Group Cohomology

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Lent 2023*

Contents

Lecture 1

1 Definitions and resolutions

Let G be a group.

Definition 1.1 (integral group ring)
The *integral group ring* $\mathbb{Z}G$ has elements being formal sums

$$\sum n_g g$$

for $n_g \in \mathbb{Z}, g \in G$, and only finitely many n_g are non-zero. This is a free abelian group under addition, i.e.

$$\left(\sum m_g g\right) + \left(\sum n_g g\right) = \sum (m_g + n_g) g$$

and we have multiplication

$$\left(\sum m_h h\right) \left(\sum n_k k\right) = \sum_g \left(\sum_{hk=g} m_h n_k\right) g$$

This is the associative ring underlying the integral representation theory of G. We will write $1 = 1e$ for the multiplicative identity in G.

*Based on lectures by Chris Brookes. Last updated March 14, 2024.

Definition 1.2 (module)
A *(left)* $\mathbb{Z}G$*-module* M is an abelian group under addition, and a map

$$\begin{aligned} \mathbb{Z}G \times M &\to M \\ (r, m) &\mapsto rm \end{aligned}$$

such that

1. $r(m+n) = rm + rn$,
2. $(r_1 + r_2)m = r_1 m + r_2 m$,
3. $(r_1 r_2)m = r_1(r_2 m)$,
4. $1m = m$

A module is *trivial* if $gm = m$ for all $g \in G$, $m \in M$. We say *the trivial module* is $\mathbb{Z}$ with trivial action $gm = m$. A *free* $\mathbb{Z}G$*-module* on X is formal sums

$$\sum r_x x$$

for $r_x \in \mathbb{Z}G, x \in X$, and finitely many r_x non-zero. We will write $\mathbb{Z}G\{X\}$ for this.

Submodules and *quotient modules* are defined as we would expect.

Definition 1.3 ($\mathbb{Z}G$-map)
An $\mathbb{Z}G$*-map (or morphism)* $\alpha : M_1 \to M_2$ is a map of abelian groups, with

$$\alpha(rm_1) = r\alpha(m_1)$$

for $r \in \mathbb{Z}G, m_1 \in M_1$.

Example 1.4
The *augmentation map* is

$$\begin{aligned} \varepsilon : \mathbb{Z}G &\to \mathbb{Z} \\ \sum n_g g &\mapsto \sum n_g \end{aligned}$$

where we treat $\mathbb{Z}G$ as a left $\mathbb{Z}G$-module, and $\mathbb{Z}$ as the trivial module. This is a $\mathbb{Z}G$ map, and it is also a ring homomorphism.

Notation 1.5. Write $\mathrm{Hom}_G(M, N)$ for the set of $\mathbb{Z}G$-maps from $M \to N$, with pointwise addition.

Example 1.6
If we regard $\mathbb{Z}G$ as a left $\mathbb{Z}G$-module, then

$$\mathrm{Hom}_G(\mathbb{Z}G, M) \cong M$$

for any left $\mathbb{Z}G$-module M, by sending φ to $\varphi(1)$. This is an isomorphism since

$$\varphi(r) = \varphi(r1) = r\varphi(1)$$

Note that $\mathrm{Hom}_G(\mathbb{Z}G, M)$ is a left $\mathbb{Z}G$-module, with

$$(s\phi)(r) = \phi(rs)$$

In particular,

$$\mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G$$

With this, $\varphi \mapsto \varphi(1)$ corresponds to multiplication on the *right* by $\varphi(1)$.

Note that G may not be abelian, so the distinction between left and right modules matters.

Definition 1.7

If $f : M_1 \to M_2$ is a $\mathbb{Z}G$-map, then the dual map is

$$f^* : \mathrm{Hom}_G(M_2, N) \to \mathrm{Hom}_G(M_1, N)$$

for any $\mathbb{Z}G$-module N. This is given by

$$f^*(\varphi) = \varphi \circ f$$

Similarly, if $f : N_1 \to N_2$, then we have

$$f_* : \mathrm{Hom}_G(M, N_1) \to \mathrm{Hom}(M, N_2)$$

with $f_*(\varphi) = f \circ \varphi$.

These are maps of abelian groups, since in general $\mathrm{Hom}_G(M, N)$ doesn't have to be a $\mathbb{Z}G$-module, as we don't have a right action.

Example 1.8 (Prototypical example)

Let $G = \langle t \rangle$ be an infinite cyclic group, and consider the graph Γ with vertices $\{v_i\}_{i\in\mathbb{Z}}$, edges $v_i \leftrightarrow v_{i+1}$. Then G acts on the set V of vertices, by $tv_i = v_{i+1}$, and also on the set E of edges. The action is transitive in both cases. The formal integral sums $\mathbb{Z}V$ and $\mathbb{Z}E$ are $\mathbb{Z}G$-modules, and are free. Fix the edge $e : v_0 \leftrightarrow v_1$. Then we have $\mathbb{Z}G$-maps

$$\begin{aligned} \mathrm{d} : \mathbb{Z}E &\to \mathbb{Z}V \\ e &\mapsto v_1 - v_0 \end{aligned}$$

and also

$$\begin{aligned} \mathbb{Z}V &\to \mathbb{Z} \\ v_0 &\mapsto 1 \end{aligned}$$

This correspond to the augmentation map.

Definition 1.9 (chain complex, exact, homology)

A *chain complex* of $\mathbb{Z}G$-modules is a sequence

$$M_s \xrightarrow{\mathrm{d}_s} M_{s-1} \xrightarrow{\mathrm{d}_{s-1}} \cdots \xrightarrow{\mathrm{d}_{t+1}} M_t$$

of $\mathbb{Z}G$-modules and maps, such that for $t < n < s$, $\mathrm{d}_n \circ \mathrm{d}_{n+1} = 0$. We will write

$$M_\bullet = (M_n, \mathrm{d}_n)_{t \le n \le s}$$

We say that $M_\bullet$ is *exact* at M_n if $\mathrm{im}(\mathrm{d}_{n+1}) = \ker(\mathrm{d}n)$. The sequence is exact if if it exact at all M_n with $t < n < s$.

The *homology* of $M_\bullet$ is given by

$$\begin{aligned} \mathrm{H}_s(M_\bullet) &= \ker(\mathrm{d}s) \\ \mathrm{H}_n(M_\bullet) &= \frac{\ker(\mathrm{d}_n)}{\mathrm{im}(\mathrm{d}_{n+1})} \text{ for } t < n < s \\ \mathrm{H}_t(M_\bullet) &= \mathrm{coker}(\mathrm{d}_{t+1}) \end{aligned}$$

A *short exact sequence* is a chain complex of the form

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$

In our prototypical example, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}E \longrightarrow \mathbb{Z}V \longrightarrow \mathbb{Z} \longrightarrow 0$$

which corresponds to

$$0 \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the map $\mathbb{Z}G \to \mathbb{Z}G$ is multiplication on the right by $t - 1$.

Definition 1.10 (projective)

A $\mathbb{Z}G$-module P is *projective* if for every surjective $\mathbb{Z}G$-map $\alpha : M_1 \to M_2$, and every $\mathbb{Z}G$-map $\beta : P \to M_2$, there exists $\overline{\beta}$ such that

$$\begin{array}{ccccc} & & P & & \\ & \overset{\overline{\beta}}{\nearrow} & \downarrow \beta & & \\ M_1 & \underset{\alpha}{\longrightarrow} & M_2 & \longrightarrow & 0 \end{array}$$

commutes.

If we have any short exact sequence

$$0 \longrightarrow N \xrightarrow{f} M_1 \xrightarrow{\alpha} M_2 \longrightarrow 0$$

Consider

$$0 \longrightarrow \mathrm{Hom}_G(P, N) \xrightarrow{f_*} \mathrm{Hom}_G(P, M_1) \xrightarrow{\alpha_*} \mathrm{Hom}_G(P, M_2) \longrightarrow 0 \qquad (*)$$

Then P is projective if and only if $(*)$ is exact. Note that in $(*)$, we always have exactness except at $\mathrm{Hom}_G(P, M_2)$.

Lecture 2

Lemma 1.11. Free modules are projective.

Proof. Let $\alpha : M_1 \to M_2$ be a surjective $\mathbb{Z}G$-map, $\beta : \mathbb{Z}G\{X\} \to M_2$. For each $x \in X$, since α is surjective, there exists $m_x \in M_1$ such that $\alpha(m_x) = \beta(x)$. We can then define $\overline{\beta} : \mathbb{Z}G\{X\} \to M_1$, by

$$\sum r_x \cdot x \mapsto \sum r_x \cdot m_x$$

□

Definition 1.12

A *projective (resp. free) resolution of the trivial module* $\mathbb{Z}$ is an exact sequence

$$\cdots \xrightarrow{\mathrm{d}_2} P_1 \xrightarrow{\mathrm{d}_1} P_0 \xrightarrow{\mathrm{d}_0} \mathbb{Z} \longrightarrow 0$$

with all the P_i projective (resp. free).

Example 1.13
When $G = \langle t \rangle$ is infinite cyclic, we have the free resolution

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{\cdot(t-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Example 1.14
If $G = \langle t \rangle$ cyclic of order n. Then

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a free resolution, where $\alpha(x) = x(t-1)$ and $\beta(x) = x(1 + t + \cdots + t^{n-1})$.

From algebraic topology, if we have a connected simplicial complex X, with $\pi_1(X) = G$, and the universal cover $\widetilde{X}$ is contractible, then we have a free resolution of $\mathbb{Z}$. The point is that the simplicial complex X contains a lot of information about G. We are trying to replicate this algebraically.

For calculation purposes, we're interested in *small resolutions*. For example, where the free modules have small rank. But for theory development purposes, we're wanting general constructions, and such resolutions tend to be large.

Definition 1.15
G is of type FP_n if $\mathbb{Z}$ has a projective resolution

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

so that $P_n, \ldots, P_0$ are finitely generated.

G is of type FP_n is $\mathbb{Z}$ has a projective resolution with all P_n finitely generated.

G is of type FP if there exists a projective resolution of $\mathbb{Z}$ with finite length and all P_n are finitely generated. That is, $P_s = 0$ for s sufficiently large.

Example 1.16
$G = \langle t \rangle$ infinite cyclic is of type FP, $G = C_n = \langle t \rangle$ finite cyclic is of type FP_∞.

These are to be regarded as finiteness conditions on the group. The topological version of FP_n (which is called F_n) would be asking for X with $\pi_1(X) = G$, having a finite n-skeleton. Note that these two conditions are not equivalent.

Now lets meet some general constructions. If we have a partial projective resolution

$$P_s \xrightarrow{d_s} P_{s-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

Then we can make it longer by setting

$$P_{s+1} = \mathbb{Z}G\{X_{s+1}\}$$

with $X_{s+1} = \ker(d_s)$, and set

$$d_{s+1} : \sum r_x x \mapsto \sum r_x x$$

Then

$$P_{s+1} \xrightarrow{d_{s+1}} P_s \xrightarrow{d_s} P_{s-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

is a longer partial projective resolution. To make P_{s+1} smaller, we could take X_{s+1} to be a $\mathbb{Z}G$-generating set for $\ker(d_s)$. This is particularly useful if $\ker(d_s)$ is finitely generated.

Continuing this, we can get a (full) projective resolution.

Definition 1.17

The *standard (or bar) resolution of* $\mathbb{Z}$, for any group G, is: Let

$$G^{(n)} = \{[g_1|\dots|g_n] \mid g_i \in G\}$$

be the set of symbols. In particular, $G^{(0)} = \{[]\}$ Let

$$F_n = \mathbb{Z}G\{G^{(n)}\}$$

be the free module on $G^{(n)}$. The differential is given by

$$\begin{aligned} \mathrm{d}_n[g_1|\dots|g_n] = {} & g_1[g_2|\dots|g_n] \\ & - [g_1g_2|g_3|\dots|g_n] \\ & + [g_1|g_2g_3|\dots|g_n] + \dots \\ & + (-1)^{n-1}[g_1|\dots|g_{n-1}g_n] \\ & + [g_1|\dots|g_{n-1}] \end{aligned}$$

With this, $\mathrm{d}_{n-1}\mathrm{d}_n = 0$, and we have a chain complex

$$\cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{[]\mapsto 1} \mathbb{Z} \longrightarrow 0$$

Remark 1.18. The bar resolution corresponds to the standard resolution in algebraic topology. Consider $n+1$ tuples G^{n+1}, and form the free abelian group $\mathbb{Z}G^{n+1}$ under addition. G acts on G^{n+1} diagonally, so

$$g \cdot (h_0, \dots, h_n) = (gh_0, \dots, gh_n)$$

So $\mathbb{Z}G^{n+1}$ is a free $\mathbb{Z}G$-module, on the basis of $n+1$ tuples with $g_0 = 1$. Then we have a correspondence

$$[g_1|\dots|g_n] \mapsto (1, g_1, g_1g_2, \dots, g_1\cdots g_n)$$

Note that removing the first entry gives $g_1(1, g_2, g_2g_3, \dots, g_2\cdots g_n)$, and removing the seconc entry in the tuple gives $(1, g_1g_2, g_1g_2g_3, \dots, g_1\cdots g_n)$.

Lemma 1.19. The bar resolution is exact.

