Symplectic topology

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Lecture 1

1 Motivation

Symplectic topology is the study of global phenomenon of *symplectic manifolds*, which are a pair (M, ω), where M is a smooth manifold, and ω is a non-degenerate closed 2-form on M. In particular, non-generacy implies the map

$$\begin{aligned} \mathrm{T}M &\to \mathrm{T}^*M \\ v &\mapsto \omega(v, \cdot) \end{aligned}$$

is an isomorphism.

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Example 1.1

$\mathbb{R}^{2n}$ with coordinates $x_1, y_1, \dots, x_n, y_n$ is a symplectic manifold, with

$$\omega_{\text{std}} = \omega_0 = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i$$

Example 1.2

Let Σ be a surface of genus g. Then any volume form makes Σ_g into a symplectic manifold.

Why study such manifolds? Some motivation from classical mechanics: Say we have a phase space X encoding position and momentum of some particles in a physical system. For example, we have a pendulum with angular coordinate θ, and angular momentum $p = m\dot{\theta}$. The phase space is then $S^1 \times \mathbb{R}$.

Given $H : X \to \mathbb{R}$, encoding the total energy of a configuration, we want a vector field $\mathcal{X}_H$ such that the trajectory $x(t)$ satisfies

$$\dot{x}(t) = \mathcal{X}_H(x(t))$$

It's not enough to know that H is constant on trajectories, i.e. $\mathrm{d}H(\mathcal{X}_H) = 0$, since this is a co-dimension 1 condition. Recall if X has symplectic form ω, then we can take $\mathcal{X}_H$ such that

$$\omega(\mathcal{X}_H, \cdot) = \mathrm{d}H \tag{1}$$

Indeed, every phase space has such a symplectic form. Note that ω is non-degenerate if and only if eq. (1) has a solution for all H, and ω being closed is that ω is preserved under the flow of $\mathcal{X}_H$.

Example 1.3 (harmonic oscillator)

In this case, $X = \mathbb{R}^2$, with coordinates p for momentum, q for position. The symplectic form is

$$\mathrm{d}p \wedge \mathrm{d}q = r\mathrm{d}r \wedge \mathrm{d}\theta$$

where (r, θ) is polar coordinates on X. This has Hamiltonian

$$H = \frac{1}{2}(p^2 + q^2) = \frac{1}{2}r^2$$

Hence

$$\mathrm{d}H = r\mathrm{d}r$$

Thus, $\mathcal{X}_H = \frac{\partial}{\partial \theta}$.

1.1 Mathematical perspective

A question to keep in mind is: How does symplectic geometry compare to e.g. Riemannian geometry, complex geometry etc.

In this context, $\mathrm{d}\omega = 0$ is a form of "flatness". Suppose (Y, g) is a Riemannian manifold, i.e. we have $g \in \mathrm{Sym}^2 \mathrm{T}^*Y$ non-degenerate. It is a fact that (Y, g) is locally isometric to $\mathbb{R}^n$ with the Euclidean metric if and only if the curvature F of g vanishes. That is, g is flat.

On the other hand, suppose (Z, J) is an almost complex manifold. That is, $J \in \mathrm{End}(\mathrm{T}Z)$, with $J^2 = -\mathrm{id}_{\mathrm{T}Z}$. One example would be $\mathbb{C}^n$, with $J(z) = iz$. One question is, when is (Z, J) locally isomorphic to $\mathbb{C}^n$? That is, when is (Z, J) a complex manifold. A difficult result in analysis, called the Newlander-Nirenberg theorem, shows that this occurs if and only if the Nijenhuis tensor,

$$\mathcal{N}_J(v, w) = [Jv, Jw] - J([v, Jw] + [Jv, w]) - [v, w]$$

vanishes

Theorem 1.4 (Darboux). A manifold (X, ω) with non-degenerate 2-form ω, is locally modeled on $(\mathbb{R}^{2n}, \omega_{std})$ if and only if $d\omega = 0$.

Finally, symplectic topology is ubiquitous in research in geometry and topology, e.g. global topology, algebraic geometry and so on.

Lecture 2

2 (Symplectic) linear algebra

Let V be a vector space, over a field $\mathbb{F}$ of charactistic 0, with $\dim(V) = n$. $V^* = \mathrm{Hom}(V, \mathbb{F})$ will denote the dual.

Define $\mathbb{F}$-algebras

$$\mathrm{T}(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$$

$$\Lambda^*(V) = \frac{\mathrm{T}(V)}{\langle v \otimes v \rangle}$$

If $\{e_1, \dots, e_n\}$ is a basis for V, then

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1 < \cdots < i_k\}$$

is a basis for $\Lambda^k V$. Using this,

$$\dim(\Lambda^k V) = \binom{n}{k}$$

Proposition 2.1. $\Lambda^k(V^*)$ is isomorphic to the space of alternating multilinear maps

$$V^{\otimes k} \to \mathbb{F}$$

Proof. The map is given by

$$e_{i_1}^\vee \wedge \cdots \wedge e_{i_k}^\vee \mapsto \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) e_{i_{\sigma(1)}}^\vee \otimes \cdots \otimes e_{i_{\sigma(k)}}^\vee$$

The rest is left as an exercise. □

In particular, $\Lambda^n(V^*) = \Lambda^{\mathrm{top}}(V^*)$ is (non-canonically) isomorphic to $\mathrm{span}\{\det\}$, or volume forms.

Definition 2.2

A *symplectic form* on V is $\omega \in \Lambda^2(V^*)$, such that $\omega : V \times V \to \mathbb{F}$ is a perfect pairing[a].

[a] That is, $\omega(v, \cdot)$ induces an isomorphism $V \cong V^*$.

Example 2.3

Let M^{4m+2} be an oriented manifold, and $V = \mathrm{H}^{2m+1}(M; \mathbb{Q})$. The pairing

$$\langle \alpha, \beta \rangle = \int_M \alpha \cup \beta$$

is a symplectic form.

Example 2.4
Let $V = \mathbb{R}^{2n}$, with basis $\{x_i, y_i\}$, with dual basis $\{e_i, f_i\}$. We can define

$$\omega = \sum_{i=1}^{n} e_i \wedge f_i$$

More generally, if $(V, \omega_V), (W, \omega_W)$ are symplectic, then so is

$$(V \oplus W, \omega_V \oplus \omega_W)$$

We can write

$$\omega(v, w) = v \cdot J_0 w$$

where

$$J_0 e_j = f_j \quad \text{and} \quad J_0 f_j = -e_j$$

If we use complex coordinates $z_j = x_j + iy_j$, then J_0 is just multiplication bu i.

Let (V, ω) be a symplectic vector space. Let $W \subseteq V$ be a subspace. Define

$$W^\omega = \{v \in V \mid \omega(w, v) = 0 \text{ for all } w \in W\}$$

Definition 2.5
W is

1. *isotropic* if $W \subseteq W^\omega$,
2. *coisotropic* if $W^\omega \subseteq W$,
3. *symplectic* if $W \cap W^\omega = 0$,
4. *Lagrangian* if $W = W^\omega$. That is, it is isotropic and coisotropic.

Example 2.6
Take $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega = \sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i$$

Then $\mathrm{span}\{x_1, x_2\}$ is isotropic. In fact, $\mathrm{span}\{x_1, \dots, x_n\}$ is a Lagrangian subspace. On the other hand, $\mathrm{span}\{x_1, y_1\}$ is a symplectic subspace.

Remark 2.7. For *any* subspace $W \subseteq V$,

$$\dim(W) + \dim(W^\omega) = \dim(V)$$

and we have that $(W^\omega)^\omega = W$.

Theorem 2.8. Given any symplectic vector space (V, ω) over $\mathbb{R}$, there exists a linear isomorphism $\Phi : V \to \mathbb{R}^{2n}$, such that $\Phi^* \omega_0 = \omega$.

