

$$\widetilde{L} = L - c^+$$

So $\widetilde{L}u = Lu - c^+u \geq 0$, and we can apply the above to $\widetilde{L}$. □

Theorem 2.10 (strong maximum principle)**.** Suppose $\Omega \subseteq \mathbb{R}^n$ is a domain, with non-empty boundary $\partial\Omega$ satisfying the interior sphere condition for all $y \in \partial\Omega$. Let L be uniformly elliptic, with

$$\sup_\Omega \left(\frac{|b| + |c|}{\lambda}\right) < \infty$$

Suppose $u \in C^2(\Omega)$, with $M = \sup_\Omega u < \infty$, and $Lu \geq 0$ on Ω. Suppose (at least) one of the following holds

(i) if $c = 0$, and $u(y) = M$ for some $y \in \Omega$.

(ii) if $c \leq 0, M \geq 0$ and $u(y) = M$ for some $y \in \Omega$,

(iii) if $M = 0$ and $u(y) = M = 0$ for some $y \in \Omega$.

Then u is constant.

Proof. Let
$$\Sigma = \{x \in \Omega \mid u(x) = M\}$$
By continuity, Σ is closed in Ω. Suppose $\Omega \setminus \Sigma \neq \varnothing$. Choose $z \in \Omega \setminus \Sigma$, such that
$$d(z, \partial\Omega) > d(z, \partial\Sigma)$$
To see this, first choose $z_1 \in \partial\Sigma \cap \Omega$. Then choose $\rho_1 > 0$ such that $B_{\rho_1}(z_1) \subseteq \Omega$. Then choose any
$$z \in B_{\rho_1/2}(z_1) \setminus \Sigma$$
Let
$$R = \sup\{\rho \mid B_\rho(z) \subseteq \Omega \setminus \Sigma\}$$
By construction, there exists $y \in \partial B_R(z) \cap \Sigma$. Since $Du(y) = 0$, this contradicts the Hopf boundary point lemma. So $\Omega \setminus \Sigma = \varnothing$, i.e. $\Omega = \Sigma$, so u is constant.

The assumption (i), (ii) and (iii) are so we can apply the Hopf boundary point lemma. □

Some corollaries of this:

Corollary 2.11 (comparison principle). Let
$$L = a^{ij}\partial_i\partial_j + b^i\partial_i + c$$
be uniformly elliptic in $\Omega \subseteq \mathbb{R}^n$, with
$$\sup_\Omega \left(\frac{|b| + |c|}{\lambda}\right) < \infty$$
Suppose $u, v \in C^2(\Omega)$, such that $Lu \geq Lv$ and $u \leq v$ in Ω. Then

- either $u = v$ on Ω,
- or $u < v$ on Ω,

Proof. $L(u - v) \geq 0$ in Ω, and $u - v \leq 0$. So if there exists $x_0 \in \Omega$, with $u(x_0) = v(x_0)$, then the strong maximum principle implies $u = v$. If not, then $u < v$ in Ω. □

Corollary 2.12 (uniqueness for the Neumann problem). Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, and $\partial\Omega$ satisfies the interior sphere condition at each point. Suppose L is uniformly elliptic, with
$$\sup_\Omega \left(\frac{|b| + |c|}{\lambda}\right) < \infty$$
and $c \leq 0$. Then if $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is such that
$$Lu_i = f \text{ in } \Omega$$
$$\frac{\partial u_i}{\partial \nu} = g \text{ on } \partial\Omega$$
for some $f : \Omega \to \mathbb{R}, g : \partial\Omega \to \mathbb{R}$. Then $u_1 - u_2$ is constant.