Proof. We'll just consider the d_n as maps of abelian groups. F_n has basis $G \times G^{(n)}$ as a free abelian group, and $G \times G^{(n)}$ is the set

$$\{g_0[g_1|\dots|g_n] \mid g_i \in G\}$$

Define $\mathbb{Z}$-linear maps $s_n : F_n \to F_{n+1}$, such that

$$\mathrm{id}_{F_n} = \mathrm{d}_{n+1}s_n + s_{n-1}\mathrm{d}_n \tag{$*$}$$

given by

$$s_n(g_0[g_1|\dots|g_n]) = [g_0|\dots|g_n]$$

We can verify that $(*)$ holds on the basis, and so if $x \in \ker(\mathrm{d}_{n+1})$, then

$$x = \mathrm{id}(x) = \mathrm{d}_{n+1}s_n(x) + s_{n-1}\mathrm{d}_n(x) = \mathrm{d}_{n+1}s_n(x) \in \mathrm{im}(\mathrm{d}_{n+1})$$

That is, s_n is a chain homotopy from the identity to zero. □

Corollary 1.20. A finite group is of type FP_∞.

Proof. The bar resolution gives such a resolution. □

Lecture 3

Definition 1.21

Consider a projective resolution

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \xrightarrow{d_n} \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. Let M be a (left) $\mathbb{Z}G$-module. Apply the functor $\mathrm{Hom}_G(\cdot, M)$, we get the *dual sequence*

$$\cdots \longleftarrow \mathrm{Hom}(P_{n+1}, M) \xleftarrow{\mathrm{d}^{n+1}} \mathrm{Hom}(P_n, M) \longleftarrow \cdots \longleftarrow \mathrm{Hom}(P_0, M) \qquad (\dagger)$$

where $\mathrm{d}^n = \mathrm{d}_n^*$. Then the *$n$-th cohomology group* $\mathrm{H}^n(G, M)$ *with coefficients in M is:*

$$\mathrm{H}^n(G, M) = \frac{\ker(\mathrm{d}^{n+1})}{\mathrm{im}(\mathrm{d}^n)} \text{ for } n \geq 1$$
$$\mathrm{H}^0(G, M) = \ker(\mathrm{d}^1)$$

Remark 1.22. We have dropped the $\mathbb{Z}$-term in $(\dagger)$.

Also, these cohomology groups are the homology groups of the chain complex

$$C_n = \mathrm{Hom}_G(P_{-n}, M)$$

with $-\infty < n \leq 0$.

Later, we will see that these cohomology groups are independent of the choice of resolution.

Example 1.23

In our prototypical example, $G = \langle t \rangle$ infinite cyclic, we have a projective resolution

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{\mathrm{d}} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $\mathrm{d}(x) = x(t-1)$. For $\phi \in \mathrm{Hom}_G(\mathbb{Z}G, M)$, $x \in \mathbb{Z}G$,

$$\mathrm{d}^1(\phi)(x) = \phi(\mathrm{d}_1(x)) = \phi(x(t-1))$$

Moreover, recall $\mathrm{Hom}_G(\mathbb{Z}G, M) \cong M$, where $i(\phi) = \phi(1)$. In particular,

$$\mathrm{d}^1(\phi) \mapsto \mathrm{d}^1\phi(1) = \phi(t-1) = (t-1)\phi(1) = (t-1)i(\phi)$$

In this case, we have

$$0 \longleftarrow M \xleftarrow[(t-1)\cdot]{} M$$

and so

$$\mathrm{H}^0(G, M) = \{m \in M \mid tm = m\} = M^G$$
$$\mathrm{H}^1(G, M) = \frac{M}{(t-1)M} = M_G$$

and $\mathrm{H}^n(G, M) = 0$ otherwise. Here, M^G is the group of *invariants*, the largest submodule with trivial G-action. Conversely, M_G is the *coinvariants*, the largest quotient module with trivial G-action.

Remark 1.24. $\mathrm{H}^0(M, G) = M^G$ is generally true. But $\mathrm{H}^1(M, G) = M_G$ is special behaviour of $G = \langle t \rangle$.

For any group of type FP, we have that $\mathrm{H}^n(G, M) = 0$ for large enough n.

Definition 1.25
G is of *cohomological dimension m over $\mathbb{Z}$* if there exists an $\mathbb{Z}G$-module M, with

$$\mathrm{H}^m(G, M) \neq 0$$

and $\mathrm{H}^n(G, M') = 0$ for any $\mathbb{Z}G$-module M', and $n > m$.

Note that for all G, $\mathrm{H}^0(G, \mathbb{Z}) = \mathbb{Z} \neq 0$.

Example 1.26
$G = \langle t \rangle$ infinite cyclic is of cohomological dimension 1 (over $\mathbb{Z}$).
An exercise is to show that if G is a free group of finite rank, then it is also of cohomological dimension 1. To see this, the Cayley graph is a tree and we can construct a resultion using this.

Remark 1.27. The converse is true. A finitely generated group of cohomological dimension 1 is free. In fact, this is true in general. See [Stallings, 1968], Swan, 1969.

Now consider the bar resolution in our definition of cohomology. Note that

$$\mathrm{Hom}_G(\mathbb{Z}G\{G^{(n)}\}, M) \cong C^n(G, M) = \left\{\text{functions } G^{(n)} \to M\right\}$$

This is the same as function $G^n \to M$. We also have that

$$C^0(G, M) = \{\text{functions } [] \to M\} = M$$

Definition 1.28
The group of *n-cochains* of G with coefficients in M is $C^n(G, M)$. The *n-coboundary map*

$$\mathrm{d}^n : C^{n-1}(G, M) \to C^n(G, M)$$

is dual to d_n in the bar resolution. For $\phi \in C^{n-1}(G, M)$,

$$\begin{aligned}(\mathrm{d}^n \phi)(g_1, \dots, g_n) &= g_1 \phi(g_2, \dots, g_n) \\ &\quad - \phi(g_1 g_2, \dots, g_n) \\ &\quad + \phi(g_1, g_2 g_3, \dots, g_n) \\ &\quad + \dots \\ &\quad + (-1)^{n-1} \phi(g_1, g_2, \dots, g_{n-1} g_n) \\ &\quad + (-1)^n \phi(g_1, g_2, \dots, g_{n-1})\end{aligned}$$

The group of *n-cocycles* is $Z^n(G, M) = \ker(\mathrm{d}^{n+1}) \subseteq C^n(G, M)$, and the *$n$-coboundaries* is $B^n(G, M) = \mathrm{im}(\mathrm{d}^n) \subseteq C^n(G, M)$.

Thus,

$$\mathrm{H}^n(G, M) = \frac{Z^n(G, M)}{B^n(G, M)}$$

Corollary 1.29. For any group G, $\mathrm{H}^0(G, M) = M^G$ is the invariants.

Definition 1.30

A *derivation of G with coefficients in M* is a function $\phi : G \to M$, such that

$$\phi(gh) = g\phi(h) + \phi(g)$$

Note that $Z^1(G, M)$ is the set of derivations. Also note that these are also called 'crossed homomorphisms'.

An *inner derivation* is ϕ of the form

$$\phi(g) = gm - m$$

for a fixed $m \in M$.

Corollary 1.31.

$$\mathrm{H}^1(G, M) = \frac{\{\text{derivations } G \to M\}}{\{\text{inner derivations} G \to M\}}$$

In particular, if M is a trivial $\mathbb{Z}G$-module, then

$$\mathrm{H}^1(G, M) = \mathrm{Hom}_{\mathrm{Group}}(G, M)$$

We will return to considering homology arising from different resolutions.

Definition 1.32

Let (A_n, α_n), (B_n, β_n) be chain complexes of $\mathbb{Z}G$-modules, a *chain map* $f = (f_n)$ are $\mathbb{Z}G$-maps $f_n : A_n \to B_n$, such that

$$\begin{array}{ccc} A_n & \xrightarrow{\alpha_n} & A_{n-1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\ B_n & \xrightarrow[\beta_n]{} & B_{n-1} \end{array}$$

commutes.

Lemma 1.33. Given a chain map (f_n), it induces a map on homology groups

$$f_* : \mathrm{H}_n(A_\bullet) \to \mathrm{H}_n(B_\bullet)$$

Lecture 4

Proof. Let $x \in \ker(\alpha_n)$. Define $f_*([x]) = [f_n(x)]$, where $[\cdot]$ denotes a homology class. Observe that

$$f_n(x) \in \ker(\beta_n)$$

since $\beta_n(f_n(x)) = f_{n-1}(\alpha_n(x)) = 0$. Moreover, if $x' = x + \alpha_{n-1}(y)$, then

$$f_n(x') = f_n(x) + f_n(\alpha_{n-1}(y)) = f_n(x) + \beta_n(f_n(y))$$

and so $[f_n(x')] = [f_n(x)]$. Moreover, this gives a map of abelian groups. □

Theorem 1.34. The definition of $\mathrm{H}^n(G, M)$ is independent of the choice of projective resolution.

Proof. Take projective resolutions (P_n, d_n) and (P'_n, d'_n) of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. We will produce

- chain maps $f_n : P_n \to P'_n$, $g_n : P'_n \to P_n$,
- chain homotopies $s_n : P_n \to P_{n+1}$, $s'_n : P'_n \to P'_{n+1}$,

That is,

$$ds + sd = gf - \text{id} \quad \text{and} \quad ds' + s'd = fg - \text{id}$$

Using this data, the f_n define chain maps

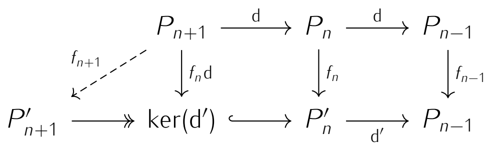
$$f^* : \text{Hom}_G(P'_n, M) \to \text{Hom}_G(P_n, M) \quad \text{and} \quad g^* : \text{Hom}_G(P'_n, M) \to \text{Hom}_G(P_n, M)$$

which induce maps between the respective homology groups. Now observe if $\phi \in \ker(\text{d}^{n+1}) \in \text{Hom}(P_n, M)$, then

$$\begin{aligned} f^*g^*(\phi)(x) &= \phi(g(f(x))) \\ &= \phi(x) + \phi(\text{d}s(x)) + \phi(s\,\text{d}(x)) \\ &= \phi(x) + s^*\text{d}\phi(x) + \text{d}s^*\phi(x) \\ &= \phi(x) + \text{d}s^*\phi(x) \end{aligned}$$

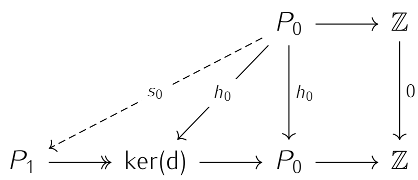
and so $f^*g^*(\phi) = \phi + \text{d}(s^*\phi)$. Hence f^*g^* induces the identity map on homology. Similarly, g^*f^* also induce the identity map. This, f^*, g^* yield isomorphisms on homology. It remains to construct the maps as above.

Consider the end of the resolutions, so we have $f_{-1} : \mathbb{Z} \to \mathbb{Z}$ which is the identity map, and $f_{-2} : 0 \to 0$ is the zero map. Suppose we have already defined f_{n-1}, f_n. We would like to construct f_{n+1}. Thus, $f_n\text{d} : P_{n+1} \to P'_n$, and $\text{d}'(f\text{d}) = f\text{dd} = 0$. Thus, the map $f\text{d}$ has image contained in $\ker(\text{d}')$. So we can define f_{n+1} as follows:



Here, by exactness the map $P'_{n+1} \to \ker(\text{d}')$ is surjective, and so as P'_{n+1} is projective, such a map exists. We can define g_n similarly.

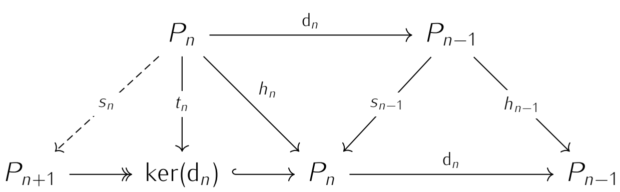
To define s_n, set $h_n = g_nf_n - \text{id}$. This is a map from P_n to itself. In particular, h_n is a chain map, with $h_{-1} = 0$. Set $s_{-1} : \mathbb{Z} \to P_0$ to be the zero map, and note $\text{d}_0h_0 = h_{-1}\text{d}_0 = 0$, and so $\text{im}(h_0) \subseteq \ker(d_0)$. As before, $\text{d}_1 : P_1 \to \ker(\text{d}_0)$ is surjective, and so we have



Inductively, suppose s_{n-1}, s_{n-2} are already defined, set $t_n = h_n - s_{n-1}\text{d}_n : P_n \to P_n$. We have that

$$\begin{aligned} \text{d}_nt_n &= \text{d}_nh_n - \text{d}_ns_{n-1}\text{d}_n \\ &= h_{n-1}\text{d}_n - (h_{n-1} - s_{n-2}\text{d}_{n-1})\text{d}_n \\ &= s_{n-2}\text{d}_{n-1}\text{d}_n \\ &= 0 \end{aligned}$$

So $\text{im}(t_n) \subseteq \ker(\text{d}_n)$. So we have[1]



Define s'_n similarly and we are done. □

[1]THIS DIAGRAM DOES NOT COMMUTE!

Remark 1.35. For any (left) $\mathbb{Z}G$-module N, we can take a projective (resp. free) resolution of N by $\mathbb{Z}G$-modules. Repeating everything we have done, and applying $\mathrm{Hom}_G(\cdot, M)$ gives homology groups, which are called

$$\mathrm{Ext}^n_{\mathbb{Z}G}(N, M)$$

and so

$$\mathrm{H}^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

As above, the Ext-groups are independent of resolution.