Proof. We induct on $\dim(V)$. Choose any $v_1 \in V$ non-zero. By non-degeneracy, there exists $w_1 \in V$ such that $\omega(v_1, w_1) = 1$. Let $V_1 = \mathrm{span}\{v_1, w_1\}$. Define $\Phi_1 : V_1 \to \mathbb{R}^2$, by $\Phi_1(v_1) = x_1$ and $\Phi_1(w_1) = y_1$. This satisfies that $\Phi_1^* \omega_0 = \omega|_{V_1}$.

Define $V_2 = V_1^\omega$. Then V_1 and V_2 are symplectic, and $\dim(V_2) = \dim(V) - 2$. By induction hypothesis, there exists $\Phi_2 : V_2 \to \mathbb{R}^{2n-2}$, with $\Phi_2^* \omega_0 = \omega|_{V_2}$. We can then define

$$\Phi = \Phi_1 \oplus \Phi_2 : (V_1, \omega|_{V_1}) \oplus (V_2, \omega|_{V_2}) \to (\mathbb{R}^{2n-2}, \omega_0) \oplus (\mathbb{R}^2, \omega_0) = (\mathbb{R}^{2n}, \omega_0)$$

□

Lecture 3

In particular, symplectic vector spaces are always even dimensional.

Corollary 2.9. $\alpha \in \Lambda^2 V^*$ is symplectic if and only if $\alpha \wedge \cdots \wedge \alpha \in \Lambda^{\text{top}} V^* = \mathbb{R}$ is non-zero.

Proof. Note that

$$\omega_{\text{std}}^n = x_1^\vee \wedge y_1^\vee \cdots \wedge x_n^\vee \wedge y_n^\vee$$

is the usual volume form.[1] □

Returning to linear algebra, we have

$$\text{Hom}(V, V) \to \text{Hom}(\Lambda^{\text{top}} V, \Lambda^{\text{top}} V) \cong \mathbb{R}$$

The composition is the *determinant* $\det : \text{Hom}_*(V, V) \to \mathbb{R}$.

2.1 Orientations

Define

$$\text{or}_V = \frac{\{\text{ordered bases for } V\}}{\text{GL}_+(V)} = \frac{\text{Fr}(V)}{\text{GL}_+(V)}$$

where we consider the $\text{GL}_+(V)$-orbits, i.e. two frames are equivalent if they are related by a transformation with positive determinant.

In this case, or_V has two elements. Given any volume form α on V, there exists an induced orientation on V, given by the sign of α on a basis. In particular, any symplectic vector space has such a canonical orientation, induced by ω^n.

2.2 Symplectomorphism

Definition 2.10

Define

$$\text{Sp}(2n) = \{\Phi \in \text{GL}(\mathbb{R}^{2n}) \mid \Phi^* \omega_{\text{std}} = \omega_{\text{std}}\}$$

if J_0 is the matrix so that $\omega(u, v) = u \cdot J_0 v$, then we have that

$$\text{Sp}(2n) = \{\Phi \in \text{GL}(\mathbb{R}^{2n}) \mid \Phi^\mathsf{T} J_0 \Phi = J_0\}$$

Example 2.11

For $\mathbb{R}^2$, ω is the standard volume form $dx \wedge dy$. So preserving ω is the same as preserving the volume form. Hence

$$\text{Sp}(2) = \text{SL}(2, \mathbb{R})$$

In general, we have an embedding

$$\text{Sp}(2n) \subseteq \text{SL}(2n, \mathbb{R})$$

since ω^n is the standard volume form[a] on $\mathbb{R}^{2n}$.

[a] maybe up to a factor of $n!$

[1] or maybe with a coefficient of $n!$ in front of it...

Note if Φ commutes with J_0, and Φ preserves the inner product, then $\Phi \in \mathrm{Sp}(2n)$. So

$$\mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) \subseteq \mathrm{Sp}(2n)$$

In fact, we have

$$\mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n) = \mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n)$$

Proposition 2.12. The set S of symplectic forms on $\mathbb{R}^{2n}$, as a subset of $\Lambda^2(V^*)$ is $\mathrm{GL}(2n, \mathbb{R}) / \mathrm{Sp}(2n)$, a homoegenous space.

Proof. $\mathrm{GL}(2n, \mathbb{R})$ acts transitively on S, with stabiliser $\mathrm{Sp}(2n)$ for any point. □

Let $\mathrm{Lag}(V)$ be the space of Lagrangian subspaces L of V.

Example 2.13

$\mathrm{Lag}(\mathbb{R}^2)$ is just the one-dimensional subspaces of $\mathbb{R}^2$, i.e. $\mathbb{RP}^1 = S^1$. More generally,

$$\mathrm{Lag}(\mathbb{R}^{2n}) = \frac{\mathrm{U}(n)}{\mathrm{O}(n)}$$

Example 2.14

Let $V = C^\infty(S^1, \mathbb{R})$, with

$$\omega(f, g) = \int_{S^1} f \mathrm{d}g$$

Using integration by parts, this is alternating.

3 Vector bundles

If $\pi : \mathcal{E} \to X$ is a vector bundle, we can define $\Lambda^* \mathcal{E} \to X$, $\Lambda^*(\mathcal{E}^*) \to X$, $\mathrm{or}_{\mathcal{E}} \to X$, and so on. We have a vector space $\Gamma(\mathcal{E})$ of global sections of $\mathcal{E}$. In particular,

$$\Omega^k(X) = \Gamma(\Lambda^k(\mathrm{T}^* X))$$

is the space of k-forms.

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Definition 3.1

$\mathcal{E}$ is *orientable* if there exists a global section of $\mathrm{or}_{\mathcal{E}}$.

Example 3.2 (non-example)

The Möbius bundle over S^1 is not orientable. More generally, for any vector bundle, $\mathcal{E} \to S^1$, $\mathcal{E}$ is trivial if and only if it is orientable.

Let M be a (connected smooth) manifold, and consider the tangent bundle $\mathrm{T}M \to M$.

Definition 3.3

M is *orientable* if $\mathrm{T}M$ is. If so, a choice of orientation for M is a choice of orientation for $\mathrm{T}M$.

Note that if M is orientable, then there exists two choices of orientation. If M represents one orientation, we will write $\overline{M}$ for the other. Moreover, if M has a boundary, then ∂M has a canonical orientation, by saying $(e_1, \dots, e_{n-1})$ is positively oriented if $(v, e_1, \dots, e_{n-1})$ is positively oriented, where v is the outward pointing normal.

For compact M, M is orientable if and only if $\mathrm{H}^n(M; \mathbb{Z}) \cong \mathbb{Z}$. A choice of orientation corresponds to a choice of generator in $\{\pm 1\}$. Note that elements of $\mathrm{Diff}(M)$ must either preserve or reverse orientation, which we can read off from $\varphi_* : \mathrm{H}^n(M; \mathbb{Z}) \to \mathrm{H}^n(M; \mathbb{Z})$.

3.1 Cohomology and integration

Theorem 3.4 (de Rham)**.** We have an isomorphism between de Rham cohomology and singular homology. Moreover, this is an isomorphism of rings, between the wedge product and the cup product.

Intuitively, the isomorphism is given by integrating a form over a chain.

Theorem 3.5 (Stokes)**.** Let M be a smooth n-manifold with boundary, and α a compactly supported $(n-1)$-form on M. Then

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha$$

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One corollary of Stokes is that we have a (perfect) pairing

$$\mathrm{H}_k(M; \mathbb{R}) \otimes \mathrm{H}^k(M; \mathbb{R}) \to \mathbb{R}$$

given by integration.

More generally, if N^k is an oriented submanifold of M, $\alpha \in \Omega^k(M)$ has $\mathrm{d}\alpha = 0$, then

$$\int_N \alpha = \langle [N], \alpha \rangle$$

where we take the pairing using real (co)homology of M. The proof of this is beyond the scope of the course.