Proof. Let $u = u_1 - u_2$. This satisfies the Neumann problem
$$Lu = 0 \text{ in } \Omega$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega$$

Let $M = \sup_{\overline{\Omega}} u \geq 0$. We can assume this since we can just take $-u$ instead. By the strong maximum principle, if $u \not\equiv M$ on Ω, then there exists $y \in \partial\Omega$, such that $u(y) = M$, and $u(x) < u(y)$ for all $x \in \Omega$. But by Hopf,

$$\frac{\partial u}{\partial \nu}(y) = 0$$

Contradiction. □

> **Remark 2.13.** This says that the trivial Neumann problem (i.e. with zero data) has solutions which are constants. Now
>
> $$LM = 0$$
>
> but if $LM = Mc(x)$ for all x, and so if c is not identically zero, then $M = 0$. This constant is only non-zero when $c = 0$[a].
>
> [a]But this is obvious, since Lu in this case only involves derivatives of u.

What happens when for non-zero right hand side? The following will be critical for Schauder theory.

> **Theorem 2.14** (maximum principle a priori estimate)**.** Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, L elliptic, $c \leq 0$, and $\beta = |b|/\lambda$ is L^∞. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $f : \Omega \to \mathbb{R}$, then
>
> (i) if $Lu \geq f$, then
>
> $$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} \left(\frac{|f|}{\lambda} \right)$$
>
> (ii) if $Lu = f$, then
>
> $$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} \left(\frac{|f|}{\lambda} \right)$$
>
> where C is a constant which only depends on β and $\operatorname{diam}(\Omega)$.

Proof. Set $d = \operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} |x - y|$. As Ω is bounded, we can contain

$$\Omega \subseteq \{x \mid a \leq x_1 \leq a + d\}$$

for some $a \in \mathbb{R}$. Without loss of generality, $a = 0$. As before, we will construct subsolutions, and use the weak maximum principle. Let

$$v(x) = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} \frac{|f|}{\lambda}$$

where α is to be determined.

We can compute Lv:

$$(a^{ij}\partial_i\partial_j + b^i\partial_i)e^{\alpha x_1} = e^{\alpha x_1}(a^{11}\alpha^2 + b^1\alpha) \geq e^{\alpha x_1}\lambda(\alpha^2 + \beta\alpha) \geq \lambda$$

if we take $\alpha = \beta + 1$. Hence

$$Lv \leq cv - \lambda \sup \frac{|f|}{\lambda} \leq -\lambda \sup_{\Omega} \frac{|f|}{\lambda}$$

as $c \leq 0, v \geq 0$. Then:

(i) if $Lu \geq f$, then

$$L(u - v) \geq f + \lambda \sup \frac{|f|}{\lambda} \geq 0$$

Lecture 6

Note that $u \leq u^+$, and $u|_{\partial\Omega} \leq v|_{\partial\Omega}$ from the definition of v. So by the weak maximum principle, we have $u \leq v$ in Ω, and

$$\sup_{\Omega} \leq u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} \frac{|f|}{\lambda}$$

where

$$C = \sup_{\Omega} \left(e^{(\beta+1)d} - e^{(\beta+1)x_1} \right)$$

For (ii), if $Lu = f$, apply (i) to $-u$. □

2.2 Hölder spaces

Fix $\Omega \subseteq \mathbb{R}^n$ open, and let $\alpha \in (0, 1]$.

Definition 2.15

we say $u : \Omega \to \mathbb{R}$ is *uniformly Hölder continuous with exponent* α, or *uniformly* α*-Hölder continuous*, if

$$[u]_{\alpha,\Omega} = \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty$$

This is the *Hölder seminorm*.

If $\alpha = 1$, this says that u is uniformly Lipschitz. If $\alpha > 1$, then u is constant, by the mean value theorem.

Definition 2.16

We say that u is *locally* α*-Hölder continuous* if for all $K \Subset \Omega$, $u|_K : K \to \mathbb{R}$ is uniformly α-Hölder continuous.