2 Low degree cohomology, group extensions

2.1 First cohomology

Recall that $\mathrm{H}^0(G, M) = M^G$ is the group of invariants under G, and that a derivation (or a 1-cocycle) is a map $\phi : G \to M$, such that

$$\phi(g_1 g_2) = g_1 \phi(g_2) + \phi(g_1)$$

We will see two interpretations of (inner) derivations. Recall that an inner derivation is one of the form $\phi(g) = gm - m$, for some fixed $m \in M$.

Let M be a $\mathbb{Z}G$-module, and consider possible $\mathbb{Z}G$-actions on the abelian group $M \oplus \mathbb{Z}$ of the form

$$g(m, n) = (gm + n\phi(g), n)$$

In this case,

$$g_1 g_2(m, n) = g_1(g_2 m + n\phi(g_2), n) = (g_1 g_2 m + n g_1 \phi(g_2) + n\phi(g_1), n)$$

On the other hand,

$$(g_1 g_2)(m, n) = (g_1 g_2 m + n\phi(g_1 g_2), n)$$

These are the same exactly when ϕ is a derivation. In particular, if M is a free $\mathbb{Z}$-module of finite rank, then we get a map $G \to \mathrm{GL}(M)$, by can write this as

$$g \mapsto \begin{pmatrix} \theta_1(g) & \mathrm{d}g \\ 0 & 1 \end{pmatrix}$$

where $\theta_1(g)$ is the action of g on M. In particular, this is a homomorphism of groups if and only if ϕ is a derivation. Moreover, ϕ is an inner derivation if and only if $(-m, 1)$ generates a $\mathbb{Z}G$-submodule which is a trivial module inside $M \oplus \mathbb{Z}$.

Lecture 5

For the second interpretation:

Definition 2.1 (semidirect product)
Let G be a group, M be a $\mathbb{Z}G$-module. We can construct their *semidirect product* $M \rtimes G$ as a group $M \times G$ with operation

$$(m_1, g_1) * (m_2, g_2) = (m_1 + g_1 m_2, g_1 g_2)$$

Here, we have an isomorphism

$$M \cong \{(m, 1) \mid m \in M\}$$

This is an abelian normal subgroup. Similarly, we have an isomorphism

$$G \cong \{(0, g) \mid g \in G\}$$

and conjugation by $\{(0, g)\}$ corresponds to the G-action on M.

Moreover,

$$\frac{M \rtimes G}{\{(m, 1) \mid m \in M\}} \cong G$$

Note that there is a group homomorphism $s : G \to M \rtimes G$, $s(g) = (0, g)$ is such that

$$G \to M \rtimes G \to G$$

is the identity. s is called a *splitting*. Now if we have another splitting $s_1 : G \to M \rtimes G$, so that the composition $G \to M \rtimes G \to G$ is the identity. Define $\psi_{s_1} : G \to M$, so that

$$s_1(g) = (\psi_{s_1}(g), g)$$

Then $\psi_{s_1} \in Z^1(G, M)$, and given two splittings s_1, s_2, $\psi_{s_1} - \psi_{s_2} \in B^1(G, M)$ if and only if there exists an m such that

$$(m, 1)s_1(g)(m_1)^{-1} = s_2(g)$$

Conversely, given $\phi \in Z^1(G, M)$, there exists a splitting s_1 such that $\phi = \psi_{s_1}$.

Theorem 2.2. $\mathrm{H}^1(G, M)$ correspond to the M-conjugacy classes of splittings.

Proof. Examples Sheet 1. □

2.2 Second cohomology

Definition 2.3 (extension)

An *extension* E of G by M for a group G, and a $\mathbb{Z}G$-module M, is an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$$

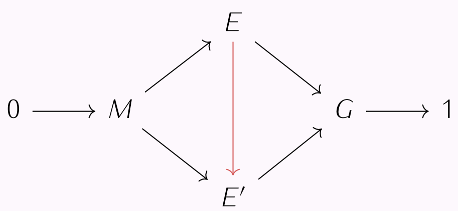
where the maps are group homomorphisms. That is, M embeds in E, so that the image M is an abelian normal subgroup. E acts on M by conjugation, and as M is abelian, we have an induced action on $E/M \cong G$. This agrees with the given G-action on M.

Example 2.4

$E = M \rtimes G$ is an extension of G by M. This is called the *split extension*.

Definition 2.5 (equivalent)

Two extensions are *equivalent* if we have a commuting diagram of group homomorphisms



Remark 2.6. The vertical map is an isomorphism of groups. However, the converse is false. On examples sheet 1, we have E, E' isomorphic groups, but inequivalent extensions.

Definition 2.7 (central extension)

A *central* extension is one where the $\mathbb{Z}G$-module is a trivial module.

Proposition 2.8. Let E be an extension of G by M. If there exists a splitting $s_1 : G \to E$, then E is equivalent to $M \rtimes G$.

Proof. Exercise. □

Theorem 2.9. Let G be a group, and M be a $\mathbb{Z}G$-module. Then there exists a bijection between $\mathrm{H}^2(G, M)$ and the equivalence classes of extensions of G by M.

Given an extenstion

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$$

a *section* is a function $s : G \to E$, such that the composition $\pi \circ s = \mathrm{id}$, where π is the map $E \to G$. Note that s need not be a group homomorphism. Suppose $s(1) = 1$. Let

$$\phi(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}$$

Then

$$\pi(\phi(g_1g_2)) = 1$$

and so $\phi(g_1, g_2) \in \ker(\pi) = M$. That is, we have a map $\phi : G^2 \to M$, which is a 2-cochain. In fact, it is a 2-cocycle.

$$\begin{aligned} s(g_1)s(g_2)s(g_2) &= \phi(g_1, g_2)s(g_1g_2)s(g_3) \\ &= \phi(g_1, g_2)\phi(g_1g_2, g_3)s(g_1g_2g_3) \end{aligned} \tag{$*$}$$

Similarly,

$$\begin{aligned} s(g_1)s(g_2)s(g_3) &= s(g_1)\phi(g_2, g_2)s(g_2g_3) \\ &= s(g_1)\phi(g_2g_3)s(g_1)^{-1}s(g_1)s(g_2g_3) \\ &= s(g_1)\phi(g_2g_2)s(g_1)^{-1}\phi(g_1, g_2g_3)s(g_1g_2g_3) \end{aligned} \tag{$**$}$$

Equating $(*)$ and $(**)$, cancelling the $s(g_1g_2g_3)$, and changing into additive notation, we get

$$\phi(g_1, g_2) + \phi(g_1g_2, g_3) = g_1\phi(g_2, g_3) + \phi(g_1, g_2g_3)$$

Hence

$$\mathrm{d}\phi(g_1, g_2, g_3) = 0$$

and so $\phi \in Z^2(G, M)$. Note ϕ is *normalised*. That is, $\phi(1, g) = \phi(g, 1) = 0$. So what we have shown is that an extension of G by M, along with a choice of section s, yields a normalised 2-cocycle $\phi \in Z^2(G, M)$.

Now take another choice of section s', with $s'(1) = 1$. Then the corresponding normalised cocycles ϕ and ϕ' differ by a coboundary, and so we have defined a map

$$\{\text{extensions}\} \to \mathrm{H}^2(G, M)$$

Lecture 6

To see this, note that $\pi(s(g)s'(g)^{-1}) = 1$, and $\psi(g) = s(g)s'(g)^{-1} \in \ker(\pi) = M$. Thus, we have a map $\psi : G \to M$. Then

$$\begin{aligned} s'(g_1)s'(g_2) &= \psi(g_1)s(g_1)\psi(g_2)s(g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_1)^{-1}s(g_1)s(g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_1)^{-1}\phi(g_1, g_2)s(g_1g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_2)^{-1}\phi(g_1, g_2)\psi(g_1g_2)^{-1}s'(g_1g_2) \end{aligned}$$

Switching to additive notation:

$$\begin{aligned} \phi'(g_1, g_2) &= \psi(g_1) + g_1\psi(g_2) + \phi(g_1, g_2) - \psi(g_1g_2) \\ &= \phi(g_1, g_2) + \mathrm{d}\psi(g_1, g_2) \end{aligned}$$

Thus, ϕ and ϕ' differ by a coboundary. For the rest of the proof, we need:

(a) to show that equivalent extensions give the same cohomology class,

(b) construct an inverse map from cohomology classes to extensions,

(c) show that the two maps are inverse to each other.

We will show (b), and leave (a) and (c). For this, we need

Lemma 2.10. Let $\phi \in Z^2(G, M)$. Then there exists a cochain $\psi \in C^1(G, M)$, such that $\phi + d\psi$ is a normalised cocycle. Hence every cohomology class can be reresented by a normalised cocycle.

Proof. Let $\psi(g) = -\phi(1, g)$. Then

$$(\phi + d\psi)(1, g) = \phi(1, g) - (\phi(1, g) - \phi(1, g) + \phi(1, 1)) = \phi(1, g) - \phi(1, 1)$$

and

$$(\phi + d\psi)(g, 1) = \phi(g, 1) - g\phi(1, 1)$$

But we know that $d\phi(1, 1, g) = d\phi(g, 1, 1) = 0$ as ϕ is a cocycle. A computation shows that both of the above are zero. □

Now take a normalised cocycle ϕ representating our cohomology class. We construct an extension as follows.

$$0 \longrightarrow M \longrightarrow E_\phi \longrightarrow G \longrightarrow 1$$

by considering $E = M \times G$, with the product structure

$$(m_1, g_1)(m_2, g_2) = (m_1 + g_1 m_2 + \phi(g_1, g_2), g_1 g_2)$$

For this to be a group operation, we use that ϕ is normalised. In this case, we do have an extension, where the map π is projection onto the second factor. Notice if we have another normalised 2-cocycle ϕ' representing our cohomology class, then $\phi - \phi' = d\psi$ is a coboundary, and we can define a map

$$\begin{aligned} E_\phi &\to E_{\phi'} \\ (m, g) &\mapsto (m + \psi(g), g) \end{aligned}$$

This gives us an equivalence of extensions.

Example 2.11 (central extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$)
Note we already know of (at least) two such. We could have

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

and we also have the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & r & m \\ & 1 & s \\ & & 1 \end{pmatrix} \,\middle|\, r, s, m \in \mathbb{Z} \right\}$$

and we have an extension given by

$$m \mapsto \begin{pmatrix} 1 & 0 & m \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & r & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mapsto (r, s)$$

Now if we write things multiplicatively, $T \cong \mathbb{Z}^2$ is generated by a, b. We have a free resolution

$$0 \longrightarrow \mathbb{Z}T \xrightarrow{\beta} (\mathbb{Z}T)^2 \xrightarrow{\alpha} \mathbb{Z}T \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where

$$\begin{aligned} \beta(z) &= (z(1 - b), z(a - 1)) \\ \alpha(x, y) &= x(a - 1) + y(b - 1) \end{aligned}$$

and ε is the augmentation map. Now applying $\mathrm{Hom}_T(\cdot, \mathbb{Z})$, we get the chain complex

$$\mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \xrightarrow{\alpha^*} \mathrm{Hom}_T((\mathbb{Z}T)^2, \mathbb{Z}) \xrightarrow{\beta^*} \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \longrightarrow 0$$

In fact, α^*, β^* are the zero maps, and so $\mathrm{H}^2(T, \mathbb{Z}) = \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \cong \mathbb{Z}$, with generator corresponding to the augmentation map $\varepsilon : \mathbb{Z}T \to \mathbb{Z}$.

To see that α^*, β^* are zero, take a $\mathbb{Z}T$-map $f : (\mathbb{Z}T)^2 \to \mathbb{Z}$ and $z \in \mathbb{Z}T$, then

$$\begin{aligned} (\beta^* f)(z) &= f(\beta(z)) \\ &= f(z(1-b), z(a-1)) \\ &= f(z - bz, 0) + f(0, za - z) \\ &= (1-b)f(z, 0) + (a-1)f(0, z) = 0 \end{aligned}$$

The proof that α^* is zero is similar.