Corollary 3.6. Let (M^{2n}, ω) be a closed symplectic manifold. Then $\mathrm{H}^{2k}(M; \mathbb{R}) \neq 0$ for all $0 \leq k \leq n$.

Proof. We know that $\int_M \omega^n > 0$, since ω^n is a volume form on M. This shows that $[\omega^n] \neq 0$ in $\mathrm{H}^{2n}(M)$. Now noting that $[\omega]^n = [\omega^n]$ gives the result. □

For example, in $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$, $\omega = \mathrm{PD}[\text{hyperplane}]$. The class of ω generates $\mathrm{H}^*(\mathbb{CP}^n)$. Note on the other hand the result is false when M is open. For example, consider $\mathbb{R}^{2n}$ with the usual symplectic form.

Suppose $\varphi : M \to M$ is a diffeomorphism. Then $\varphi^*\omega$ is another symplectic form. To see this,

$$\mathrm{d}\varphi^*\omega = \varphi^*\mathrm{d}\omega = 0$$

The fact that $\varphi^*\omega$ is non-degenerate is clear, as $\mathrm{d}\varphi$ is an isomorphism on tangent spaces.

We'll say that $\omega, \varphi^*\omega$ are *diffeomorphic*. One question is: are all symplectic forms on M diffeomorphic? The answer is no for some M, but we only know the result for very few M.

Definition 3.7

Define

$$\mathrm{Symp}(M, \omega) = \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*\omega = \omega\}$$

This is the *symplectomorphism group* of (M, ω).

Clearly, we must have that $\mathrm{Symp}(M, \omega) \subseteq \mathrm{Diff}_+(M, \omega)$, as symplectomorphisms preserve orientation.

4 Vector fields and isotopies

Let M^n be any manifold.

Definition 4.1
An *isotopy* of M is a smooth map $\varphi : M \times \mathbb{R} \to M$, such that

1. for all $t \in \mathbb{R}$, $\varphi_t = \varphi(\cdot, t) : M \to M$ is a diffeomorphism,
2. $\varphi_0 = \mathrm{id}_M$.

Equivalently, we can replace $\mathbb{R}$ in the above with $(-\varepsilon, \varepsilon)$.

Example 4.2
Define φ_t to be the rotation through angle t through a fixed axis on S^2.

Any isotopy defines a time-dependent vector field X_t, via

$$X_t(m) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \varphi_s(m)$$

For compact M, we have a correspondence

$$\{\text{time dependent vector fields } X_t\} \leftrightarrow \{\text{isotopies } \varphi_t\}$$

For M non-compact, a flow might blow up in finite time. We are interested in how isotopies will act on tensors.

Let $X \in V(M)$ be a vector field, with associated flow φ_t defined for all $t \in \mathbb{R}$. Then for α, β are covariant and contravariant tensors respectively. Define the *Lie derivative*

$$\mathcal{L}_X \alpha = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_{-t})_* \alpha$$

and

$$\mathcal{L}_X \beta = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_t)^* \beta$$

We have the following properties:

Proposition 4.3.
1. if Y is a vector field, then $\mathcal{L}_X Y = [X, Y]$,
2. if α is a k-form, then we have *Cartan's (magic) formula*

   $$\mathcal{L}_X \alpha = \mathrm{d}(\iota_X \alpha) + \iota_X(\mathrm{d}\alpha)$$

3. if $f \in C^\infty(M)$, then $\mathcal{L}_X f = X \cdot f = \mathrm{d}f(X)$. This gives a map $V(M) \to \mathrm{Der}_{\mathbb{R}}(\mathbb{C}^\infty(M))$, which is a homomorphism of Lie algebras.
4. From Cartan, $\mathcal{L}_X(\mathrm{d}\alpha) = \mathrm{d}\mathcal{L}_X \alpha$,
5. $\mathrm{d}\alpha(X, Y) = X\alpha(y) - Y\alpha(X) - \alpha([X, Y])$ when α is a 1-form,
6. if X_t is a time dependent vector field, with flow φ_t, then

   $$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t^* \alpha = \varphi_t^* (\mathcal{L}_X \alpha)$$

Lecture 6

5 Interlude - Examples of symplectic manifolds

Example 5.1
Any oriented surface with dimension 2, with a volume form is a symplectic manifold.

On S^2, we have a standard volume form given by the embedding $S^2 \hookrightarrow \mathbb{R}^3$. In particular,

$$\omega_u(v, w) = u \cdot (v \times w)$$

On T^2, we can take $\mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2$, where θ_1, θ_2 are the coordinates from S^1.

Example 5.2
Products of symplectic manifolds is a symplectic manifold.

Example 5.3
Let Q be a smooth manifold, then T^*Q is a symplectic manifold, with the canonical symplectic form

$$\omega = \mathrm{d}\lambda$$

On T^*Q, we have local coordinates $(q_1, \dots, q_k, p_1, \dots, p_k)$,

$$\lambda = \sum_i p_i \mathrm{d}q_i$$

6 Lie derivatives continued

Let (M, ω) be a symplectic form. When does an isotopy $\varphi_t : M \to M$ preserve ω? Let X_t be the associated time-dependent vector field, then φ_t preserves ω if and only if $\mathcal{L}_{X_t}\omega = 0$. In turn, by Cartan's formula, this is true if and only if $\mathrm{d}(\iota_{X_t}\omega) = 0$, i.e. $\iota_{X_t}\omega$ is closed.

When if $\iota_{X_t}\omega = \mathrm{d}H_t$, for some $H_t : M \to \mathbb{R}$? We recover Hamiltonian dynamics. In this case, we call φ_t an *Hamiltonian isotopy*.

Note that if M is compact, for any Hamiltonian function, we have an associated Hamiltonian isotopy associated to the vector field X_{H_t} given by

$$\iota_{X_{H_t}}\omega = \mathrm{d}H_t$$

Note that if $\mathrm{H}^1(M; \mathbb{R}) = 0$, then all symplectic isotopy is a Hamiltonian isotopy, as every closed one-form is exact. In particular, we have a map given by the time 1 flow

$$C^\infty(M) \to \mathrm{Ham}(M) \subseteq \mathrm{Symp}_0(M)$$

Locally, the kernel of this map is the space of constant functions. Equivalently, if we quotient out the constant functions, then the map is locally injective.

Lecture 7

Theorem 6.1 (Moser). Let $(\omega_t)_{t\in[0,1]}$ be a smooth family of symplectic forms on M, M compact, satisfying $[\omega_t] = [\omega_0] \in \mathrm{H}^2(M; \mathbb{R})$ for all t. Then there exists $\psi \in \mathrm{Diff}(M)$ such that $\varphi^*\omega_1 = \omega_0$.

Note that it is not true if $[\omega_t]$ is allowed to vary.

Proof (Moser's trick). We seek an isotopy φ_t such that $\varphi_t^*\omega_t = \omega_0$. Now since $[\omega_t]$ is constant,

$$\left[\frac{\mathrm{d}\omega_t}{\mathrm{d}t}\right] = \frac{\mathrm{d}}{\mathrm{d}t}[\omega_t] = 0$$

and so

$$\frac{\mathrm{d}\omega_t}{\mathrm{d}t} = \mathrm{d}\sigma_t$$

for some $\sigma_t \in \Omega^1(M)$. Now if φ_t is an isotopy, with associated vector field X_t, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^*\omega_t) &= \varphi_t^*\left(\frac{\mathrm{d}\omega_t}{\mathrm{d}t} + \mathcal{L}_{X_t}\omega_t\right) \\ &= \varphi_t^*\left(\mathrm{d}\sigma_t + \mathrm{d}\iota_{X_t}\omega_t\right)\end{aligned}$$

We would like this to be zero, since this would imply that $\varphi_t^*\omega_t = \omega_0$. We can find X_t such that

$$\sigma_t + \iota_{X_t}\omega_t = 0$$

since ω_t is non-degenerate. Since M is compact, letting φ_t be the flow of X_t gives the result. □

Theorem 6.2 (Darboux's theorem)**.** For any $p \in (M, \omega)$, there exists a chart $f : U \hookrightarrow M$, sending 0 to p, such that $f^*\omega = \omega_{\text{std}}|_U$.