Let $k \in \mathbb{N} \cup \{\infty\}$. Recall for a multi-index $\beta \in \mathbb{N}^n$, we have

$$|\beta| = \sum_i \beta_i$$

and

$$C^k(\Omega) = \{u : \Omega \to \mathbb{R} \mid \mathrm{D}^\beta u \text{ exists and is continuous for all } |\beta| \leq k\}$$

Definition 2.17

The *Hölder space* $C^{k,\alpha}(\Omega)$ is the space

$$C^{k,\alpha}(\Omega) = \{u \in C^k(\Omega) \mid \mathrm{D}^\beta u \text{ is locally } \alpha\text{-Hölder continuous for all } |\beta| \leq k\}$$

and

$$C^{k,\alpha}(\overline{\Omega}) = \{u \in C^k(\Omega) \mid \mathrm{D}^\beta u \text{ is uniformly } \alpha\text{-Hölder continuous for all } |\beta| \leq k\}$$

For $\alpha \in (0, 1)$, we will write

$$C^\alpha(\Omega) = C^{0,\alpha}(\Omega)$$

and

$$C^\alpha(\overline{\Omega}) = C^{0,\alpha}(\overline{\Omega})$$

Moreover, we have

$$C^{k,0}(\Omega) = C^k(\Omega) \quad \text{and} \quad C^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega})$$

Remark 2.18. On the other hand, note that $C^{k+1}(\Omega) \neq C^{k,1}(\Omega)$, since Lipschitz continuity does not imply C^1. On the other hand, Lipschitz functions are differentiable almost everywhere.

Finally, define

$$C_0^{k,\alpha}(\Omega) = C_c^{k,\alpha}(\Omega) = \{u \in C^{k,\alpha}(\Omega) \mid \operatorname{supp}(u) \text{ compact}\}$$

To define norms on these spaces: for $k \in \mathbb{N}, u \in C^k(\overline{\Omega})$, define

$$[u]_{k,\Omega} = [\mathrm{D}^k u]_{0,\Omega} = \sup_{|\beta|=k} [\mathrm{D}^\beta u]_{0,\Omega} = \sup_{|\beta|=k} \sup_{x\in\Omega} \left|\mathrm{D}^\beta u(x)\right|$$

For $u \in C^{k,\alpha}(\overline{\Omega})$, define

$$[u]_{k,\alpha;\Omega} = [\mathrm{D}^k u]_{\alpha,\Omega} = \sup_{|\beta|=k} [\mathrm{D}^k u]_{\alpha,\Omega}$$

Note that these are seminorms. To get norms, set

$$\|u\|_{C^k(\Omega)} = |u|_{k,\Omega} = |u|_{k,0,\Omega} = \sum_{j=0}^{k} |D^j u|_{0,\Omega}$$

and

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = |u|_{k,\alpha,\Omega} = |u|_{k,\Omega} + [\mathrm{D}^k u]_{\alpha,\Omega}$$

With these norms, C^k and $C^{k,\alpha}$ become Banach spaces. Since we will be using sequences, it is important to understand the compactness properties.

Theorem 2.19 (Arzelà-Ascoli for Hölder spaces)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $k \in \mathbb{N}$, and $\alpha \in (0, 1]$. If $(u_j)_j$ is a sequence in $C^{k,\alpha}(\Omega)$ such that

$$\sup_j |u_j|_{k,\alpha,\Omega'} < \infty$$

for all $\Omega' \Subset \Omega$, then there exists $u \in C^{k,\alpha}(\Omega)$, and a subsequence $(u_{j'})_{j'}$ such that $u_{j'} \to u$ in $C^k(\overline{\Omega'})$.

Remark 2.20. Nothing is said about convergence in $C^{k,\alpha}$.

Proof. Examples sheet 2. □

We have two more ingredients before starting Schauder theory. The first is interpolation.