Next, we will interpret $\mathrm{H}^2(T, \mathbb{Z})$ in terms of cocycles from the bar resolution. We will construct a chain map

$$\begin{array}{ccccccccc} \mathbb{Z}T\{T^{(2)}\} & \longrightarrow & \mathbb{Z}T\{T^{(1)}\} & \longrightarrow & \mathbb{Z}T\{T^{(0)}\} & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle =} & & \\ \mathbb{Z}T & \xrightarrow[\beta]{} & (\mathbb{Z}T)^2 & \xrightarrow[\alpha]{} & \mathbb{Z}T & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

Note that for degrees -1 and 0, we can just take the identity map. To construct $f_1 : \mathbb{Z}T\{T^{(1)}\} \to (\mathbb{Z}T)^2$, we need that $\alpha f_1 = \mathrm{d}$. So we just need to define the image of the symbols $[a^r b^s]$, for $r, s \in \mathbb{Z}$. Say

$$f_1([a^r y^s]) = (x_{r,s}, y_{r,s})$$

and we need

$$\alpha(x_{r,s}, y_{r,s}) = \mathrm{d}^{a^r b^s} = a^r b^s - 1 = (a^r - 1)b^s + (b^s - 1)$$

Define

$$S(a, r) = \begin{cases} 1 + a + \cdots + a^{r-1} & r > 0 \\ -a^{-1} - \cdots - a^r & r \leq 0 \end{cases}$$

We note that $S(a, r)(a - 1) = a^r - 1$ in both cases. Then

$$\alpha(S(a, r)b^s, S(b, s)) = S(a, r)b^s(a - 1) + S(b, s)(b - 1) = \mathrm{d}^{a^r b^s}$$

as required. So we can define

$$f_1[a^r b^s] = (S(a, r)b^s, S(b, s))$$

Lecture 7

Example 2.12 (continued)

We need to define f_2. For each $[a^r b^s \mid a^t b^u]$, we find

$$z_{rstu} \in \mathbb{Z}T$$

such that

$$f_1 \mathrm{d}_2[a^r b^s \mid a^t b^u] = \beta(z_{rstu})$$

Note that

$$\begin{aligned} f_1 \mathrm{d}_2[a^r b^s \mid a^t b^u] &= f_1(a^r b^s [a^t b^u] - [a^{r+t} b^{s+u}] + [a^r b^s]) \\ &= a^r b^s S(a, t) b^u - S(a, r+t) b^{s+u} + S(a, r) b^s - a^r b^s S(b, u) - S(b, s+u) + S(b, s) \end{aligned}$$

Note that

$$z_{rstu} = S(a, r)b^s S(b, u)$$

works. So we define

$$f_2[a^r b^s \mid a^t b^u] = S(a, r)b^s S(b, u)$$

Now we find a cochain $\phi : T^2 \to \mathbb{Z}$ representing the cohomology class $p \in \mathbb{Z} = \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) = \mathrm{H}^2(T, \mathbb{Z})$. Such a cochain is given by the composition

$$T^2 \xrightarrow{f_2} \mathbb{Z}T \xrightarrow{p\varepsilon} \mathbb{Z}, \qquad \phi = p\varepsilon \circ f_2$$

Since $\varepsilon(S(a, r)) = r$, we find that

$$\phi(a^r b^s, a^t b^u) = p\varepsilon(z_{rstu}) = pru$$

The group structure on $\mathbb{Z} \times T$ correspnding to ϕ is

$$(m, a^r b^s)(n, a^t b^u) = (m + n + pru, a^{r+t} b^{s+u})$$

This corresponds to the group of matrices

$$\left\{ \begin{pmatrix} 1 & pr & m \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \middle| \, r, s, m \in \mathbb{Z} \right\}$$

2.3 Group extensions by presentations

Another approach to consider (central) extensions is to use a partial resolution arising from generators and relations.

Let G be a group, and let X be a generating set. Then we have a canonical map $F = F(X) \twoheadrightarrow G$. Let R be the kernel. Then we have a short exact sequence

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

which is the presentation of G. The subgroup R can be thought of as the set of relations. Now $R \trianglelefteq F$, and so F acts on R by conjugation. Often we take a set of generators of R as a normal subgroup for F.

Let $R_{\mathrm{ab}} = R/[R, R]$ be the abelianisation of R. It inherits an action of F, but R acts trivially on R_{ab}, and so we ahve an induced action by $G = F/R$. Thus, R_{ab} is a $\mathbb{Z}G$-module, called the *relation module*.

Associated to this is an extension

$$1 \longrightarrow R_{\mathrm{ab}} \longrightarrow F/[R, R] \longrightarrow G \longrightarrow 1$$

For a central extension, we need

$$1 \longrightarrow R/[R, F] \longrightarrow F/[R, F] \longrightarrow G \longrightarrow 1$$

Unfortunately, there isn't a largest (or univeral) central extension, since we can always form direct products with an abelian group. The central extension above does have good properties.

Theorem 2.13 (MacLane)**.** Given a presentation of G, let M be a left $\mathbb{Z}G$-module. Then there is an exact sequence

$$\mathrm{H}^1(F, M) \longrightarrow \mathrm{Hom}_G(R_{\mathrm{ab}}, M) \longrightarrow \mathrm{H}^2(G, M) \longrightarrow 0$$

Thus, any equivalence class of extensions of G by M corresponding to a cohomology class, arises from a $\mathbb{Z}G$-map $R_{\mathrm{ab}} \to M$.

Note that M is a $\mathbb{Z}F$-module, via the map $F \to G$.

Corollary 2.14. In the above, if M is a trivial module, then we have

$$\mathrm{Hom}(F, M) \longrightarrow \mathrm{Hom}_G(R/[R, F], M) \longrightarrow \mathrm{H}^2(G, M) \longrightarrow 0$$

Proof. M is a trivial $\mathbb{Z}F$-module, and so $\mathrm{H}^1(F, M) = \mathrm{Hom}(F, M)$ which are the group homomorphisms to an abelian group, which factors through the abelianisation. So $\mathrm{H}^1(F, M) = \mathrm{Hom}(F_{\mathrm{ab}}, M)$.

Similarly, $\mathrm{Hom}_G(R_{\mathrm{ab}}, M) = \mathrm{Hom}_G(R/[R, F], M)$. □

There is also a connection with homology groups. Given a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, instead of applying $\mathrm{Hom}(\cdot, M)$, we can apply the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} \cdot$. We get a chain complex, and we can define the associated homology groups.

The homology groups don't depend on the choice of resolution. We will write the homology groups as $\mathrm{H}_n(G; \mathbb{Z})$.

Definition 2.15 (Schur multiplier)
The *Schur multiplier (or multiplicator)* is $\mathrm{M}(G) = \mathrm{H}_2(G; \mathbb{Z})$.

This is important for studying central extensions.

Theorem 2.16 (universal coefficient)**.** Let G be a group, and M be a trivial $\mathbb{Z}G$-module. Then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(G_{\mathrm{ab}}, M) \longrightarrow \mathrm{H}^2(G, M) \longrightarrow \mathrm{Hom}(\mathrm{M}(G), M) \longrightarrow 0$$

where $\mathrm{Ext}^1(G_{\mathrm{ab}}, M)$ arises by applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot, M)$ to a projective resolution of an abelian group G_{ab}.

Corollary 2.17. Suppose $G = [G, G]$. Then $G_{\mathrm{ab}} = 1$, and we get that

$$\mathrm{H}^2(G, M) \cong \mathrm{Hom}(\mathrm{M}(G), M)$$

Remark 2.18. Sometimes people define the Schur multiplier to be $\mathrm{H}^2(G, \mathbb{C}^*)$ instead of $\mathrm{H}_2(G; \mathbb{Z})$. Schur was considering projective representations, $G \to \mathrm{PGL}(\mathbb{C})$. Such a map lifts to a linear representation of a central extension of G.

There is a formula:

Theorem 2.19 (Hopf)**.** Given a presentation $G = F/R$,

$$\mathrm{M}(G) = \frac{[F, F] \cap R}{[R, F]}$$

Note that this is not necessarily all of $F/[R, F]$.

Remark 2.20. This shows that $([F, F] \cap R)/[R, F]$ is independent of the choice of presentation.

Lecture 8

Define $I_F = \ker(\varepsilon : \mathbb{Z}F \to \mathbb{Z})$, and $\overline{I_R} = \ker(\mathbb{Z}F \to \mathbb{Z}G)$, where the map $\mathbb{Z}F \to \mathbb{Z}G$ is induced by the map $F \to G$. These are ideals in $\mathbb{Z}F$.

Proposition 2.21. We have an exact sequence

$$\overline{I_R}/\overline{I_R}^2 \xrightarrow{\mathrm{d}_2} I_F/(\overline{I_R}I_F) \xrightarrow{\mathrm{d}_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 1$$

where d_1 is induced by the map $\mathbb{Z}F \to \mathbb{Z}G$, and d_2 is induced by the inclusion $\overline{I_R} \hookrightarrow I_F$. Furthermore, $I_F/(\overline{I_R}I_F)$ and $\overline{I_R}/\overline{I_R}^2$ are free left $\mathbb{Z}G$-modules.

Finally, $\mathrm{im}(\mathrm{d}_2) = \overline{I_R}/(\overline{I_R}I_F)$, which is isomorphic to R_{ab} as $\mathbb{Z}G$-modules.

Remark 2.22. Here, R_{ab} is a $\mathbb{Z}G$-module via the action induced on R by conjugation by F.

From Geometric Group Theory, we know that subgroups of free groups are free, and so R is a free group. Thus, R_{ab} is a free abelian group, on the same alphabet as R.

This partial resolution can be completed to a full resolution, called the *Gruenberg resolution*.

In practice, when wanting to deduce information about second (co)homology, it is enough to know about the image of d^2.

Lemma 2.23. Let G be a group, and M is a (left) $\mathbb{Z}G$-module. Then

(i) $I_G = \ker(\varepsilon : \mathbb{Z}G \to \mathbb{Z})$ is a free abelian group under addition, on the basis $\{g - 1 \mid g \in G \setminus 1\}$,

(ii) $I_G/I_G^2 \cong G_{\mathrm{ab}}$,

(iii) $\mathrm{Der}(G, M) \cong \mathrm{Hom}_G(I_G, M)$.

Proof. For (i), the kernel of $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ are of the form

$$\sum n_g g$$

where

$$\sum n_g = 0$$

In this case, we can write

$$\sum n_g g = \sum n_g(g-1) + \sum n_g = \sum n_g(g-1)$$

Clearly anything of this form is in the kernel of ε. Also, the set $\{g - 1 \mid g \in G \setminus 1\}$ forms a basis, since if

$$\sum n_g(g-1) = 0$$

Then by reading off the coefficient of g, we see that $n_g = 0$.

For (ii), define a group homomorphism

$$\begin{aligned} \theta : I_G &\to G_{\mathrm{ab}} \\ g - 1 &\mapsto g[G, G] \end{aligned}$$

But $(g_1 - 1)(g_2 - 1) = (g_1g_2 - 1) - (g_1 - 1) - (g_2 - 1)$, and so $I_G^2 \subseteq \ker(\theta)$. Hence we have a natural map $I_G/I_G^2 \to G_{\mathrm{ab}}$.

Conversely, define $\phi : G \to I_G/I_G^2$, by sending $g \to (g-1) + I_G^2$. This is a group homomorphism. Since I_G/I_G^2 is abelian, we have an induced map $\overline{\phi} : G_{\mathrm{ab}} \to I_G/I_G^2$. Then we can check that these two maps define inverses.

For (iii), the map sends a derivation ϕ to $\theta \in \mathrm{Hom}_G(I_G, M)$, where $\theta(g - 1) = \phi(g)$. We can check that θ is a $\mathbb{Z}G$-map. Conversely, given $\theta \in \mathrm{Hom}_G(I_G, M)$, define ϕ by $\phi(g) = \theta(g - 1)$. □

Lemma 2.24. (i) Let F be a gree group on X. Then I_F is a free $\mathbb{Z}F$-module, on $\widetilde{X} = \{x - 1 \mid x \in X\}$.

(ii) If R is a normal subgroup of F, then R is free on a set Y, then $\overline{I_R}$ is a free (left) $\mathbb{Z}F$-module on $\widetilde{Y} = \{y - 1 \mid y \in Y\}$.

Proof. For (i), let $\alpha : \widetilde{X} \to M$ be a function, to a $\mathbb{Z}F$-module M. By definition of freeness, it suffices to show that α extends to a $\mathbb{Z}F$-map $I_F \to M$.

First let $\alpha' : F \to M \rtimes F$, defined by $\alpha'(x) = (\alpha(x-1), x)$. This is a group homomorphism as F is free. Thus for each $f \in F$, $\alpha'(f) = (a, f)$ for some $a \in M$. Then we have a function $\overline{\alpha} : F \to M$, sending f to a. Hence

$$\alpha'(f) = (\overline{\alpha}(f), f)$$

Note that

$$\begin{aligned}\alpha(f_1 f_2) &= \alpha(f_1)\alpha(f_2) \\ &= (\overline{\alpha}(f_1), f_1)(\overline{\alpha}(f_2), f_2) \\ &= (\overline{\alpha}(f_1) + f_1\overline{\alpha}(f_2), f_1 f_2)\end{aligned}$$

Hence $\overline{\alpha}(f_1 f_2) = \overline{\alpha}(f_1) + f_1\overline{\alpha}(f_2)$. Hence $\overline{\alpha}$ is a derivation $F \to M$. By the previous lemma, we have a corresponding $\mathbb{Z}F$-map $I_F \to M$.

For (ii), suppose

$$\sum r_y(y-1) = 0$$

with $r_y \in \mathbb{Z}F$. Choose a transversal T to the cosets of R in F. We can write

$$r_y = \sum_{t \in T} t s_{t,y}$$

with $s_{t,y} \in \mathbb{Z}R$. So

$$\sum_{t \in T}\sum_{y \in Y} t s_{t,y}(y-1) = 0$$

Since $\mathbb{Z}F$ is free abelian, for each t,

$$\sum_{y \in Y} s_{t,y}(y-1) = 0$$

But I_R is a free $\mathbb{Z}R$-module on $\{y - 1 \mid y \in Y\}$ by (i), and so $s_{t,y} = 0$ for all $y \in Y, t \in T$. □

Proof of proposition 2.21. By (i) of the preceding lemma, I_F is a free $\mathbb{Z}F$-module on $\{x - 1 \mid x \in X\}$. Hence $I_F/(\overline{I_R}I_F)$ is a free left $\mathbb{Z}G$-module, on the basis $\{x - 1 \mid x \in X\}$.

Now $\overline{I_R}$ is a free left $\mathbb{Z}F$-module on $\{y-1 \mid y \in Y\}$. So $\overline{I_R}/\overline{I_R}^2$ is a free left $\mathbb{Z}G$-module on $\{y-1 \mid y \in Y\}$.