That is, there is no local symplectic geometry.

Proof. Fix any chart $h : U \hookrightarrow M$, with $h(0) = p$. Moreover, we can require that $h^*\omega \in \Omega^2(U)$ is equal to ω_{std} at p, since this is just a linear algebra condition.

Let $U' \subseteq U$ be a neighbourhood of 0, such that $\omega_t = \omega_{\text{std}} + t(h^*\omega - \omega_{\text{std}})$ is symplectic for $t \in [0, 1]$. This exists since being non-degenerate is an open condition.

By Moser's trick,

$$\frac{\mathrm{d}\omega_t}{\mathrm{d}t} = h^*\omega - \omega_{\text{std}} = \mathrm{d}\sigma_t$$

This exists since we can choose U' to be a disc, and so $\mathrm{H}^1(U') = 0$. Let X_t be such that

$$\iota_{X_t}\omega_t = \mathrm{d}\sigma_t$$

Let φ_t be the flow of X_t. In particular there exists $U'' \subseteq U'$ open, such that for all $t \in [0, 1]$,

$$\varphi(U'') \subseteq U'$$

In particular,

$$\varphi_t^*\omega_t = \omega_{\text{std}}$$

Now $h \circ \varphi : U'' \to M$ satisfies the requirements. □

Lecture 8

7 Almost complex manifolds

Definition 7.1 (almost complex structure)

Let X be a smooth manifold. Then $J \in \mathrm{End}(\mathrm{T}X)$ is an *almost complex structure* if $J^2 = -\mathrm{id}$.

Example 7.2

When $X = \mathbb{C}^n$, we have a canonical identification $\mathrm{T}_zX \cong \mathbb{C}^n$. On this, J is multiplication by i. We'll denote this by $J_{\text{std}} \in \mathrm{End}(\mathrm{T}\mathbb{C}^n)$.

Definition 7.3 (integrable)

An almost complex structure J is *integrable* if it is the pullback of J_{std} from an atlas of charts with holomorphic transition functions. That is, if X is a complex manifold, and J is multiplication by i.

Definition 7.4 (compatible)
Let (M, ω) be a symplectic manifold. We say that an almost complex structure J is *compatible* with ω if

- $\omega(v, w) = \omega(Jv, Jw)$ for all v, w,
- $\omega(v, Jv) > 0$ for all v non-zero.

Equivalently, $\omega(\cdot, J\cdot)$ is a Riemannian metric.

Note that in this convention, we have

$$\omega_{\mathrm{std}} = g(J\cdot, \cdot)$$

In this case, (g, J, ω) is a compatible triple.

Proposition 7.5. Any symplectic manifold (M, ω) admits a compatible almost complex structure J. Moreover, the set $\mathcal{J}(M, \omega)$ of all such J is contractible. So J is unique up to contractible choice.

Proof. Pick any Riemannian metric g on M. Then we have two isomorphisms

$$\widetilde{g}, \widetilde{\omega} : \mathrm{T}M \to \mathrm{T}^*M$$

Let

$$A = \widetilde{g}^{-1}\widetilde{\omega} : \mathrm{T}M \to \mathrm{T}M$$

If we have a compatible, triple, then A is an almost complex structure. Now

$$\omega(u, v) = -\omega(v, u) = -g(Av, u) = -g(u, Av)$$

and so we obtain that $A^* = -A$. Here, * denote the transpose. Hence $AA^* = -A^2$. Now

$$g(AA^*v, v) = g(A^*v, A^*v) > 0$$

So AA^* is symmetric and positive definite. In particular, it is diagonalisable, with positive eigenvalues.

Let $J = \left(\sqrt{AA^*}\right)^{-1} A$. Then

$$J^2 = -A^{-2}A^2 = -\mathrm{id}$$

where we use the fact that everything commutes. Moreover,

$$\omega(Ju, Jv) = g(AJu, Jv) = g(JAu, Jv) = g(Au, v) = \omega(u, v)$$

and

$$\omega(u, Ju) = g(-JAu, v) = g(AA^*u, u) = g(A^*u, A^*u) > 0$$

So $\mathcal{J}(M, \omega)$ is non-empty. Choose $J_0 \in \mathcal{J}(M, \omega)$. Define

$$F : \mathcal{J}(M, \omega) \times [0, 1] \to \mathcal{J}(M, \omega)$$
$$F(J, t) = J_t$$

where J_t is the almost complex structure associated to the Riemannian metric

$$g_t = tg_{J_0} + (1 - t)g_J$$

Here, we are using the convexity of the space of Riemannian metrics. Then F defines a retraction of $\mathcal{J}(M, \omega)$ onto J_0. ☐

Definition 7.6 (Lagrangian submanifold)
A submanifold $L^n \subseteq (M^{2n}, \omega)$ is a *Lagrangian submanifold* if $\omega|_L = 0$.

For example, $\mathbb{R}^n \subseteq \mathbb{C}^n$, $\mathrm{T}^n \subseteq \mathbb{R}^{2n}$, $\mathbb{RP}^n \subseteq \mathbb{CP}^n$. If X is a projective variety defined over $\mathbb{R}$, then $X(\mathbb{R}) \subseteq X(\mathbb{C})$ is Lagrangian.

Example 7.7

If $\phi : M \to M$ is a diffeomorphism, then it is a symplectomorphism if and only if the graph

$$\Gamma_\phi \subseteq (M \times M, \omega \oplus -\omega)$$

is Lagrangian.

Theorem 7.8 (Weinstein neighbourhood)**.** If $L \subseteq (M^{2n}, \omega_M)$ is a compact Lagrangian, then there exists a neighbourhood $U(L)$ of the zero section in T^*L, and a symplectic embedding $U(L) \hookrightarrow M$, sending the zero section to L.

Lecture 9

Proof. Fix an almost complex structure J compatible with ω_M. This gives us an orthogonal decomposition

$$\mathrm{T}M|_L = \mathrm{T}L \oplus J \cdot \mathrm{T}L$$

This is orthogonal with respect to the metric g_J. Let $\Phi : \mathrm{T}^*M \to \mathrm{T}M$ be the lowering operator given by the metric. Now define

$$\begin{aligned} \varphi &: U(L) \to M \\ \varphi(q, f) &\mapsto \exp_q(J \cdot \Phi(f)) \end{aligned}$$

Here, exp is the exponential map with respect to the metric g_J. We can assume that φ is a diffeomorphism onto its image, by compactness of L and by shrinking $U(L)$.

Consider $L \subseteq \mathrm{T}^*L$ embedded as the zero section. Note we have a canonical identification

$$\mathrm{T}_{q,0}(\mathrm{T}^L) = \mathrm{T}_qL \oplus \mathrm{T}^*_qL$$

With this,

$$\mathrm{D}\varphi(v, f) = v + J \cdot \Phi(f) \in \mathrm{T}L \oplus J \cdot \mathrm{T}L$$

This is by the definition of the exponential map.