If we have Banach spaces $X \Subset Y \subseteq Z$, then we can bound the norm in Y by X and Z norms. Interpolation is the exchange of sizes of the X and Z norms. Here, we have

$$C^{k,\alpha}(\overline{\Omega}) \Subset C^k(\overline{\Omega}) \subseteq C^k(\overline{\Omega})$$

Theorem 2.21 (interpolation inequality for Hölder spaces)**.** Let $\varepsilon > 0$, $\ell \in \mathbb{N}$, $\alpha \in (0, 1]$. Then there exists $C = C(n, \ell, \alpha, \varepsilon)$, such that if $u \in C^{\ell,\alpha}(\overline{B_R(x_0)})$, then

$$R^k |D^k u|_{0,B_R(x_0)} \leq \varepsilon R^{\ell+\alpha} [D^\ell u]_{\alpha,B_R(x_0)} + C|u|_{0,B_R(x_0)}$$

for all $0 \leq k \leq \ell$.

Sketch proof. For details, see examples sheet 2. By rescaling and shifting, it suffices to consider the case $R = 1$. Then argue by contradiction using Arzelà-Ascoli. □

The second ingredient is the following lemma.

Theorem 2.22 (Simon's absorbing lemma)**.** Let $B_R(x) \subseteq \mathbb{R}^n$ be fixed, and let S be a nonnegative, subadditive function on the collection of sub-balls of $B_R(x)$. That is, if

$$B_\rho(y) \subseteq \bigcup_{j=1}^{N} B_{\rho_j}(y_j) \subseteq B_R(x)$$

then

$$S(B_\rho(y)) \leq \sum_{j=1}^{N} S(B_{\rho_j}(y_j))$$

Let $\lambda \geq 0, \theta \in (0, 1)$, then there exists $\delta = \delta(n, \lambda, \theta) \in (0, 1)$ such that: Suppose that for *all* balls $B_\rho(y) \subseteq B_R(x)$, we have

$$\rho^\lambda S(B_{\theta\rho}(y)) \leq \delta \rho^\lambda S(B_\rho(y)) + \gamma$$

for some fixed γ. Then

$$R^\lambda S(B_{\theta R}(x)) \leq C\gamma$$

for some $C = C(n, \theta, \lambda)$.

Remark 2.23. This says that if there exists a local bound on S, then we can "absorb" the S-term on the right hand side to get a global bound.

Lecture 7

Proof of Simon's absorbing lemma. Let

$$Q = \sup_{B_\rho(y) \subseteq B_R(x)} \rho^\lambda S(B_{\theta\rho}(y))$$

Recall we have that

$$\rho^\lambda S(B_{\theta\rho}(y)) \le \delta \rho^\lambda S(B_\rho(y)) + \gamma$$

By subadditivity of S, we have that

$$Q \le R^\lambda S(B_R(x)) < \infty$$

Fix any $B_\rho(y) \subseteq B_R(x)$. Cover $B_{\theta\rho}(y)$ by a collection of balls

$$\{B_{(1-\theta)\theta^2\rho}(z_j)\}_{j=1}^N$$

with $N \le C(\theta, n)$, which is independent of ρ and y. Moreover, $z_j \in B_{\theta\rho}(y)$. To do this: Choose a maximal pairwise disjoint collection of balls

$$\{B_{(1-\theta)\theta^2\rho/2}(z_j)\}_{j=1}^N$$

where $z_j \in B_{\theta\rho(y)}$. We claim that these z_j's work. If not, then there exists

$$z \in B_{\theta\rho}(y) \setminus \bigcup_{i=1}^N B_{(1-\theta)\theta^2\rho}(z_j)$$

and so $d(z, z_j) \ge (1-\theta)\theta^2\rho$ for all j. In particular,

$$B_{(1-\theta)\theta^2\rho/2}(z) \cap B_{(1-\theta)\theta^2\rho/2}(z_j) = \varnothing$$

This contradicts maximality.

For the bound on N, note that from considering radii,

$$\bigcup_{j=1}^N B_{(1-\theta)\theta^2\rho/2}(z_j) \subseteq B_{(1-\theta)\theta^2\rho/2+\theta\rho}(y) \subseteq B_\rho(y)$$

Since the balls on the left hand side are disjoint, there exists a volume bound

$$N\omega_n \left(\frac{(1-\theta)\theta^2\rho}{2}\right)^n \le \omega_n \left(\frac{(1-\theta)\theta^2\rho}{2} + \theta\rho\right)$$

Which is independent of ρ and y.