The image of d_2 is $\overline{I_R}/(\overline{I_R}I_F)$. Consider $\overline{I_R}$ as a *right* $\mathbb{Z}F$-module. By the analogous version of the lemma with right modules, this is a free right $\mathbb{Z}F$-module on $\{y - 1 \mid y \in Y\}$. So $\overline{I_R}/(\overline{I_R}I_F)$ is a free abelian group on $\{y - 1 \mid y \in Y\}$, and so is isomorphic to R_{ab} as they are free abelian groups on the same basis.

For the left $\mathbb{Z}G$-action, we have that

$$\begin{aligned}g(y-1) &\equiv (gyg^{-1} - 1)g \pmod{\overline{I_R}I_F} \\ &\equiv (gyg^{-1} - 1) \pmod{\overline{I_R}I_F}\end{aligned}$$

So this left $\mathbb{Z}G$-action corresponds to the conjugation action of G on R_{ab}.

Exactness is basically clear from definitions, since $\operatorname{im}(d_1) = I_G = \ker(\varepsilon)$ and $\ker(d_1) = \overline{I_R}/(\overline{I_R}I_F) = \operatorname{im}(d_2)$. □

Lecture 9

Lemma 2.25. Given a projective resolution

$$\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow \mathbb{Z}$$

Let $J_n = \operatorname{im}(d_n) \subseteq P_{n-1}$, and let $\psi : P_n \to J_n$ be the induced map. Then

(i) For a left $\mathbb{Z}G$-module M, we have an exact sequence

$$\operatorname{Hom}_G(P_{n-1}, M) \longrightarrow \operatorname{Hom}_G(J_n, M) \longrightarrow \mathrm{H}^n(G, M) \longrightarrow 0$$

where the first map is by restriction.

(ii) There is an exact sequence

$$0 \longrightarrow \mathrm{H}_n(G, \mathbb{Z}) \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} J_n \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} P_{n-1}$$

Proof. For (i), we have

$$\begin{array}{ccccccc} P_{n+1} & \longrightarrow & P_n & \xrightarrow{\psi} & J_n & \longrightarrow & 0 \\ & & & \searrow^{d_n} & \downarrow \iota & & \\ & & & & P_{n-1} & & \end{array}$$

where the first row is exact. So we have

$$\begin{array}{ccccccc} \mathrm{Hom}_G(P_{n+1}, M) & \xleftarrow{d^{n+1}} & \mathrm{Hom}_G(P_n, M) & \xleftarrow{\psi^*} & \mathrm{Hom}_G(J_n, M) & \longleftarrow & 0 \\ & & & \nwarrow^{d^n} & \uparrow \iota^* & & \\ & & & & \mathrm{Hom}_G(P_{n-1}, M) & & \end{array}$$

where the first row is exact. So $\mathrm{im}(\psi^*) = \ker(d^{n+1}), \ker(\psi^*) = 0$. With this, $\mathrm{im}(\psi^*) \cong \mathrm{Hom}_G(J_n, M)$.

But also $\mathrm{im}(\mathrm{d}^n) = \mathrm{im}(\psi^* \circ \iota^*) \cong \mathrm{im}(\iota^*)$. Thus,

$$\mathrm{H}^n(G, M) = \frac{\ker(\mathrm{d}^{n+1})}{\mathrm{im}(\mathrm{d}^n)} = \frac{\mathrm{im}(\psi^*)}{\mathrm{im}(\iota^*)} = \frac{\mathrm{Hom}_G(J_n, M)}{\mathrm{im}(\iota^*)}$$

(ii) is similar. □

Proof of theorem 2.13. Applying the previous lemma to our partial resolution, we get an exact sequence

$$\mathrm{Hom}_G\left(\frac{I_F}{I_R I_F}\right) \longrightarrow \mathrm{Hom}_G(R_{\mathrm{ab}}, M) \longrightarrow \mathrm{H}^2(G, M) \longrightarrow 0$$

But

$$\begin{aligned} \mathrm{Hom}_G\left(\frac{I_F}{I_R I_F}, M\right) &= \mathrm{Hom}_F\left(\frac{I_F}{I_R I_F}, M\right) \\ &= \mathrm{Hom}_F(I_F, M) \end{aligned}$$

since M is a $\mathbb{Z}G$-module, and so it is trivial as a $\mathbb{Z}R$-module. Thus, any $\mathbb{Z}F$-map $I_F \to M$ factors through $I_F / I_R I_F$. But

$$\mathrm{Hom}_F(I_F, M) = \mathrm{Der}(F, M)$$

by lemma 2.23 (iii). Recall an inner derivation is of the form

$$f \mapsto (f - 1)m$$

In particular, any inner derivation sends $r \mapsto 0$, where $r \in R$. Thus, the restriction map

$$\mathrm{Hom}_G\left(\frac{I_F}{I_R I_F}\right) \to \mathrm{Hom}_G(R_{\mathrm{ab}}, M)$$

sends inner derivations to zero. Hence we have an induced map

$$\mathrm{H}^1(F, M) \to \mathrm{Hom}_G(R_{\mathrm{ab}}, M)$$

and we are done. □

Remark 2.26. We will see this again with the five term exact sequence.

Proof of theorem 2.19. Apply the lemma again, we get

$$0 \longrightarrow \mathrm{H}_2(G, \mathbb{Z}) \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} R_{\mathrm{ab}} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} \frac{I_F}{I_R I_F}$$

But $\mathbb{Z} \otimes_{\mathbb{Z}G} \cdot$ is the same as taking the coinvariants, and so

$$\mathbb{Z} \otimes_{\mathbb{Z}G} R_{\mathrm{ab}} = \frac{R}{[R, F]}$$
$$\mathbb{Z} \otimes_{\mathbb{Z}G} \frac{I_F}{I_R I_F} = \frac{I_F}{I_F^2} = \frac{F}{[F, F]} = F_{\mathrm{ab}}$$

The kernel of the map

$$\frac{R}{[R, F]} \to \frac{F}{[F, F]}$$

is

$$\frac{[F, F] \cap R}{[R, F]}$$

□

Example 2.27

Let $G = V = (\mathbb{Z}/2\mathbb{Z})^2$ be the Klein 4 group. Say x, y are generators, $F = F_2$ free group on x, y, R is generated as a normal subgroup of F by $x^2, y^2, [x, y]$. So

$$G = \langle x, y \mid x^2, y^2, [x, y] \rangle$$

Note another presentation is $G = \langle x, y \mid x^2, y^2, (xy)^2 \rangle$. Let

$$R_D = \langle\langle x^2, y^2, (xy)^4 \rangle\rangle$$

Note that $R_D \subseteq R$, and so we have a map $D_8 = F/R_D \to F/R = V$. Note that the kernel of this map is the centre of D_8. So $[R, F] \subseteq R_D$, and $[R, F] \subsetneq [F, F]$. In Hopf's formula, $[F, F]$ is generated as a normal subgroup by $[x, y]$, and so $[F, F] \subseteq R$. So in the formula, $[F, F] \cap R = [F, F]$.

Observe $1 \equiv [x, y^2] \equiv [x, y]y[x, y]y^{-1} \equiv [x, y]^2 \pmod{[R, F]}$. Hence $[F, F]/[R, F]$ is generated by $[x, y]$ as a $\mathbb{Z}F$ module, and is a trivial module, and is killed by multiplication by 2. So

$$\mathrm{M}(G) = \frac{[F, F]}{[R, F]}$$

is either 0 or $\mathbb{Z}/2$. But $[R, F] \neq [F, F]$, so $\mathrm{M}(G) = \mathbb{Z}/2$.

Remark 2.28. The universal coefficient theorem and the corollary will be left unproved.

3 Brauer groups and Galois cohomology

Definition 3.1 (central simple algebra)

A *central simple algebra* A over a field k is a finite dimensional k-vector space, with associative multiplication. Moreover, $Z(A) = k$ and the only two-sided ideals are 0 and A.

Example 3.2

$\mathrm{Mat}_n(k)$ is a central simple algebra.

Example 3.3

Let $k = \mathbb{R}$. Then the quaternions $\mathbb{H}$ is a central simple algebra. This is in fact a division ring, any non-zero element has a multiplicative inverse.

We would like to classify central simple k-algebras, for a field k.

Theorem 3.4 (Artin-Wedderburn)**.** A finite dimensional simple k-algebra A is isomorphic to $\mathrm{Mat}_n(D)$, where D is a division ring over k.

We will prove this later.

Remark 3.5. Note that $Z(A) = \{\lambda I \mid \lambda \in Z(D)\}$.

We will define an equivalence relation on central simple k-algebras: We say that $A \sim B$ if

$$A \otimes_k \mathrm{Mat}_n(k) \cong B \otimes_k \mathrm{Mat}_m(k)$$

for some m, n. From Artin-Wedderburn, $A \cong M_n(D)$, and so $[A] = [D]$.

Definition 3.6 (Brauer group)

The *Brauer group* $\mathrm{Br}(k)$ is the set of equivalence classes, with

$$[A][B] = [A \otimes_k B]$$

Recall that

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$$

This has identity element

$$[k] = [\mathrm{Mat}_n(k)]$$

and

$$[A]^{-1} = [A^{\mathrm{op}}]$$

Lecture 10

Definition 3.7 (opposite ring)

A^{op} is the k-algebra with the same underlying k-vector space, but with

$$a \cdot_{\mathrm{op}} b = b \cdot a$$

Remark 3.8. A right (resp. left) A-module is the same as a left (resp. right) A^{op}-module.

Lemma 3.9. $A \otimes A^{\mathrm{op}} \cong \mathrm{Mat}_n(k)$, where $n = \dim_k(A)$.

We will prove this later.

Example 3.10

If k is algebraically closed, $\mathrm{Br}(k) = 1$ since any division algebra D which is finite dimensional over k is k itself, since any non-zero element is algebraic over k, and so in k.

Example 3.11

$\mathrm{Br}(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\}$.

Definition 3.12

The subgroup $\mathrm{Br}(L/k)$ of $\mathrm{Br}(k)$ is the group of equivalence classes represented by a central simple k-algebra A, such that $A \otimes_K L \cong \mathrm{Mat}_n(L)$ for some n, and L/k is (finite) Galois. In this case, we say that A is *split by* L.

We will prove

Theorem 3.13. There exists a homomorphism of abelian groups

$$\mathrm{H}^2(\mathrm{Gal}(L/k), L^\times) \to \mathrm{Br}(L/k)$$

which sends $[\phi]$ to $[A(L, G, \phi)]$, where $G = \mathrm{Gal}(L/k)$. $A(L, G, \phi)$ is a crossed product which we will construct.

Remark 3.14. In fact, this homomorphism is an isomorphism.

Remark 3.15. We have a directed union

$$\mathrm{Br}(k) = \bigcup_{L/k \text{finite Galois}} \mathrm{Br}(L/k)$$

and so we can express $\mathrm{Br}(k)$ as a directed union of second cohomology groups.

Example 3.16

When $k = \mathbb{R}$, we know that there are only two Galois extensions, namely $\mathbb{R}$ and $\mathbb{C}$. We know $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic of order 2, generated by complex conjugation. In this case, $\mathrm{H}^2(G, \mathbb{C}^*) = \mathbb{Z}/2\mathbb{Z}$, generated by ϕ say.

In our prototypical example, $\mathbb{H} = A(\mathbb{C}, C_2, \phi)$. In this case, we can write

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i \subseteq \mathbb{H} \oplus \mathbb{C} \oplus \mathbb{C}j$$

$\mathbb{C}$ is a maximal subfield of $\mathbb{H}$, and we have a basis $e_1 = 1, e_\phi = j$ over $\mathbb{C}$.

We have a map $\phi : G \times G \to \mathbb{C}^*$, given by

$$e_\sigma e_\tau = \phi(\sigma, \tau) e_{\sigma\tau}$$

Here, $\phi(\sigma, \tau) \in \mathbb{C}^*$ and ϕ is a normalised 2-cocycle.

Our general construction will start from a Galois extension L/k, and constructing an algebra with a 2-cocycle telling us how to multiply basis elements.

Before that, we need to prove some things:

Proof of theorem 3.4. Consider the sum I of the minimal non-zero right A-submodules of A_A, which is A thought of as a right A-module.

Thus, I is a sum of simple right A-modules.

Let M be a simple right A-submodule of A_A, consider the map

$$\begin{aligned} \theta_a : M &\to aM \\ m &\mapsto am \end{aligned}$$

θ_a is a right A-module map. Since M is simple, θ_a is zero or injective. Thus, aM is either zero or itself a simple submodule of A_A, isomorphic to M.

Now consider

$$\sum_{a \in A} aM$$

This is a two-sided ideal of A, and it is the sum of simple right A-modules isomorphic to M. Thus,

$$A_A = \sum_{a \in A} aM$$

We need

Lemma 3.17 (Schur's lemma). $\mathrm{End}_A(M) \cong D$ is a division algebra, for a simple right A-module M.

Proof. Any module map $\phi : M \to M$ is either zero or an isomorphism, by considering the options of $\ker(\phi)$ and $\mathrm{im}(\phi)$. Thus, any non-zero element of $\mathrm{End}_A(M)$ is invertible. □

Now A_A is a sum of simple right A-submodules, isomorphic to M. An easy induction shows that we may take A_A to be a direct sum. Thus,

$$A_A = \bigoplus_i M_i$$

where $M_i \cong M$. Now

$$\mathrm{End}_A(A_A) \cong \mathrm{Mat}_n(\mathrm{End}_A(M))$$

But as k-algebras, $A \cong \mathrm{End}_A(A_A)$, since an endomorphism is determined by the image of 1. Thus, $A \cong \mathrm{Mat}_n(D)$. □

Corollary 3.18. Every finitely generated right A module V is isomorphic to a direct sum of copies of M (as above), and any two A-modules with the same dimension are isomorphic.