We claim that $\varphi^*\omega_M = \omega_L$ on $L \subseteq \mathrm{T}^*L$. Indeed,

$$\begin{aligned} \varphi^*\omega_M((v, f), (v', f')) &= \omega_M((v + J\Phi(f), v' + J\Phi(f'))) \\ &= \omega_M(v, J\Phi(f')) - \omega_M(v', J\Phi(f)) \\ &= f'(v) - f(v') \\ &= \mathrm{d}\lambda_{\mathrm{can}}((v, f), (v', f')) \end{aligned}$$

Now apply Moser's trick. We can write

$$\varphi^*\omega_M - \omega_{\mathrm{can}} = \mathrm{d}\sigma$$

for some $\sigma \in \Omega^1(\mathrm{T}^*L)$, with $\sigma|_L = 0$. The time-1 flow then gives the associated symplectic embedding. Here, we flow along the vector field X_t so that

$$\iota_{X_t}\omega_{\mathrm{can}} = \sigma$$

□

Remark 7.9. We've used that $J : \mathrm{T}L \to \nu_{L/M}$ is an isomorphism. This tell us that (assuming L is oriented), the intersection number $[L] \cdot [L] = (-1)^{n+1}\chi(L)$.

In this case, if $\chi(L) \neq 0$, then $[L] \neq 0$.

7.1 First Chern class

Definition 7.10

A *complex vector bundle* $\mathcal{E} \to X$ is a real vector bundle, with am assocated endomorphism $J \in \operatorname{End}(\mathcal{E})$, with $J^2 = -\mathrm{id}$.

Definition 7.11

A *symplectic vector bundle* $\mathcal{E} \to X$ is a vector bundle, with $\omega \in \Gamma(\Lambda^2\mathcal{E}^*)$, such that ω_x is a symplectic form on $\mathcal{E}_x$.

On each sympelctic vector bundle, we have a canonical choice of a complex structure J, up to homotopy. Moreover, since we have a homotopy equivalence $\mathrm{Sp}(2n) \simeq \mathrm{U}(n)$, classification of symplectic vector bundles and complex vector bundles are the same.

Remark 7.12. Note that classification of complex vector bundles is *not* the same as classification of holomorphic vector bundles.

Example 7.13

The tautological line bundle $\mathcal{L}_{\mathrm{taut}} = \mathcal{O}(-1)$ over $\mathbb{CP}^n$ is a complex vector bundle.

The *canonical bundle* $\mathcal{O}(1) = \mathcal{L}^*_{\mathrm{taut}}$ is also a complex vector bundle.

Theorem 7.14. There exists a correspondence

$$\{\text{complex line bundles on } X\} \leftrightarrow \mathrm{H}^2(X;\mathbb{Z})$$

by sending $\mathcal{L}$ to $c_1(\mathcal{L})$.

Lecture 10

We have functors

$$X \mapsto \{\text{vector bundles on } X\}$$

and

$$X \mapsto \mathrm{H}^2(X,\mathbb{Z})$$

In this case, the first functor is given by the functor

$$X \mapsto \frac{\{f : X \to \mathbb{CP}^\infty\}}{\text{homotopy}}$$

where we pullback the tautological line bundle using a continuous map. Define $c_1(\mathcal{L}) = f^*x$, where $\mathrm{H}^*(\mathbb{CP}^\infty) = \mathbb{Z}[x]$. Note also $K(\mathbb{Z},2) = \mathbb{CP}^\infty$, and so $\mathbb{CP}^\infty$ represents $\mathrm{H}^2(\cdot,\mathbb{Z})$. Moreover, note that $\mathbb{CP}^\infty = B\mathrm{U}(1)$, so it represents principal $\mathrm{U}(1)$ bundles.

We can also define c_1 in terms of Chern-Weil theory, for example see Griffiths-Harris. In fact, for this it is just the curvature of a unitary connection.

For a more general complex vector bundle $\mathcal{E}$, we can define $c_1(\mathcal{E}) = c_1(\det(E))$, where $\det(E) = \Lambda^{\mathrm{top}}\mathcal{E}$ is the *determinant line bundle*.

In practice, for a line bundle over X satisfying Poincaré duality, $c_1(\mathcal{L})$ is the Poincaré dual of the zero locus of a generic section of $\mathcal{L}$. Note that the zero locus lives in $\mathrm{H}_{n-2}(\mathcal{L})$, since $\mathcal{L}$ is a complex line bundle.

For an almost complex manifold X, we will define $c_1(X) = c_1(\mathrm{T}X)$.

Example 7.15

For $X = S^2$, consider rotation about a fixed axis. This has two zeroes. So $c_1(S^2) = \mathrm{PD}(2 \text{ points}) = 2 \in \mathrm{H}^2(S^2;\mathbb{Z})$.

More generally, if Σ_g is the surface of genus g, $c_1(\Sigma_g)$ is the number of zeroes of a generic vector field, which is $\chi(\Sigma_g) = 2 - 2g$.

- $c_1(\mathcal{E})$ is invariant under homotopy of the almost complex structure J, so $c_1(M)$ is well defined for a symplectic manifold M,
- $c_1(f^*\mathcal{E}) = f^* c_1(\mathcal{E})$
- $c_1(\mathcal{E} \otimes \mathcal{F}) = \operatorname{rank}(\mathcal{E}) c_1(F) + \operatorname{rank}(\mathcal{F}) c_1(\mathcal{E})$,
- $c_1(\mathcal{L}^*) = -c_1(\mathcal{L})$,
- $c_1(\mathcal{L}_{\text{taut}} \to \mathbb{CP}^n) = -\operatorname{PD}(H) \in \mathrm{H}^2(\mathbb{CP}^n; \mathbb{Z})$, where H is a hyperplane (class).
- $c_1(\mathbb{CP}^n) = (n+1)\operatorname{PD}(H)$. Note here, $\mathrm{T}\mathbb{CP}^n = \mathcal{O}(n+1)$.
- if
$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$
is a short exact sequence of complex vector bundles, then $c_1(\mathcal{B}) = c_1(\mathcal{A}) + c_1(\mathcal{C})$.

7.2 Four manifolds

Let (X^4, J) be an almost complex 4 manifold. Let $C^2 \subseteq X$ be an almost complex curve. So J restricts to an endomorphism $\mathrm{T}C \to \mathrm{T}C$.

Proposition 7.16.
$$-\chi(C) = 2g(C) - 2 = -c_1(X) \cdot [C] + [C]^2$$

Here, in dimension 4 we have a pairing $\mathrm{H}_2(X) \otimes \mathrm{H}_2(X) \to \mathbb{Z}$.

Proof. We have a short exact sequence of vector bundles
$$0 \longrightarrow \mathrm{T}C \longrightarrow \mathrm{T}X|_C \longrightarrow \nu_{C/X} \longrightarrow 0$$
Hence
$$c_1(\mathrm{T}X|_C) = c_1(\mathrm{T}C) + c_1(\nu_{C/X})$$
Taking intersection with $[C]$, the right hand side is
$$\chi(C) + [C] \cdot [C]$$
and the left hand side is $c_1(X) \cdot [C]$. □

Lecture 11

Let X^4 be a 4-manifold, with an almost complex structure.

Theorem 7.17 (Hirzebruch signature)**.** Let X^4 be an almost complex manifold. Then
$$c_1(X)^2 = 2\chi(X) + 3\sigma(X)$$
Here, $\mathrm{H}^2(X; \mathbb{R})$ has a non-degenerate symmetric pairing from the intersection form/cup product. Thus, it has a signature which we denote by $\sigma(X) = b_+ - b_-$.

We note that $b_+ + b_- = b_2(X)$, and that $c_1(X)^2 \equiv \sigma(x) \pmod 8$.

Proposition 7.18. If X^4 admits an almost complex structure, then $X \# X$ does not.

In particular, $\mathbb{P}^2 \# \mathbb{P}^2$ is not a complex manifold.

Proof. Note that $b_1(X \# X) = 2b_1(X)$ and $b_\pm(X \# X) = 2b_\pm(X)$. If X admits an almost complex structure, then $1 - b_1 + b_+$ must be even. □

In fact, a connect sum of n copies of $\mathbb{P}^2$ has an almost complex structure if and only if n is odd.