To conclude, by subadditivity,

$$\begin{aligned}
\rho^\lambda S(B_{\theta\rho}(y)) &\le \rho^\lambda \sum_{j=1}^N S(B_{(1-\theta)\theta^2\rho}(z_j)) \\
&\le ((1-\theta)\theta)^{-\lambda} \sum_{j=1}^N \left(\delta\left((1-\theta)\theta\rho\right)^\lambda S(B_{(1-\theta)\theta\rho}(z_j)) + \gamma\right) \\
&\le \delta((1-\theta)\theta)^{-\lambda} NQ + N_\gamma((1-\theta)\theta)^{-\lambda}
\end{aligned}$$

Now taking the supremum over all $B_{\rho(y)} \subseteq B_R(x)$,

$$Q \le \delta C_1 Q + C_2\gamma$$

where C_1, C_2 depends on n, θ, λ. Choose δ sufficiently small such that $\delta C_1 < 1$. Taking $\delta = 1/(2C_1)$,

$$Q \le 2C_2\gamma$$

□

3 Schauder theory

3.1 Interior Schauder estimates

We will first prove interior estimates in the unit ball, and then extend them to more general domains. The main point is: if the coefficients of L are α-Hölder continuous, then any $C^{2,\alpha}$-solution of $Lu = f$ can be bounded in $C^{2,\alpha}$ on a smaller ball by $|u|_0$ and f.

Theorem 3.1 (unit scale interior Schauder estimates). Let $\alpha \in (0,1), \beta > 0$, and suppose $a^{ij}, b^i, c \in C^{0,\alpha}(B_1(0))$, with

$$\left|a^{ij}\right|_{0,\alpha;B_1(0)} + \left|b^i\right|_{0,\alpha;B_1(0)} + |c|_{0,\alpha;B_1(0)} \leq \beta$$

Suppose L is strictly elliptic, so there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $x \in B_1(0), \xi \in \mathbb{R}^n$. Then if $u \in C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$, and $f \in C^{0,\alpha}(B_1(0))$ satisfies $Lu = f$ in $B_1(0)$, then

$$|u|_{2,\alpha;B_{1/2}(0)} \leq C\left(|u|_{0;B_1(0)} + |f|_{0,\alpha;B_1(0)}\right)$$

for some constant $C = C(n, \lambda, \alpha, \beta)$.

Remark 3.2.

- We can never take $\alpha = 0$ or $\alpha = 1$. The theorem is false in these cases.
- Strict ellipticity gives a lower bound for λ, and upper bound on $|a^{ij}|_{0,\alpha}$ gives an upper bound on Λ. So Λ/λ is bounded, and so we have uniform ellipticity.
- Note that we can control two derivatives of u using no derivatives on u or f.
- We will in fact strengthen this to

  $$|u|_{2,\alpha;B_\theta(0)} \leq C\left(|u|_{0;B_1(0)} + |f|_{0,\alpha;B_1(0)}\right)$$

  for all $\theta \in (0,1)$, $C = C(n, \lambda, \alpha, \beta, \theta)$.
- We may no assumptions, and state no conclusions, about the regularity on the boundary.
- The Schauder estimate gives a compactness property for the space of solutions to $Lu = f$. If $(u_k) \subseteq C^{2,\alpha}(B_1(0)) \cap C^0(\overline{B_1(0)})$ solves $Lu_k = f$ in $B_1(0)$, and

  $$\gamma = \sup_k \sup_{B_1(0)} |u_k| < \infty$$

  Then

  $$|u_k|_{2,\alpha;B_\theta(0)} \leq C(\gamma, n, \theta, \beta, \lambda, f)$$

  and so by Arzela-Ascoli, there exists a subsequence $(u_{k'})$, with $u \in C^{2,\alpha}(B_1(0))$, such that $u_{k'} \to U_k$ in $C^2(B_\theta(0))$ for all $\theta \in (0,1)$. Passing to the limit, $Lu = f$.