Moreover, $\mathrm{End}_A(V) \cong \mathrm{Mat}_r(D)$ for some r.

Proof. Let M be a simple submodule of A_A, $v_1, \dots, v_s$ be a set of generators of V as a right A-module. Then the map

$$(a_1, \dots, a_s) \mapsto \sum_i a_i v_i$$

shows V as a quotient of a sum of copies of A_A. But A_A is a direct sum of copues of M, and so V is a quotient of a direct sum of copies of M.

Induction shows that $V \cong \oplus_i M_i$, where M_i is isomorphic to M. In particular,

$$\mathrm{End}_A(V) \cong \mathrm{Mat}_r(D)$$

where $D = \mathrm{End}_A(M)$ and the dimension of V as a k-vector space determines r. □

Lecture 11

Definition 3.19

Let V be a finite dimensional k-vector space, with basis $\{e_i\}_{i \in I}$. For $v \in V$, define its *support*

$$J(v) = \{i \in I \mid a_i \neq 0\}$$

where

$$v = \sum_i a_i e_i$$

For W a subspace of V, $w \in W$ non-zero is *primordial* with respect to the basis if

1. $J(w)$ is minimal amongst $\{J(w') \mid w' \in W\}$,
2. $a_i = 1$ for some i, when $w = \sum a_i e_i$.

Lemma 3.20. (i) For $w, w' \in W$, with $J(w)$ minimal, $J(w') \subseteq J(w)$ if and only if $w' = cw$ for some $c \in k$. If so, $J(w) = J(w')$.

(ii) The primordial elements span W.

Proof. (i) is clear. For (ii), induct on $|J(w)|$. Let

$$w = |i \in J(w)| a_i e_i$$

Amongst non-zero elements $w' \in W$ with $J(w') \subseteq J(w)$, choose one with $|J(w')|$ minimal. Then $w_0 = cw'$ for some $c \in k^\times$ is primordial. Now

$$w = a_j w_0 + (w - a_j w_0)$$

with $w - a_j w_0 \in W$, and $\left|J(w - a_j w_0)\right| < |J(w)|$.

By the inductive hypothesis, $w - a_j w_0$ is a linear combination of primordial elements. □

Remark 3.21. All of the same applies for vector spaces over division algebras.

Lemma 3.22. Let A be a k-algebra, D a division algebra over k, with centre k. Then every two-sided ideal I of $A \otimes_k D$ is generated as a left D-module by $I \cap (A \otimes 1)$.

Proof. The left D-module structure on $A \otimes_k D$ is given by

$$\delta(a \otimes \delta') = a \otimes (\delta\delta')$$

The ideal I is a D-submodule of $A \otimes_k D$.

Let $\{e_i\}_{i \in I}$ be a basis for A as a k-vector space, then $\{e_i \otimes 1\}_{i \in I}$ is a basis for $A \otimes_k D$ as a left D-module, Let $r \in I$ be primordial with respect to this basis, say

$$r = \sum_{i \in J(r)} \delta_i (e_i \otimes 1) = \sum_{i \in J(r)} e_i \otimes \delta_i$$

For any non-zero $\delta \in D$, $r\delta \in I$, and

$$r\delta = \sum_i \delta_i \delta (e_i \otimes 1)$$

In particular, $J(r\delta) = J(r)$, and so $r\delta = \delta' r$ for some $\delta' \in D$.

Since r is primordial, we have that $\delta_i = 1$ for some i, $\delta = \delta'$, and so each $\delta_i \in Z(D) = k$, and so $r \in A \otimes 1$. Hence every primordial element of I is in $A \otimes 1$. But by the previous lemma, the primordial elements span. □

Proposition 3.23. The tensor product of two finite dimensional simple k-algebras, at least one of whicb is central, is simple.

Proof. By Artin-Wedderburn, we may assume that one of the algebras is $\mathrm{Mat}_n(D)$, where D is a division ring over k and central. Let A be the other simple algebra. By the lemma, $A \otimes_k D$ is simple. So by Artin-Wedderburn, $A \otimes_k D \cong \mathrm{Mat}_{n'}(D')$ for some division algebra D' over k, and thus,

$$A \otimes_k \mathrm{Mat}_n(D) \cong \mathrm{Mat}_n(A \otimes_k D) \cong \mathrm{Mat}_n(\mathrm{Mat}_{n'}(D')) = \mathrm{Mat}_{nn'}(D')$$

and this is simple. □

Corollary 3.24. The tensor product of two central simple k-algebras is a central simple k-algebra.

Proof. Use the proposition, along with the fact that

$$Z(A \otimes_k B) = Z(A) \otimes_k Z(B)$$

□

Thus, the Brauer group is defined. For the inverses, consider thr ring homomorphism

$$\begin{aligned} A \otimes_k A^{\text{op}} &\to \text{End}(V) \\ a \otimes a' &\mapsto (v \mapsto ava') \end{aligned}$$

Here, V is the vector space underlying A and A^{op}. The map is injective, since $A \otimes A^{\text{op}}$ is simple, and $1 \otimes 1$ is mapped to the identity. By dimension counting, it is a linear isomorphism.

Definition 3.25

Let $G = \text{Gal}(L/k)$, where L/k is a finite Galois extension, and $\phi : G \times G \to L^\times$ is a normalised 2-cocycle. We will define $A = A(L, G, \phi)$.

As a L-vector space, let A have basis $\{e_\sigma\}_{\sigma \in G}$. The multiplication is

$$\left(\sum_{\sigma \in G} \lambda_\sigma e_\sigma\right)\left(\sum_{\tau \in G} \mu_\tau e_\tau\right) = \sum_{\sigma, \tau \in G} \lambda_\sigma \sigma(\mu_\tau) \phi(\sigma, \tau) e_{\sigma, \tau}$$

This has multiplciative identity e_1, since ϕ is normalised. The multiplication is associative, since ϕ is a 2-cocycle.

The centre of $A(L, G, \phi)$ is k. To see this, assume

$$x = \sum \lambda_\sigma e_\sigma \in Z(A)$$

Then for $\beta \in L$, $(\beta e_1)x = x(\beta e_1)$. That is,

$$\sum_\sigma (\beta \lambda_\sigma) e_\sigma = \sum_\sigma \lambda_\sigma \sigma(\beta) e_\sigma$$

Thus, $\beta = \sigma(\beta)$ for any $\beta \in L, \sigma \in G$ with $\lambda_\sigma \neq 0$. Thus, $\lambda_\sigma = 0$ for $\sigma \neq 1$, and hence $x = \lambda_1 e_1$. But this has to commute with all e_τ, and so

$$Z(A) = ke_1 \cong k$$

Next, we note that A is simple. Let I be a two-sided dieal, and $x \in I$ non-zero. Say

$$x = \lambda_{\sigma_1} e_{\sigma_1} + \cdots + \lambda_{\sigma_m} e_{\sigma_m}$$

with m minimal[2]. If $m > 1$, we can find $\beta \in L^\times$ such that $\sigma_m(\beta) \neq \sigma_{m-1}(\beta)$. Then

$$y = x - \sigma_m(\beta) x \beta^{-1} \in I$$

The coefficient of e_{σ_m} in y is zero, and this contradicts minimality. So $x = \lambda_\sigma e_\sigma$ for some $\sigma \in G$. This is a unit, and has inverse $x^{-1} = \sigma^{-1}(\lambda_\sigma^{-1}) e_{\sigma^{-1}}$. Hence $I = A$.

Lecture 12

Suppose $[\phi] = [\phi']$, where ϕ, ϕ' are normalised 2-cocycles. Then ϕ, ϕ' differ by a coboundary. So

$$\phi'(\sigma, \tau) = \phi(\sigma, \tau) \sigma(u_\tau) u_{\sigma\tau}^{-1} u_\sigma$$

where we write things multiplicatively, and $u : G \to L^\times$. Define a L-linear map

$$\begin{aligned} F : A(L, G, \phi') &\to A(L, G, \phi) \\ F(e'_\sigma) &= u_\sigma e_\sigma \end{aligned}$$

[2] We choose x such that m is minimal.

With this,

$$F(e'_\sigma)F(e'_\tau) = F(e'_\sigma e'_\tau)$$

and so F is an algebra homomorphism $A(L, G, \phi') \to A(L, G, \phi)$. But these are simple algebras, and the equality of dimension on both sides means that F is an isomorphism.

Thus, the map $\mathrm{H}^2(G, L^\times) \to \mathrm{Br}(k)$ only depends on the cohomology class. Recall that every cohomology class is represented by a(t least one) normalised cocycle. It remains to show our map

$$\mathrm{H}^2(\mathrm{Gal}(L/k), L^\times) \to \mathrm{Br}(k)$$

is a homomorphism of abelian groups.

Lemma 3.26. For normalised 2-cocycles ϕ, ϕ',

$$A(L, G, \phi + \phi') \sim A(L, G, \phi) \otimes A(L, G, \phi')$$

Proof. Let $A = A(L, G, \phi), B = A(L, G, \phi'), C = A(L, G, \phi + \phi')$. Regard A, B as left L-vector spaces. Define $V = A \otimes_L B$. Note that

$$V = \frac{A \otimes_k B}{\langle (la) \otimes b - a \otimes (lb) \mid a \in A, b \in B, l \in L \rangle}$$

V has a right $A \otimes_k B$-structure, given by

$$(a' \otimes_L b')(a \otimes_k b) = (a'a) \otimes_L (b'b)$$

and V has a left C-module structure,

$$(le''_\sigma)(a \otimes_L b) = (le_\sigma a) \otimes_L (e'_\sigma b)$$

where (e_σ) is a basis of A, (e'_σ) is a basis of B, and (e''_σ) is a basis of C.

The two actions commute, and so the right action of $A \otimes_k B$ on V defines a homomorphism

$$f : (A \otimes_k B)^{\mathrm{op}} \to \mathrm{End}_C(V)$$

Note here we need the opposite ring since we think of End_C as endomorphisms as a left-module. f is injective, since $A \otimes_k B$ is simple.

Claim 3.27. The two algebras have the same dimension, and so f is an isomorphism.

We'll assume this for now. Then note that

$$\mathrm{End}_C(V) \sim C^{\mathrm{op}}$$

since we swap from a left C-module to a right C^{op}-module. Thus,

$$(A \otimes_k B)^{\mathrm{op}} \sim C^{\mathrm{op}}$$

and so $A \otimes_k B \sim C$, and $[A][B] = [A \otimes_k B] = [C]$ in $\mathrm{Br}(k)$.

Proof of claim. We know that C is a simple algebra, and so C^{op} is as well. V is a left C-module, and so it is a right C^{op}-module. Moreover,

$$V \cong \bigoplus_{i=1}^{r} M$$

where M is a simple C^{op}-module. Thus,

$$\mathrm{End}_C(V) \cong \mathrm{End}_{C^{\mathrm{op}}}(V) \cong \mathrm{Mat}_r(D)$$

for some division algebra $D = \mathrm{End}_{C^{\mathrm{op}}}(M)$. But

$$C^{\mathrm{op}} \cong \bigoplus_{i=1}^{m} M$$

by Artin-Wedderburn, which in turn is isomorphic to $\mathrm{Mat}_m(D)$, with $\dim(M) = m\dim(D)$.

Now consider dimensions,

$$\begin{aligned}\dim(V) &= \dim(M)\\ \dim(C) &= \dim(C^{\mathrm{op}}) = m^2\dim(D)\\ \dim(\mathrm{End}_C(V)) &= r^2\dim(D)\end{aligned}$$

and so

$$\dim(\mathrm{End}_C(V))\dim(C) = \dim(V)^2$$

□

□

Remark 3.28. The dimension count can also be done using the double centraliser theorem.

Theorem 3.29 (double centraliser)**.** Let A be a central simple k-algebra, with simple subalgebra B. Then

(i) the centraliser $C_A(B)$ is also simple,

(ii) $\dim(B)\dim(C_A(B)) = \dim(A)$

(iii) $C_A(C_A(B)) = A$.

(iv) if B is central, then $C_A(B)$ is central and $A = B \otimes_k C_A(B)$.

Proof. Exercise. □

In fact, the image of the map $\mathrm{H}^2(\mathrm{Gal}(L/k), L^\times) \to \mathrm{Br}(k)$ is contained in $\mathrm{Br}(L/k)$. See the examples sheet.

Remark 3.30. (i) By a theorem of Wedderburn, if k is a finite field, then $\mathrm{Br}(k)$ is trivial. This is becuase finite division algebras are fields.