8 Kähler manifolds

Suppose X is a complex manifold. Let J be the almost complex structure. Define

$$\mathrm{T}_{\mathbb{C}}X = \mathrm{T}X \otimes \mathbb{C}$$

Then J extends to an endomorphism of $\mathrm{T}_{\mathbb{C}}X$. But now $J^2 + \mathrm{id} = 0$, and so J is diagonalisable, and $\mathrm{T}_{\mathbb{C}}X$ splits into

$$\mathrm{T}^{1,0}X \oplus \mathrm{T}^{0,1}X$$

which are the i and $-i$ eigenspaces of J respectively. Write

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + i\frac{\partial}{\partial y_i}\right)$$

Then $\mathrm{T}^{1,0}X$ is the span of the $\frac{\partial}{\partial z_i}$ and $\mathrm{T}^{0,1}X$ is the span of the $\frac{\partial}{\partial \bar{z}_i}$.

In terms of the cotangent bundle, we have a splitting

$$\mathrm{T}^*_{\mathbb{C}}X = (\mathrm{T}^{1,0})^*X \oplus (\mathrm{T}^{0,1})^*X$$

which again are the i and $-i$ eigenspaces of J. Then $(\mathrm{T}^{1,0})^*X$ is the span of the $\mathrm{d}z_i$ and $(\mathrm{T}^{0,1})^*X$ is the span of the $\mathrm{d}\bar{z}_i$.

We define

$$\Lambda^{\ell,k}X = \Lambda^{\ell}(\mathrm{T}^{1,0}X)^* \otimes \Lambda^k(\mathrm{T}^{0,1}X)^*$$

and let $\Omega^{\ell,k} = \Gamma(\Omega^{\ell,k})$ be the space of sections. Locally, elements are of the form[2]

$$f_{I,J}\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

where $|I| = \ell$ and $|J| = k$.

Note if $f : X \to \mathbb{C}$ is a smooth function, then we have

$$\mathrm{d}f = \frac{\partial f}{\partial z_i}\mathrm{d}z_i + \frac{\partial f}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j$$

We write

$$\partial f = \frac{\partial f}{\partial z_i}\mathrm{d}z_i \quad \text{and} \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j$$

for the holomorphic and antiholomorphic parts. In particular, $\partial f \in \Omega^{1,0}(X)$, $\bar{\partial} f \in \Omega^{0,1}(X)$, and f is holomorphic if and only if $\bar{\partial} f = 0$.

Lecture 12

With this all in mind, we have

$$\Omega^k(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X)$$

where

$$\Omega^{p,q}(X) = \Gamma(\Lambda^p(\mathrm{T}^{1,0}X)^* \otimes \Lambda^q(\mathrm{T}^{0,1}X)^*)$$

Since J is integrable, the exterior derivative $\mathrm{d} : \Omega^k(X) \to \Omega^{k+1}(X)$ splits as

$$\mathrm{d} : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X) \oplus \Omega^{p,q+1}(X)$$

We write $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ and $\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$.

In particular, as $\mathrm{d}^2 = 0$,

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$$

and we can define the Dolbeault cohomology.

Definition 8.1 (Kähler)

X is *Kähler* if there exists a closed form

$$\omega = \frac{i}{2}\sum_{i,j} h_{ij}\mathrm{d}z_i \wedge \mathrm{d}\overline{z_j} \in \Omega^{1,1}(X)$$

[2] Summation convention applies.

such that $H = (h_{ij})$ is a positive definition Hermitian matrix.

In particular, $\omega \in \Omega^2_{\mathbb{R}}(X) \cap \Omega^{1,1}(X)$, and

$$\omega^n = \left(\frac{i}{2}\right)^n \det(H) \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z_1} \wedge \cdots \wedge \mathrm{d}\overline{z_n}$$

Now $\det(H)$ is non-vanishing, and so ω is non-degenerate. Thus, ω defines a symplectic form.

Remark 8.2. Complex submanifolds of Kähler manifolds are Kähler when we restrict ω.

Note that an alternative definition is:

Definition 8.3 (Kähler)
(X, J, ω) is Kähler if J is integrable, and ω is compatible with J, and so

$$g(u, v) = \omega(u, Jv)$$

defines a Riemannian metric.

Definition 8.4 (plurisubharmonic)
$\rho : X \to \mathbb{R}$ is *plurisubharmonic* if the matrix

$$\left(\frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}\right)$$

is positive definite.

If ρ is plurisubharmonic, then

$$\omega = \frac{i}{2}\partial\overline{\partial}\rho$$

is a Kähler form, and we call ρ a *Kähler potential*.

Example 8.5
On $\mathbb{C}$, $\rho = |z|^2 = \sum_i z_i \overline{z}_i$, and the associated Kähler form is the standard symplectic form.

Proposition 8.6. $\mathbb{CP}^n$ is Kähler.

Proof. We will use homoegeneous coordinates $[z_1 : \cdots : z_{n+1}]$ on $\mathbb{CP}^n$. Recall that $\mathbb{CP}^n$ is covered by standard affines

$$U_j = \{z_j \neq 0\}$$

and on this, we have coordinates

$$\frac{z_1}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_{n+1}}{z_j}$$

Define ρ_j on $U_j \cong \mathbb{C}^n$ by

$$\rho_j(w) = \log\left(1 + |w|^2\right)$$

We claim that ρ_j is plurisubharmonic on U_j, and that $\partial\overline{\partial}\rho_i = \partial\overline{\partial}\rho_j$ on $U_i \cap U_j$. Then we will have a globally defined form ω. We leave these computations.

For plurisubharmonic, note that we can compute at $0 \in \mathbb{C}^n$ and use $\mathrm{U}(n)$ equivariance. □

We call the above form ω the *Fubini-Study form*.

Corollary 8.7. All quasi-projective varieties are Kähler.

By GAGA, we have a correspondence between closed complex submanifolds of $\mathbb{P}^n$ and projective varieties over $\mathbb{C}$. On the other hand, there exists closed Kähler manifolds which are not projective, for example a generic K3 surface.

Theorem 8.8 (Hodge)**.** If X is compact Kähler, then there exists isomorphisms

$$\mathrm{H}^k(X) \cong \bigoplus_{p+q=n} \mathrm{H}^{p,q}_{\overline{\partial}}(X)$$

Moreover, we have a natural isomorphism

$$\mathrm{H}^{p,q}_{\overline{\partial}}(X) \cong \overline{\mathrm{H}^{q,p}_{\overline{\partial}}(X)}$$

In particular, the odd Betti numbers are even.

Theorem 8.9 (Lefschetz)**.** The map

$$\cdot \wedge [\omega]^k : \mathrm{H}^{n-k}(X) \to \mathrm{H}^{n+k}(X)$$

is an isomorphism. Moreover, this respects the bidegree decomposition.

Lecture 13

Let $F_d = \mathbb{V}(z_0^d + \cdots + z_n^d)$ be the Fermat hypersurface. Then

$$[F_d] = d[H] \in \mathrm{H}_{2n-2}(\mathbb{CP}^n, \mathbb{Z})$$

Moreover, all degree d hypersurfaces in $\mathbb{P}^n$ are symplectomorphic. To see this, note that they are all diffeomorphic as we have a path of smooth hypersurfaces connecting any two degree d hypersurfaces. We'll see the fact that they are symplectomorphic later.

We can get more complex submanifolds, by taking transverse intersection of smooth hypersurfaces.

9 Symplectic blowups

Let $Z = \{(z, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid z \in \ell\}$. We call this $\mathrm{Bl}_0 \mathbb{C}^n$, the blowup of $\mathbb{C}^n$ at the origin. We have two projection maps:

$$\pi : Z \to \mathbb{C}^n$$
$$p : Z \to \mathbb{P}^{n-1}$$

Note that Z is also the total space of the tautological bundle $\mathcal{O}(-1)$. Note that for z non-zero, $\pi^{-1}(z) = \{(z, \ell_z)\}$ is a single point. On the other hand, we have the *exceptional divisor*

$$E = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$$

π is an isomorphism away from E.