Lecture 8

Proof. Omitted. □

Lecture 9

We now give some corollaries.

Corollary 3.3 (scale invariant Schauder estimate). Suppose $B_R(x_0) \subseteq \mathbb{R}^n$, and $a^{ij}, b^i, c \in C^{0,\alpha}(B_R(x_0))$, with

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

for some $\lambda > 0$. Suppose also that

$$\left|a^{ij}\right|_{0,B_R(x_0)} + R^\alpha[a^{ij}]_{\alpha,B_R(x_0)} + R\left(\left|b^i\right|_{0,B_R(x_0)} + R^\alpha[b^i]_{\alpha,B_R(x_0)}\right) + R^2\left(|c|_{0,B_R(x_0)} + R^\alpha[c]_{\alpha,B_R(x_0)}\right) \leq \beta$$

for some $\beta > 0$. Suppose $u \in C^{2,\alpha}(B_R(x_0)) \cap C^0(\overline{B_R(x_0)})$, satisfying $Lu = f \in C^{0,\alpha}(B_R(x_9))$. Then

$$|u|'_{2,\alpha;B_{R/2}(x_0)} \leq C\left(|u|_{0,B_R(x_0)} + R^2|f|_{0,B_R(x_0)} + R^{2+\alpha}[f]_{\alpha,B_R(x_0)}\right)$$

where

$$|u|'_{k,\alpha;B_\rho(y)} = \sum_{j=0}^{k} \rho^j \left|D^j u\right|_{0,B_\rho(y)} + \rho^{k+\alpha}[D^k u]_{\alpha,B_\rho(y)}$$

and $C = C(n, \lambda, \alpha, \beta)$ is independent of u and of R.

Proof. Apply theorem 3.1 with $x \to x_0 + Rx$. □

Corollary 3.4 (interior Schauder estimates in general domains). Let $\alpha \in (0,1), \Omega \subseteq \mathbb{R}^n$ open and bounded. Suppose we have $a^{ij}, b^i, c \in C^{0,\alpha}(\Omega)$, with

$$\left|a^{ij}\right|_{0,\alpha;\Omega} + \left|b^i\right|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta$$

with

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for some $\lambda > 0$. Suppose $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ satisfies $Lu = f \in C^{0,\alpha}(\Omega)$. Then for all open $\widetilde{\Omega} \Subset \Omega$,

$$|u|_{2,\alpha;\widetilde{\Omega}} \leq C\left(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega}\right)$$

where $C = C(n, \alpha, \beta, \lambda, d(\widetilde{\Omega}, \partial\Omega))$.

Proof. Let

$$d = d(\widetilde{\Omega}, \partial\Omega) = \sup\{r > 0 \mid \left(\widetilde{\Omega}\right)_r \subseteq \Omega\}$$

where

$$(\widetilde{\Omega})_r = \bigcup_{x \in \widetilde{\Omega}} B_r(x)$$

is the r-neighbourhood of $\widetilde{\Omega}$. Then for all $x \in \widetilde{\Omega}$, we have that $B_d(x) \subseteq \Omega$, and so

$$\left|a^{ij}\right|'_{0,\alpha;B_d(x)} + d\left|b^i\right|'_{0,\alpha;B_d(x)} + d^2|c|'_{0,\alpha;B_d(x)} \leq C(d)\beta$$