(ii) for a non-Archimedean local field k, $\mathrm{Br}(k) \cong \mathbb{Q}/\mathbb{Z}$,

(iii) for a number field K, we have a short exact sequence

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_v \mathrm{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

4 General theory

4.1 Long exact sequence of cohomology

Proposition 4.1 (long exact sequence of cohomology)**.** Let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be a short exact sequence, then we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^n(G, M_1) \longrightarrow \mathrm{H}^n(G, M_2) \longrightarrow \mathrm{H}^n(G, M_3) \longrightarrow \mathrm{H}^{n+1}(G, M_1) \longrightarrow \mathrm{H}^{n+1}(G, M_2) \longrightarrow \mathrm{H}^{n+1}(G, M_3) \longrightarrow \cdots$$

Lemma 4.2 (snake). Let

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

be a short exact sequence of chain complexes, then there exists connecting maps $\delta : \mathrm{H}_{n+1}(C_\bullet) \to \mathrm{H}_n(A_\bullet)$, such that we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_{n+1}(C_\bullet) \longrightarrow \mathrm{H}_n(A_\bullet) \longrightarrow \mathrm{H}_n(B_\bullet) \longrightarrow \mathrm{H}_n(C_\bullet) \longrightarrow \cdots$$

Proof of proposition 4.1. Consider a projective resolution $P_\bullet$ of $\mathbb{Z}$. Since the modules in the resolution are projective, we have a short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{Hom}_G(P_\bullet, M_1) \longrightarrow \mathrm{Hom}_G(P_\bullet, M_2) \longrightarrow \mathrm{Hom}_G(P_\bullet, M_3) \longrightarrow 0$$

Apply the snake lemma. □

4.2 Cup product

Definition 4.3 (cup product)
Given $[u] \in \mathrm{H}^p(G, M), [v] \in \mathrm{H}^q(G, N)$, define their *cup product* $[u] \smile [v] = [u \smile v] \in \mathrm{H}^{p+q}(G, M \otimes_{\mathbb{Z}} N)$ by defining it on cochains:

The *diagonal action* of G on $M \otimes_{\mathbb{Z}} N$ is given by

$$g(m \otimes n) = (gm) \otimes (gn)$$

Let $u \in C^p(G, M), v \in C^q(G, N)$, define $u \smile v \in C^p(G, M \otimes N)$ by

$$(u \smile v)(g_1, \ldots, g_{p+q}) = (-1)^{pq} u(g_1, \ldots, g_p) \otimes \big(g_1 \cdots g_p v(g_{p+1}, \ldots, g_{p+q})\big)$$

This induces a cup product on cohomology classes.

In degree zero,

$$\mathrm{H}^0(G, M) \to \mathrm{H}^0(G, N) \to \mathrm{H}^0(G, M \otimes N)$$

corresponds the map $M^G \otimes N^G \to (M \otimes N)^G$ induced by inclusions.

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Moreover, there exists $1 \in \mathrm{H}^0(G, \mathbb{Z})$ which is the unit for the cup product. For associativity, if we have $u_i \in \mathrm{H}^*(G, M_i)$, then

$$(u_1 \smile u_2) \smile u_3 = u_1 \smile (u_2 \smile u_3) \in \mathrm{H}^*(G, M_1 \otimes M_2 \otimes M_3)$$

In addition, this is graded commutative. For $u \in \mathrm{H}^p(G, M), v \in \mathrm{H}^q(G, N)$, then

$$u \smile v = (-1)^{pq} \alpha_*(v \smile u)$$

where $\alpha : M \otimes N \to N \otimes M$ is the canonical isomorphism. This means that $\mathrm{H}^*(G, \mathbb{Z})$ is graded commutative.

Remark 4.4. Graded commutative does not imply commutative. But if we restrict outselves to cohomology of even degree only, then it is commutative.

Moreover, we have a natural graded $\mathrm{H}^*(G, \mathbb{Z})$-module structure on $\mathrm{H}^*(G, M)$.

Now suppose $\alpha : H \to G$ is a group homomorphism. Then $\alpha^*(u \smile v) = \alpha^* u \smile \alpha^* v$, for $u \in \mathrm{H}^*(G, M)$ and $v \in \mathrm{H}^*(G, N)$. Thus,

$$\alpha^* : \mathrm{H}^*(G, \mathbb{Z}) \to \mathrm{H}^*(H, \mathbb{Z})$$

is a ring homomorphism.

Remark 4.5. Recall that we can take a projective resolution of M, and apply $\mathrm{Hom}_G(\cdot, N)$, for $\mathbb{Z}G$-modules M, N, to get $\mathrm{Ext}^n(M, N)$. Taking $M = N$, we get $\mathrm{Ext}^n(M, M)$, which is a graded $\mathrm{H}^*(G, \mathbb{Z})$-module. So we can study modules M by considering these modules for $\mathrm{H}^*(G, \mathbb{Z})$.

See Benson-Carlson.

5 Lyndon-Hochschild-Serre spectral sequence

Let G be a group, H be a normal subgroup, $Q = G/H$. We would like to calculate the cohomology of G from that for H and Q. In low degree, we get the *five term exact sequence*. We use a general method for calculating (co)homology of double complexes using filtrations.

For the Lyndon-Hochschild-Serre spectral sequence, we have a particular double complex. Let $X^\bullet$ be a $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$. amd $Y^\bullet$ be a $\mathbb{Z}Q$-projective resolution of $\mathbb{Z}$. Then $X^\bullet$ is also a $\mathbb{Z}H$-projective resolution of $\mathbb{Z}$. Let M be a $\mathbb{Z}G$-module. Then G acts on $\mathrm{Hom}_H(X^\bullet, M)$ by

$$(gf)(x) = g(f(g^{-1}x))$$

Since H acts trivially under the above action, we may view $\mathrm{Hom}_H(X^\bullet, M)$ as a $\mathbb{Z}Q$-module. We then form a double complex

$$A = \mathrm{Hom}_Q(Y^\bullet, \mathrm{Hom}_H(X^\bullet, M))$$

with differentials

$$\mathrm{d}' = \mathrm{Hom}_Q(\mathrm{d}_y, \mathrm{id})$$
$$\mathrm{d}'' = \mathrm{Hom}_Q(\mathrm{id}, \mathrm{d}_X^*)$$

In this case, $A^{p,q}$ is zero outside of the first quadrant. In general, we have a double complex $A^{p,q}$, and differentials $\mathrm{d}', \mathrm{d}''$ of degree $(1, 0)$ and $(0, 1)$ respectively. Here, we require

$$(\mathrm{d}')^2 = (\mathrm{d}'')^2 = 0$$

and

$$\mathrm{d}'\mathrm{d}'' + \mathrm{d}''\mathrm{d}' = 0$$

Thus, if we set $\mathrm{d} = \mathrm{d}' + \mathrm{d}''$, then $\mathrm{d}^2 = 0$. In our case, we don't have this unless if we put in alternating signs in either of the differentials. We'll follow the convention from Cartan-Eilenberg, which is that we don't write the signs, but remember that they are implied. We imply a $(-1)^p$ where p is the grading on X. Write

$$A^n = \bigoplus_{p+q=n} A^{p,q}$$

Consider the cohomology of the *total complex* (A^n, d).

The strategy is to filter the total cohomology by using subcomplexes F^pA, whose components to the left of the p-th column are zero. Then

$$(F^pA)^n = \bigoplus_{p' \geq p} A^{p', n-p'}$$

Note $(F^0A)^n = A^n$, and $(F^pA)^n = 0$ for $p > n$. The inclusions $F^pA \hookrightarrow A$ induces a map $\mathrm{H}^n(F^pA) \to \mathrm{H}^n(A)$. We set

$$F^p\mathrm{H}^n(A) = \mathrm{im}(\mathrm{H}^n(F^pA) \to \mathrm{H}^n(A))$$

Thus, we get a filtration

$$\mathrm{H}^n(A) = F^0\mathrm{H}^n(A) \supseteq F^1\mathrm{H}^n(A) \supseteq \cdots \supseteq 0 \tag{$\dagger$}$$

The first spectral sequence allows us to calculate

$$\bigoplus_{m=0}^{n} \frac{F^m\mathrm{H}^n(A)}{F^{m+1}\mathrm{H}^n(A)}$$

There is a graded action of $\mathrm{H}^n(A)$. Thus, we know $\mathrm{H}^n(A)$ as an extension of abelian groups.

Alternatively, we could consider the subcomplex obtained by removing rows from the bottom of our double complex. This will give $\mathrm{H}^n(A)$ up to extensions of abelian groups in a (potentially) different way. Fortunately, with our case of the Lyndon-Hochschild-Serre spectral sequence, this second sequence actually only has one non-trivial factor, and so we get that

$$\mathrm{H}^n(A) = \mathrm{H}^n(G, M)$$

Thus, the first sequence is giving $\mathrm{H}^n(G, M)$, up to extensions of abelian groups. So how do we calculate (†)?

Back to our initial filtration. Let

$$C_r^{p,q} = \{x \in (F^pA)^{p+q} \mid \mathrm{d}x \in (F^{p+r})^{p+q+1}\}$$

Define for $r \geq 2$,

$$E_r^{p,q} = \frac{C_r^{p,q} + (F^{p+1}A)^{p+q}}{\mathrm{d}\left(C_{r-1}^{p-r+1,q+r-2}\right) + (F^{p+1}A)^{p+q}}$$

This is called the E_r*-page*.

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In this case, d induces a map $\mathrm{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$, satisfying $\mathrm{d}_r^2 = 0$. In practice, we have the E_2-page with d_2, and then we compute the cohomology

$$\mathrm{H}(E_2, \mathrm{d}_2) = E_3$$

in general,

$$\mathrm{H}(E_r, \mathrm{d}_r) = E_{r+1}$$

Note that this process must stabilise for a particular coordinate (p, q). For large enough r, the map d_r is going to have image zero when applied to an element with coordinate (p, q). Similarly, for large enough r, the elements whose image under d_r has coordinate (p, q) must be zero. Thus, when we take the cohomology, we are just dealing with zero maps. What we end up is denoted by $E_\infty^{p,q}$. Note we don't necessarily have an r where $E_\infty = E_r$, since the r for which the terms stabilise depend on (p, q).

5.1 The E_2-page

In the first spectral sequence, we consider $\mathrm{H}'(\mathrm{H}''(A))$, where H' and H' denote cohomology with respect to d' and d'' respectively. Since $\mathrm{d}'\mathrm{d}'' + \mathrm{d}''\mathrm{d}' = 0$, the horizontal differential d' induces a differential on $\mathrm{H}''(A)$. We can then calculate $\mathrm{H}'(\mathrm{H}''(A))$.

Note that for the second spectral sequence, we consider $\mathrm{H}''(\mathrm{H}'(A))$ instead. Consider how to compute $\mathrm{H}'(\mathrm{H}''(A))$. Start in the (p, q)-th position. Let $a^{p,q}$ be a vertical cocycle. So $\mathrm{d}''a^{p,q} = 0$. This defines a class in $\mathrm{H}''(A)$, modulo the image under d'' of eleents in $(p, q-1)$-th position. For $a^{p,q}$ to represent a horizontal cocycle in $\mathrm{H}''(A)$ under d', it must be that $d'a^{p,q}$ (which has coordinates $(p+1, q)$), is the image under d'' of an element $a^{p+1,q-1}$, with coordinates $(p+1, q-1)$. Thus,

$$\mathrm{d}\left(a^{p,q} - a^{p+1,q-1}\right) = -\mathrm{d}'a^{p+1,q-1} \in A^{p+2,q-1}$$

Here, we are using $\mathrm{d} = \mathrm{d}' + \mathrm{d}''$. Hence $a^{p,q} - a^{p+1,q-1}$ is a cocycle under d modulo everything two-steps to the right of the (p, q)-th position. Similarly, if $a^{p,q}$ represents coboundary in $\mathrm{H}''(A)$ under d', then there are elements $b^{p-1,q}$ and $b^{p,q-1}$ such that

$$\begin{cases} \mathrm{d}''b^{p-1,q} &= 0 \\ \mathrm{d}'b^{p-1,q} &= \mathrm{d}''b^{p,q-1} + a^{p,q} \end{cases}$$

Thus,

$$\mathrm{d}\left(b^{p-1,q} - b^{p,q-1}\right) \equiv a^{p,q}$$

modulo eveything everything two to the right of $(p-1, q)$. Thus, the cohomology classes in $\mathrm{H}'(\mathrm{H}''(A))$ gives elements of the E_2-page, and vice versa. The upshot is that the E_2-page is $\mathrm{H}'(\mathrm{H}''(A))$, together with d_2 which is induced by d.

Suppose $a \in A^n$ is a cocycle with respect to d, starting in $A^{p,q}$, where $p + q = n$. That is, $a \in (F^pA)^n \setminus (F^{p+1}A)^n$. Then $\mathrm{d}a = 0$, and so we get an element in $E_r^{p,q}$ for all r. Since d_r is induced by d, $\mathrm{d}_r a = 0$. Thus, a determines an element of $E_\infty^{p,q}$, and thus we have a map

$$F^p\mathrm{H}^{p+q}(A) \to E_\infty^{p,q}$$

This map is surjective, and so

$$E_\infty^{p,q} = \bigoplus_{p,n} \frac{F^p\mathrm{H}^n(A)}{F^{p+1}\mathrm{H}^n(A)}$$

This is a graded version of $\mathrm{H}^*(A)$. In the particular case of the Lyndon-Hochschild-Serre sequences, $A = \mathrm{Hom}_Q(Y^\bullet, \mathrm{Hom}_H(X^\bullet, M))$. Here,

$$\mathrm{d}' = \mathrm{Hom}_Q(d_Y, \mathrm{id})$$
$$\mathrm{d}'' = \mathrm{Hom}_Q(\mathrm{id}, \mathrm{d}_X^*)$$

Recall we have suppressed the alternating sign in d''. Now

$$\mathrm{H}''(A) = \mathrm{Hom}_Q(Y, \mathrm{H}^*(\mathrm{Hom}_H(X^\bullet, M)))$$

Since terms of $Y^\bullet$ are $\mathbb{Z}Q$-projective, and so $\mathrm{Hom}_Q(Y, \cdot)$ preserves exactness. Thus,

$$E_2 = \mathrm{H}'\mathrm{H}''(A) = \mathrm{H}^*(\mathrm{Hom}_Q(Y^\bullet, \mathrm{H}^*(H, M))) = \mathrm{H}^*(Q, \mathrm{H}^*(H, M))$$

Recall $\mathrm{H}^*(H, M)$ is a $\mathbb{Z}Q$-module so this makes sense. The d_2 is induced from d.