In general, if X is a complex manifold. Then we can form $\widetilde{X} = \mathrm{Bl}_p X$, by choosing local charts about p, and replace a neighbourhood of p with a neighbourhood of 0 in Z.

Why consider blowups?

- in symplectic geometry, we can use blowups to construct new symplectic manifolds,
- in algebraic geometry, we can use blowups to resolve singularities.

Now in symplectic geometry:
Let (M, ω) be a symplectic manifold. Let

$$Z_\delta = \pi^{-1}(B_\delta(0)) \subseteq Z$$

and

$$\omega_\lambda = \pi^*\omega_c + \lambda^2 p^*\omega_{\mathbb{P}^{n-1}}$$

Then $(Z_\delta, \omega_\lambda)$ is a symplectic manifold.

Lemma 9.1. $(Z_\delta \setminus E, \omega_\lambda)$ is symplectomorphic to $(B(\sqrt{\lambda^2+\delta^2}) \setminus B(\lambda), \omega_{\text{std}})$.

Let Ψ be the symplectomorphism above. Thus, to construct the symplectic blowup, suppose we have a symplectic embedding

$$\varphi : B(\sqrt{\lambda^2+\delta^2}) \hookrightarrow M$$

Let

$$\widetilde{M} = M \setminus \varphi(B(\lambda)) \cup_\Psi Z_\delta$$

Using this, we can glue to get $(\widetilde{M}, \widetilde{\omega_\lambda})$.

Proof of lemma 9.1. Let $\Phi : \mathbb{C}^n \setminus 0 \to \mathbb{P}^{n-1}$ be the quotient map. Then

$$\Phi^*\omega_{\mathbb{P}^{n-1}} = \frac{i}{2}\partial\bar{\partial}|z|^2$$

Let $\mu_\lambda \in \Omega^2(\mathbb{C}^n \setminus 0)$ be defined by

$$\mu_\lambda = \frac{i}{2}\partial\bar{\partial}\left(|z|^2 + \lambda^2 \log\left(|z|^2\right)\right)$$

We can check that

$$|z|^2 + \lambda^2 \log\left(|z|^2\right)$$

is plurisubharmonic, and that

$$\pi^*\mu_\lambda = \omega_\lambda$$

is a symplectic form on $Z_\delta \setminus E \cong B(\delta) \setminus 0$. Now define

$$F : \mathbb{C}^n \setminus 0 \to \mathbb{C}^n \setminus B(\lambda)$$
$$F(z) = z\sqrt{1 + \frac{\lambda^2}{|z|^2}}$$

A computation shows that

$$F^*\omega_{\text{std}} = \mu_\lambda$$

□

Remark 9.2. Symplectic blowup is volume decreasing, as we remove a ball of radius λ.

We would like to find $[\omega_\lambda] \in \mathrm{H}^2(M)$. We use Mayer-Vietoris,

$$\widetilde{M} = M \setminus B(\lambda) \cup_U Z_\delta$$

Here, U is homotopic to S^{2n-1}. So we get (for $n \geq 2$)

$$0 \longrightarrow \mathrm{H}^2(\widetilde{M}) \longrightarrow \mathrm{H}^2(M \setminus B(\lambda)) \oplus \mathrm{H}^2(Z_\delta) \longrightarrow 0$$

Now $\mathrm{H}^2(M \setminus B(\lambda)) \cong \mathrm{H}^2(M)$. Under the above isomorphism, $[\omega_\lambda] = [\omega] - \pi\lambda^2\,\mathrm{PD}[E]$.

Lecture 14

Another way to see that ω_λ is symplectic is as follows. Note that $\pi^*\omega_{\mathbb{C}^n}$ and $p^*\omega_{\mathbb{P}^{n-1}}$ are both compatible with respect to the standard complex structure J. In particular, $\pi^*\omega_{\mathbb{C}^n}$ is symplectic away from E.

On E, we have

$$0 \longrightarrow \mathrm{T}E \longrightarrow \mathrm{T}Z|_E \longrightarrow \nu_E \longrightarrow 0$$

Note that $\nu_E = \ker(\mathrm{d}p)$ which includes into $\mathrm{T}Z|_E$, which gives a splitting of the short exact sequence. Thus, we have a pointwise splitting

$$(\mathrm{T}Z|_E)_p = \mathrm{T}E \oplus \nu_E$$

On the first factor, we have the non-degenerate form $p^*\omega_{\mathbb{P}^{n-1}}$, and on the second factor we have the non-degenerate form $\pi^*\omega_{\mathbb{C}^n}$.

On the other hand,

$$\pi\lambda^2 \operatorname{PD}[E] = [\omega_\lambda] \in \mathrm{H}^2(Z_\delta)$$

Next, note that we have isomorphisms

$$\begin{aligned} &\mathrm{H}^2(Z_\delta) \\ &\quad\cong \mathrm{H}_{2n-2}(Z_\delta, \partial Z_\delta) \\ &\quad\cong \mathrm{H}_{2n-2}(Z_\delta) \\ &\quad\cong \mathrm{H}_{2n-2}(\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1))) \\ &\quad\cong \mathrm{H}_{2n-2}(\mathbb{P}^{n-1}) \\ &\quad= \mathbb{Z}[E] \end{aligned}$$

and for a line L in $\mathbb{P}^{n-1}$, we have $\mathrm{H}_2(Z_\delta) = [L]$, and so

$$[E] \cdot [L] = c_1(\mathcal{O}_{\mathbb{P}^{-1}}(-1)) \cdot [L] = -1$$

On the other hand,

$$\int_L \omega_\lambda = \lambda^2 \int_L \omega_{\mathbb{P}^{n-1}} = \pi$$

> **Remark 9.3.** Suppose we started off with an almost complex structure J. Then we would like to construct a $\widetilde{J}$ on the blowup $\widetilde{M}$, and so on. See McDuff-Salamon.

> **Remark 9.4.** In general, we can blow up along complex submanifolds, symplectic submanifolds, subschemes... Here, we replace the submanifold with the projectivisation of its normal bundle.

Consider the rational map

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^1 \\ [x : y : z] &\mapsto [x : y] \end{aligned}$$

In an affine, this sends (x, y) to x/y. Note this is not defined at $p_0 = [0 : 0 : 1]$. But, if we blow up at p_0, then we have a well defined map. Consider the neighbourhood

$$\{((x, y), [t_1 : t_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid xt_2 = yt_1\}$$

The projection to $\mathbb{P}^1$ defines a well defined map, which is an extension of the above map away from E. In this case, we get a Hirzebruch surface $\mathbb{F}_1$. Thus, we have a $\mathbb{P}^1$-bundle $\mathbb{F}_1 \to \mathbb{P}^1$. More generally, we can write this as

$$\mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}) \to \mathbb{P}^1$$

9.1 Rational elliptic surface

Let f, g define smooth cubics in $\mathbb{P}^2$. Let $V_f = \mathbb{V}(f)$ and $V_g = \mathbb{V}(g)$, and assume that $V_f \pitchfork V_g$, and so $\mathcal{B} = V_f \cap V_g$ is 9-points. So now we have a pencil of cubics, which is a $\mathbb{P}^1$-family of cubics $sf + tg$, where $[s : t] \in \mathbb{P}^1$. Thus, we get

$$V_{[s:t]} = \mathbb{V}(sf + tg)$$

Consider the rational map $[f : g] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. This is not defined at $\mathcal{B}$. Let

$$S = \mathrm{Bl}_{\mathcal{B}} \mathbb{P}^2$$

Then we have a well defined map $\phi : S \to \mathbb{P}^1$. Moreover, each point $E_i \in \mathcal{B}$ defines a section of ϕ. Lecture 15

Remark 9.5. If π is locally a submersion, $F = \pi^{-1}(p)$ is a smooth fibre, and ν_F is trivial as a complex fibre bundle.