Then by corollary 3.3, we have an estimate

$$\begin{aligned} |u|_{0;B_{d/2}(x)} + d|\mathrm{D}u|_{0,B_{d/2}(x)} + d^2\left|\mathrm{D}^2u\right|_{0,B_{d/2}(x)} + d^{2+\alpha}[\mathrm{D}^2u]_{\alpha,B_{d/2}(x)} &\leq C\left(|u|_{0,B_d(x)} + d^2|f|_{0;B_d(x)} + d^{2+\alpha}[f]_{\alpha,B_d(x)}\right) \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}) \end{aligned} \tag{a}$$

where $C = C(n, \lambda, \alpha, \beta, d)$. In particular,

$$|u(x)| + |\mathrm{D}u(x)| + \left|\mathrm{D}^2u(x)\right| \leq C\left(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega}\right)$$

for all $x \in \widetilde{\Omega}$. So

$$|u|_{2,\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right) \tag{b}$$

But also, by (a),

$$\sup_{x,y \in \widetilde{\Omega}, |x-y|<d/2} \frac{\left|\mathrm{D}^2u(x) - \mathrm{D}^2u(y)\right|}{|x-y|^\alpha} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right)$$

On the other hand, of $|x - y| \geq d/2$, then

$$\frac{\left|\mathrm{D}^2u(x) - \mathrm{D}^2u(y)\right|}{|x-y|^\alpha} \leq \left(\frac{d}{2}\right)^{-\alpha} |u|_{2,\widetilde{\Omega}} \leq \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right)$$

by (b). Hence

$$[\mathrm{D}^2 u]_{\alpha;\tilde{\Omega}} \leq \left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}\right)$$

and so combining this with (b), we get the required result. □

3.2 Boundary Schauder estimates

Write

$$\begin{aligned}
\mathbb{R}^n_+ &= \left\{(x', x^n) \mid x' \in \mathbb{R}^{n-1}, x^n > 0\right\} \\
\mathbb{R}^n_- &= \left\{(x', x^n) \mid x' \in \mathbb{R}^{n-1}, x^n < 0\right\} \\
B^{\pm}_R(y) &= B_R(y) \cap \mathbb{R}^n_{\pm} \\
B^{\pm}_R &= B^{\pm}_R(0) \\
S_R(y) &= B_R(y) \cap \{x^n = 0\}
\end{aligned}$$

Theorem 3.5 (boundary Schauder estimates in a unit ball)**.** As before, $0 < \alpha < 1$, $a^{ij}, b^i, c \in C^{0,\alpha}(B_1^+)$, and

$$\left|a^{ij}\right|_{0,\alpha;B_1^+} + \left|b^i\right|_{0,\alpha,B_1^+} + |c|_{0,\alpha;B_1^+} \leq \beta$$

and

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $x \in B_1^+$. Suppose $u \in C^{2,\alpha}(B_1^+)$ satisfies

$$\begin{cases} Lu = f & \text{in } B_1^+ \\ u = 0 & \text{on } S_1 \end{cases}$$

where $f \in C^{0,\alpha}$ and $\varphi \in C^{2,\alpha}$. Then

$$|u|_{2,\alpha,B^+_{1/2}} \leq C\left(|u|_{0;B_1^+} + |f|_{0,\alpha;B_1^+} + |\varphi|_{2,\alpha;B_1^+}\right)$$

Proof. By considering $v = u - \varphi$, suffices to consider the case when $\varphi = 0$, since $L\varphi \in C^{0,\alpha}(B_1^+)$.
The rest of the proof is as in theorem 3.1, which we will omit. □

Proposition 3.6 (reflection principle for harmonic functions)**.** Let Ω^+ be an open subset of $\mathbb{R}^n_+$, and let $T = \partial\Omega^+ \cap \{x^n = 0\}$. Let Ω^- be the reflection of Ω^+ in $\{x^n = 0\}$. Let $v \in C^2(\Omega^+) \cap C^0(\Omega^+ \cup T)$, and let $\overline{v}$ be the odd reflection of v in T. That is, $\overline{v} : \Omega^+ \cup T \to \Omega^- \to \mathbb{R}$, where

$$\overline{v}(x', x^n) = \begin{cases} v(x', x^n) & (x', x^n) \in \Omega^+ \cup T \\ -v(x', -x^n) & (x', x^n) \in \Omega^- \end{cases}$$

Then if $\Delta v = 0$ in Ω^+ and $v|_T = 0$, then $\overline{v} \in C^2(\mathrm{Int}(\Omega^+ \cup T \cup \Omega^-))$ and $\Delta\overline{v} = 0$.