For the second spectral sequence, we have that the E_2-page is given by $\mathrm{H}''(\mathrm{H}'(A))$, and we have

$$\mathrm{H}'(\mathrm{Hom}_Q(Y^\bullet, \mathrm{Hom}_H(X^\bullet, M))) = \mathrm{H}^*(Q, \mathrm{Hom}_H(X^\bullet, M))$$

Lemma 5.1. $\mathrm{H}^p(Q, \mathrm{Hom}_H(X^\bullet, M)) = 0$ for $p > 0$.

We'll prove this later. Thus, $\mathrm{H}'(A)$ is concentrated on the line $p = 0$. On this line,

$$\mathrm{H}^0(Q, \mathrm{Hom}_H(X^\bullet, M)) = \mathrm{Hom}_H(X^\bullet, M)^Q = \mathrm{Hom}_G(X^\bullet, M)$$

Hence

$$\mathrm{H}''(\mathrm{H}'(A)) = \mathrm{H}^*(\mathrm{Hom}_G(X^\bullet, M)) = \mathrm{H}^*(G, M)$$

Thus, for the second spectral sequence, the E_2-page is concentrated on the line $p = 0$. It follows $E_r = E_\infty$ for $r \geq 2$. So E_∞ is also concentrated on the line $p = 0$. Hence the filtration of $\mathrm{H}^n(A)$ has only one non-trivial factor. Thus, we get

$$\mathrm{H}^n(A) = \mathrm{H}^n(G, M)$$

for each n.

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Proof of lemma 5.1. Since each X^q is $\mathbb{Z}G$-projective, it is a direct summand of a free module. It suffices to prove the lemma for $\mathbb{Z}G$. Let $\widetilde{M}$ be M as an additive group, but with trivial $\mathbb{Z}G$-module structure. We claim that there is an isomorphism of $\mathbb{Z}G$-modules

$$\mathrm{Hom}_H(\mathbb{Z}G, M) \cong \mathrm{Hom}_H(\mathbb{Z}G, \widetilde{M})$$

where G acts on the left hand side by $(gf)(x) = g(f(g^{-1}x))$, and on the right hand side as in the coinduced module on examples sheet 1. That is, $(gf)(x) = f(xg)$.

To see this, for $f \in \mathrm{Hom}_H(\mathbb{Z}G, M)$, define $f' \in \mathrm{Hom}_H(\mathbb{Z}G, \widetilde{M})$ by

$$f'(x) = xf(x^{-1})$$

for $x \in G$. We leave the check that f' is well defined, $(f')' = f$ and this gives a $\mathbb{Z}G$-module isomorphism.

This claim allows us to use Shapiro's lemma, and noting that we have an isomorphism

$$\mathrm{Hom}_H(\mathbb{Z}G, \widetilde{M}) = \mathrm{Hom}(\mathbb{Z}Q, \widetilde{M})$$

since H acts trivially. Thus,

$$\mathrm{H}^p(Q, \mathrm{Hom}_H(\mathbb{Z}G, M)) \cong \mathrm{H}^p(Q, \mathrm{Hom}(\mathbb{Z}Q, \widetilde{M})) \cong \mathrm{H}^p(1, \widetilde{M}) = 0$$

for $p > 0$, where Shapiro's lemma says that

$$\mathrm{H}^p(H, N) = \mathrm{H}^p(G, \mathrm{coind}_G^H(M))$$

□

Example 5.2

Let $G = S_3$, viewed as the extension

$$1 \longrightarrow C_3 \longrightarrow S_3 \longrightarrow C_2 \longrightarrow 1$$

Thus, we need to consider

$$\mathrm{H}^p(C_2, \mathrm{H}^q(C_3, \mathbb{Z}))$$

Here, C_2 acts on C_3 by conjugation, i.e.

$$(1\ 2)(1\ 2\ 3)(1\ 2)^{-1} = (1\ 3\ 2)$$

which is the inversion map. Since this is a group homomorphism, the induced map on $\mathrm{H}^*(C_3, \mathbb{Z})$ is a ring homomorphism. As a ring,

$$\mathrm{H}^*(C_3, \mathbb{Z}) = \frac{\mathbb{Z}[c]}{\langle 3c \rangle}$$

where $\deg(c) = 2$. Note that this is a commutatitve ring, and

$$\mathrm{H}^*(C_3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/3\mathbb{Z} & * > 0 \text{ even} \\ 0 & * \text{ odd} \end{cases}$$

The action of C_2 on $\mathrm{H}^2(C_3, \mathbb{Z})$ is multiplication by -1. We found a generator on examples sheet 1. Then the action of C_2 on H^{4k} is trivial, and on H^{4k+2} is multiplication by -1. Thus, we have

$$\mathrm{H}^0(C_2, \mathrm{H}^{4k+2}(C_3, \mathbb{Z})) = 0$$
$$\mathrm{H}^0(C_2, \mathrm{H}^{4k}(C_3, \mathbb{Z})) = \mathbb{Z}/3\mathbb{Z}$$

Also from the examples sheet, $\mathrm{H}^p(C_2, \mathbb{Z}/3\mathbb{Z}) = 0$ for any $p \geq 1$. So the E_2-page of the first spectral sequence is as follows:

$$\begin{array}{cccccc}
\vdots & & & & & \\
\mathbb{Z}/3\mathbb{Z} & & & & & \\
0 & & & & & \\
0 & & & & & \\
0 & & & & & \\
\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} & \cdots
\end{array}$$

Note that d_2 goes down 1 and right 2. Hence all differentials are zero maps. To work out E_∞, we need to take the homology of the E_2-page to get the E_3-page, which is the same, and $d_3 = 0$ and so on. Thus, we have a picture of E_∞. Thus, we get

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{H}^{4k}(A) \longrightarrow \mathbb{Z}/3\mathbb{Z} \longrightarrow 0$$

which forces $\mathrm{H}^{4k}(A) = \mathbb{Z}/6\mathbb{Z}$, and we deduce that

$$\mathrm{H}^n(S_3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \pmod 4 \\ \mathbb{Z}/6\mathbb{Z} & n \equiv 0 \pmod 4, n > 0 \end{cases}$$

Remark 5.3. On examples sheet 3, we will work this out for $G = Q_8$ in two ways. Once from a well-chosen resolution, and once from spectral sequences. This shows us that the spectral sequence only gives us cohomology up to extension.

Remark 5.4. Calculations are harder when we have to think about $\mathrm{d}_2, \mathrm{d}_3$ and so on.

Remark 5.5. Spectral sequences behave well with products, and we can use them to work out the product structure on $\mathrm{H}^*(G, \mathbb{Z})$.

Now let us think about low degree cohomology.

Proposition 5.6 (five term exact sequence)**.** The first spectral sequence of a double complex A, with $A^{p,q} = 0$ unless $p, q \geq 0$, gives an exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow \mathrm{H}^1(A) \longrightarrow E_2^{0,1} \xrightarrow{\mathrm{d}_2} E_2^{2,0} \longrightarrow \mathrm{H}^2(A)$$

Corollary 5.7 (five term exact sequence of group cohomology)**.** Let $H \trianglelefteq G$, and M be a $\mathbb{Z}G$-module. Then there is an exact sequence

$$0 \longrightarrow \mathrm{H}^1(Q, M) \longrightarrow \mathrm{H}^1(G, M) \longrightarrow \mathrm{H}^1(H, M)^Q \longrightarrow \mathrm{H}^2(Q, M^H) \longrightarrow \mathrm{H}^2(G, M)$$

Proof of proposition 5.6. On the E_2-page, we have

$$\begin{array}{ccccc} E_2^{-1,1} & & & & \\ & \searrow & & & \\ & & E_2^{1,0} & & \\ & & & \searrow & \\ & & & & E_2^{3,-1} \end{array}$$

and so $E_3^{1,0} = E_2^{1,0}$. Repeating, we see that $E_\infty^{1,0} = E_2^{1,0}$. But this is the bottom factor of out filtration of $\mathrm{H}^1(A)$, and so it injects into $\mathrm{H}^1(A)$ with cokernel $E_\infty^{0,1}$. Thus, we get

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow \mathrm{H}^1(A) \longrightarrow E_\infty^{0,1} \longrightarrow 0$$

Now returning to the E_2-page, we have

$$E_2^{-2,2} \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow E_2^{4,-1}$$

and $E_2^{2,0}$ is not-necessarily zero. This, $E_3^{0,1} = \ker(E_2^{0,1} \to E_2^{2,0})$. On the E_3-page, the differential d_3 has

$$E_3^{-3,3} \longrightarrow E_3^{0,1} \longrightarrow E_3^{3,-1}$$

and we have that $E_\infty^{0,1} = E_3^{0,1}$.

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Moreover, $E_\infty^{2,0} \hookrightarrow \mathrm{H}^2(A)$ since it is the bottom factor in the filtration of $\mathrm{H}^2(A)$. Sticking everything together, we get the result. □

If we apply this to the Lyndon-Hochschild-Serre spectral sequence, we get corollary 5.7.

Recall the $\mathbb{Z}Q$-module structure on $\mathrm{H}^n(H, M)$ is as follows:

Let G act on cochains $C^n(H, M)$ by

$$(g \cdot \phi)(h_1, \ldots, h_n) = g\phi(g^{-1}hg, \ldots, g^{-1}h_n g)$$

This descends to an action of G on $\mathrm{H}^n(H, M)$. Under this, H acts trivially, and so we have a Q action. Here, we're only interested in $\mathrm{H}^1(H, M)$, and so if $\phi : H \to M$ is a derivation, representing the cohomology class $[\phi]$, then

$$[\phi] \in \mathrm{H}^1(H, M)^Q \iff [\phi] = [g\phi]$$

for all $g \in G$. If M is a trivial $\mathbb{Z}G$-module, then $\mathrm{H}^1(H, M) = \mathrm{Hom}(H_{\mathrm{ab}}, M)$, and the fixed points are the $\mathbb{Z}Q$-maps.

For the maps in corollary 5.7, we have restriction maps $\mathrm{H}^n(G, M) \to \mathrm{H}^n(H, M)^Q$, where we define the maps on cochains by

$$f \in C^n(G, M) \mapsto \mathrm{res}\, f = f|_{H^n} \in C^n(H, M)$$

which then induces a map on cohomology.

Next, we have inflation maps $\mathrm{H}^n(Q, M^H) \to \mathrm{H}^n(G, M)$, defined on cochain as follows:

$$f \in C^n(Q, M^H) \mapsto \inf f \in C^n(G, M)$$

given by

$$G^n \longrightarrow Q^n \xrightarrow{\ f\ } M^H \longrightarrow M$$

Next, we have the transgression map $T_g : \mathrm{H}^1(H, M)^Q \to \mathrm{H}^2(Q, M^H)$, which corresponds to d_2 on the E_2-page.

For general M, let $s : Q \to G$ be a set-theoretic section, with $s(1) = 1$. Define

$$\rho : G \to H$$
$$\rho(g) = gs(gH)^{-1}$$

Then we take the cohomology class represented by the derivation $f : H \to M$, and define

$$T_g(f) : G^2 \to M$$
$$T_g(f)(g_1, g_2) = f(\rho(g_1)\rho(g_2)) - f(\rho(g_1 g_2))$$

Note that changing g_1, g_2 by multiplication by elements of H does not change tthe value. So we can define a cochain $T_g(f) : Q^2 \to M$. The image lies in M^H and it is a cocycle. Thus, we have a map

$$[f] \in \mathrm{H}^1(H, M)^Q \mapsto [T_g(f)] \in \mathrm{H}^2(Q, M^H)$$

For a special case, assume the action of H on M is trivial. We have the short exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

and we have an extension

$$1 \longrightarrow H_{\mathrm{ab}} \longrightarrow G/[H, H] \longrightarrow G/H \longrightarrow 1$$

Let $\varepsilon \in \mathrm{H}^2(G/[H, H], H_{\mathrm{ab}})$ correspond to this extension. Take $[f] \in \mathrm{H}^1(H, M)^{G/H} = \mathrm{Hom}(H_{\mathrm{ab}}, M)^{G/H}$. That is, f is a $\mathbb{Z}(G/H)$-map $H_{\mathrm{ab}} \to M$. Thus, we have

$$f_* : \mathrm{H}^2(G/H, H_{\mathrm{ab}}) \to \mathrm{H}^2(G/H, M)$$

and then $T_g(f) = f_*(\varepsilon)$.

Corollary 5.8. Given a presentation $G - F/R$, M is a left $\mathbb{Z}G$-module, then we have

$$0 \longrightarrow \mathrm{H}^1(G, M) \longrightarrow \mathrm{H}^1(F, M) \longrightarrow \mathrm{Hom}_G(R_{\mathrm{ab}}, M) \longrightarrow \mathrm{H}^2(G, M) \longrightarrow 0$$

Proof. Use corollary 5.7. □

This fills in the left hand side of MacLane's theorem.

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