10 Fibre connect sums of symplectic manifolds

Theorem 10.1 (symplectic neighbourhood theorem)**.** Let $Q \subseteq (X, \omega)$ be a symplectic submanifold, then a neighbourhood of Q is symplectically determined by (Q, ω_Q) and $\nu_{Q/X} \to Q$ as a symplectic vector bundle.

Note we have an exact sequence

$$0 \longrightarrow TQ \longrightarrow TX|_Q \longrightarrow \nu_{Q/X} = TQ^\omega \longrightarrow 0$$

and so the symplectic structure on $\nu_{Q/X}$ is determined by the symplectic structure on $TX|_Q$.

Corollary 10.2. For $Q \subseteq X^4$, a symplectic neighbourhood is determined by

- $\int_Q \omega$,
- $\chi(Q)$ (corresponding to c_1 of TQ),
- $Q \cdot Q$ (corresponding to c_1 of the normal bundle)

Sketch proof of theorem 10.1. Say we have symplectic embeddings $(Q, \omega) \hookrightarrow (X_1, \omega_1)$ and into (X_2, ω_2), with normal bundles $\nu_1 \cong \nu_2$. Then use the tubular neighbourhood theorem to get a diffeomorphism φ between the neighbourhoods of Q in X_1, X_2 respectively, which is the identity on Q.

Note that $\varphi^* \omega_2|_{TQ} = \omega_1|_{TQ}$. Then use Moser's trick. □

Corollary 10.3. Say we have a symplectic embedding $(Q^{2n-2}, \Omega) \hookrightarrow (M^{2n}, \omega)$. This extends to a map

$$h : (Q^{2n-2} \times D^2(\varepsilon), \Omega \oplus \omega_{\text{std}}) \hookrightarrow (M^{2n}, \omega)$$

if and only if $c_1(\nu_{Q/M}) = 0$.

Remark 10.4. Note h is not determined topologically from this data, since framings are classified by homotopy classes of maps $[Q, SO(2)] = [Q, S^1]$.

Fact: There exists a symplectomorphism "turning the annulus inside out". Here, the annulus is $B^2(\varepsilon) \setminus 0$, and the map is

$$\Phi(r, \theta) = \left(\sqrt{\varepsilon^2 - r^2}, -\theta\right)$$

Suppose Q has two codimension 2 symplectic embeddings $Q \hookrightarrow M_1, Q \hookrightarrow M_2$, with ν_{Q/M_1} is trivial. Fix extensions $h_i : Q \times B^2(\varepsilon) \to M_i$ as above. Define the *symplectic fibre sum*

$$M_1 \#_Q M_2 = (M_1 \setminus Q) \cup_{\text{id} \times \Phi} (M_2 \setminus Q)$$

where $\omega = \omega_i$ on M_i, which agrees on the overlaps as we are gluing by a symplectomorphism.

Remark 10.5. Framings matter topologically, and ε matters symplectically. We note that

$$\text{vol}(M_1 \#_Q M_2) < \text{vol}(M_1) + \text{vol}(M_2)$$

Fact: This construction is 'local', so in fact we can relax the condition of $c_1(\nu_i) = 0$ to allow $c_1(\nu_1) = -c_1(\nu_2)$.

Example 10.6
If $C^2 \subseteq (X^4, \omega)$ is such that $g(C) = 0$, $C \cdot C = -1$. Then we can form

$$X^4 \underset{\{C=L\}}{\#} (\mathbb{P}^2, r\omega_{\mathrm{FS}})$$

where L is a line in $\mathbb{P}^2$, and r is chosen so that

$$\int_L r\omega_{\mathrm{FS}} = \int_C \omega_X$$

This is the *blow down* of Q.

Example 10.7
Say $C \subsetneq (X^4, \omega)$ is such that $g(C) = 0$, with $C \cdot C = -4$. Take $A \subsetneq \mathbb{P}^2$ be a smooth conic, and the blow down is

$$X \underset{\{C=A\}}{\#} \mathbb{P}^2$$

Again, we need to scale the symplectic form on $\mathbb{P}^2$. In this case, $A \cdot A = (2L) \cdot (2L) = 4$ and $g(A) = 0$, so everything works out.

For examples in symplectic geometry: Start off with a complex Kähler manifold, say a projective manifold. Then perform operations such as blow down, fibre connect sum and so on.

Lecture 16

We will now construct a symplectic manifold with no integrable complex structure.

Recall $\pi : E(1) \to \mathbb{P}^1$ is given by blowing up the base points of a pencil of cubics. In this case, $E(1)$ is Kähler. Let F be a smooth fibre.

Fact: $\pi_1(E(1) \setminus F) = 0$.

Also note that $c_1(\nu F) = 0$. Let $(T^4, \omega = \omega_{\mathrm{std}} + \varepsilon \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_3)$. For small ε, ω is symplectic. Let

$$C_1 = T^2 \times \{*\}$$
$$C_2 = (S^1 \times \{*\}) \times (S^1 \times \{*\})$$

These are now both symplectic submanifolds of T^4. Now νC_1 and νC_2 are trivial, and so there exists a symplectic form on

$$Y = (T^4 \#_{C_1=F} E(1)) \#_{C_2=F} E(1)$$

(choose ω scaled appropriately).

Proposition 10.8. $\pi_1(Y) \cong \pi_1(T^4)/\langle \pi_1(C_1), \pi_1(C_2)\rangle \cong \mathbb{Z}$.

Proof. Seifert-van Kampen. □

In particular, $\mathrm{H}^1(Y; \mathbb{Z}) = \mathbb{Z}$, and so $b^1(Y) = 1$. Thus, Y cannot be Kähler.

11 Fibrations of complex manifolds

Let $f : X \to Y$ be a holomorphic map of Kähler manifolds, with $\dim_{\mathbb{C}}(X) \geq \dim_{\mathbb{C}}(Y)$, and assume f is surjective. Let

$$S = \{x \in X \mid \mathrm{rank}(\mathrm{d}f_x) < \dim_{\mathbb{C}}(Y)\}$$

In this case, S is a complex submanifold of complex dimension at most $\dim_{\mathbb{C}}(Y) - 1$. The fact that it is a complex submanifold follows by the implicit function theorem. Thus, $f(S)$ has codimension at least 1 in Y. Then we have

$$f : U = X \setminus f^{-1}(f(S)) \to W = Y \setminus f(S)$$

By Ehresmann's theorem, $f : U \to W$ is a fibre bundle. But note that W is connected, since it is Y with a smooth real codimension at least 2 subset removed. Such $f : X \to Y$ are often called fibrations.

For $w_1, w_2 \in W$, $[f^{-1}(w_1)] = [f^{-1}(w_2)]$ in homology. By Moser, if we set $F_w = f^{-1}(w)$, then we have a symplectomorphism

$$(F_{w_1}, \omega_X|_{F_{w_1}}) \cong (F_{w_2}, \omega_X|_{F_{w_2}})$$

Since we can smoothly trivialise over a path connecting w_1, w_2 in W.

> **Remark 11.1.** We have not used the "horizontal" directions at all.

We can do better. On $f : U \to W$, we can split at $x \in U$, say $x \in F_y$,

$$\mathrm{T}_x X = \mathrm{T}_x F_y \oplus (T_x F_y)^\omega$$

This gives an Ehresmann connection. Given γ a path in W, with $\gamma(0) = w_0$ and $\gamma(1) = w_1$, this gives a parallel transport map $\varphi_t : F_{w_0} \to F_{\gamma(t)}$ with respect to the above connection.

If $p_t = \varphi_t(p_0)$, then $p_t' \in (\mathrm{T}_{\gamma(t)} F)^\omega$.

> **Proposition 11.2.** For all $t \in [0, 1]$, φ_t is a symplectomorphism.

Thus, we have a *canonical* symplectomorphism between fibres connected by a path.

For a more concrete example, consider $E(1) \to \mathbb{P}^1$.