Proof. Use the mean value property. See examples sheet 2. □

Remark 3.7. This is trivial if $T = \varnothing$, since Ω^+ and Ω^- are disjoint. The important part of this theorem is that $\overline{v}$ is C^2 across T.

Proposition 3.8 (boundary absorbing lemma)**.** Given $\theta \in (0, 1), \mu \in \mathbb{R}$, then there exists $\delta = \delta(n, \theta, \mu)$, and $C = C(n, \theta, \mu)$ such that:

Fix $R > 0$, and let

$$\mathcal{B} = \left\{B_\rho(y) \mid B_\rho(y) \subseteq B_R^+(0)\right\}$$
$$\mathcal{B}^+ = \left\{B_\rho^+(y) \mid y^n = 0, B_\rho^+(y) \subseteq B_R^+(0)\right\}$$

Suppose $S : \mathcal{B} \cup \mathcal{B}^+ \to \mathbb{R}$ is a nonnegative subadditve function, such that

$$\rho\mu S(B_{\theta\rho}^+(y)) \leq \delta\rho\mu S(B_\rho^+(y)) + \gamma$$

for all $B_\rho^+(y) \in \mathcal{B}^+$, and

$$\rho\mu S(B_{\theta\rho}(y)) \leq \gamma$$

for $B_\rho(y) \in \mathcal{B}$. Then

$$R^\mu S(B_{\theta R}^+(0)) \leq C\gamma$$

Proof. Examples sheet 2. □

Lecture 10

Let (H) denote the following hypothesis: "Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain, and $0 < \alpha < 1$. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$, with

$$\left|a^{ij}\right|_{0,\alpha;\Omega} + \left|b^i\right|_{0,\alpha;\Omega} + |c|_{0,\alpha;\Omega} \leq \beta$$

and suppose there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

"

Theorem 3.9 (boundary Schauder estimates in general domains). Suppose (H) holds, ad Ω is a $C^{2,\alpha}$ domain. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that if $u \in C^{2,\alpha}(\overline{\Omega})$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$ such that

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Then for all $x \in \partial\Omega$,

$$|u|_{2,\alpha;B_\varepsilon(x)\cap\Omega} \leq C\left(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$

Remark 3.10. We need Ω to be $C^{2,\alpha}$ for u to be $C^{2,\alpha}$ on $\partial\Omega$.

Proof. Omitted. □

3.3 Global Schauder estimates

We can combine the interior and boundary estimates to get:

Theorem 3.11 (global Schauder estimates). Suppose (H) holds, and suppose that Ω is a $C^{2,\alpha}$ domain. If $u \in C^{2,\alpha}(\Omega)$, $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C^{2,\alpha}(\Omega)$ is such that

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

Then

$$|u|_{2,\alpha;\Omega} \leq C\left(|u|_{0,\Omega} + |f|_{0,\alpha;\Omega} + |\varphi|_{2,\alpha;\Omega}\right)$$

where $C = C(n, \lambda, \alpha, \beta, \Omega)$.

Lecture 11

4 Solvability of the Dirichlet problem

Given $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$, the *Dirichlet problem* for L is: Given $f \in C^{0,\alpha}(\overline{\Omega})$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$, is there a solution $u \in C^2(\overline{\Omega})$, to

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

If so, is it unique?

Theorem 4.1. Let $\alpha \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ is a bounded $C^{2,\alpha}$ domain. Suppose $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$, with $c \leq 0$ in Ω. Then the following are equivalent:

(i) for any given $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$, the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

has a solution $u \in C^{2,\alpha}(\overline{\Omega})$.

(ii) For any given $f \in C^{0,\alpha}(\overline{\Omega})$ and $\varphi \in C^{2,\alpha}(\overline{\Omega})$, the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

has a solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Proof. Omitted. □