Toric varieties

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Contents

Lecture 1

In this course, we do everything over $\mathbb{C}$. In particular, $\times$ means fibre product over $\mathrm{Spec}(\mathbb{C})$.

1 What is a toric variety?

Definition 1.1 (affine variety)
An *affine variety* is $V = \mathrm{Spec}(R)$, where R is a finitely generated integral[a] $\mathbb{C}$-algebra.

[a] R is an integral domain.

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Fact: Given such an R, we can choose generators and relations, and write

$$R = \frac{\mathbb{C}[x_1, \dots, x_n]}{I}$$

where I is a prime ideal. This gives an affine embedding. i.e. a closed embedding

$$V = \mathbb{V}(I) = \mathbb{A}^n$$

Note that the affine embedding is not unique.

Definition 1.2 ((complex) variety)

A *(complex) variety* is a scheme X which is integral (reduced and irreducible), separated, and of finite type.

We can avoid schemes, and glue varieties out of affine varieties. We need

1. a finite collection $\{V_\alpha\}_{\alpha \in A}$ of affine varieties, say $V_\alpha = \operatorname{Spec}(R_\alpha)$,
2. for $\alpha, \beta \in A$, we need Zariski opens $V_{\alpha\beta} \subseteq V_\alpha$,
3. algebraic transition maps $g_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$, satisfying the usual cocycle conditions,
4. check that the resulting

$$X = \frac{\coprod_\alpha V_\alpha}{\sim}$$

is irreducible and separated (Hausdorff).

Definition 1.3 (torus)

An *(algebraic complex) torus of dimension n* is

$$\mathrm{T}^n = (\mathbb{C}^*)^n = \operatorname{Spec}(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]) = (\mathbb{A}^1 \setminus \{0\})^n$$

Note that this is not the same as the usual torus $(S^1)^n$. In particular, we have an affine embedding

$$\mathrm{T}^n = \operatorname{Spec}\left(\frac{\mathbb{C}[z_0, \dots, z_n]}{\langle z_0 \cdots z_n - 1\rangle}\right) = \mathbb{V}(z_0 \cdots z_n - 1) \subseteq \mathbb{A}^{n+1}$$

Exercise: Check that we have an isomorphism of rings

$$\mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \cong \frac{\mathbb{C}[z_0, \dots, z_n]}{\langle z_0 \cdots z_n - 1\rangle}$$

The torus is also a *commutative group scheme*, that is, we have algebraic morphisms

$$\begin{aligned} e &: \bullet \to \mathrm{T}^n \\ m &: \mathrm{T}^n \times \mathrm{T}^n \to \mathrm{T}^n \\ i &: \mathrm{T}^n \to \mathrm{T}^n \end{aligned}$$

making T^n into an abelian group with identity e, multiplication m and inverse i. On the level of coordinate rings, we have the *identity section*,

$$\begin{aligned} e^\sharp : \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] &\to \mathbb{C} \\ t_i^{\pm 1} &\mapsto 1 \end{aligned}$$

The multiplication map is given by pointwise operations, i.e.

$$m((x_1, \dots, x_n), (y_1, \dots, y_n)) = (x_1 y_1, \dots, x_n y_n)$$

With this, we have *comultiplication*

$$\begin{aligned} m^\sharp : \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] &\to \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \otimes \mathbb{C}[y_1^{\pm 1}, \dots, y_n^{\pm 1}] \\ t_i &\mapsto x_i \otimes y_i \end{aligned}$$

and the inverse map is given by

$$\begin{aligned} i^\sharp : \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] &\to \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \\ t_i &\mapsto t_i^{-1} \end{aligned}$$

Definition 1.4 (toric variety)
A *toric variety* is a variety X with a dense open $\mathrm{T}^n \subseteq X$, and an action (i.e. algebraic map)

$$a : \mathrm{T}^n \times X \to X$$

which extends the multiplication on X.

Example 1.5
T^n is itself a toric variety.

Cohomology

Example 1.6
$\mathbb{A}^n = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n])$, with the torus

$$\mathrm{Spec}(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$$

with

$$a((t_1, \dots, t_n), (x_1, \dots, x_n)) = (t_1 x_1, \dots, t_n x_n)$$

Example 1.7
Projective space

$$\mathbb{P}^n = \mathrm{Proj}(\mathbb{C}[z_0, \dots, z_n]) = \{(z_0 : \cdots : z_n) \mid (z_0, \dots, z_n) \neq (0, \dots, 0)\}$$

is also a toric variety, with torus

$$\mathrm{T}^n = \{(z_0 : \cdots : z_n) \mid z_0 \cdots z_n \neq 0\}$$

with action given coordinate-wise again, i.e.

$$a((t_0 : \cdots : t_n), (z_0 : \cdots : z_n)) = (t_0 z_0 : \cdots : t_n z_n)$$

in non-homogeneous coordinates (i.e. in the affine $z_0 \neq 0$) of the torus, we can define the action by

$$(t_1, \dots, t_n) \cdot (z_0 : \cdots : z_n) = (z_0 : t_1 z_1 : \cdots : t_n z_n)$$

We can check that T^n is contained in all of the standard opens $U_i = D(z_i)$, and if we choose the usual affine coordinates, then T^n embeds into affine space as above.

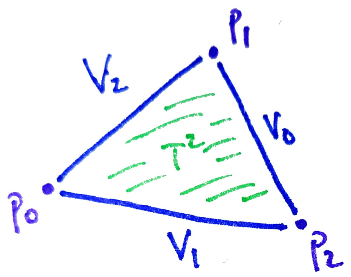
Toric varieties are determined by the dimension n of the torus, and the *toric boundary* $X \setminus \mathrm{T}^n$. For example, $\mathbb{P}^2 \setminus \mathrm{T}^2$ has three components,

1. $V_0 = \mathbb{V}(z_0)$,

2. $V_1 = \mathbb{V}(z_1)$,

3. $V_2 = \mathbb{V}(z_2)$

The V_i intersect pairwise, but not all three at the same time. Each V_i is invariant under the action of the torus, so $\mathbb{P}^2$ can be represented as



In particular, this gives us a stratification of $\mathbb{P}^2$, with strata T^2, $\mathbb{C}^* = T^1$ corresponding to V_i with the vertices removed, and the three vertices.

Lecture 2

2 Characters, cocharacters and lattices

Recall from the last lecture that $\mathrm{T}^n = (\mathbb{C}^*)^n$ is an algebraic group.

Definition 2.1 (character)
A *character* χ of T^n is a morphism $\chi : \mathrm{T}^n \to \mathbb{C}^*$ of algebraic groups. That is, $\chi : \mathrm{T}^n \to \mathbb{C}^*$ is a morphisms of schemes, such that

$$\chi(e) = 1 \quad \text{and} \quad (\chi^\sharp \otimes \chi^\sharp) \circ m^\sharp = m^\sharp \circ \chi^\sharp$$

We will write $\mathrm{Hom}_{\text{alg-gp}}(\cdot, \cdot)$ for the set of morphisms of algebraic groups.

Theorem 2.2. The characters of T^n are

$$M = \mathrm{Hom}_{\text{alg-gp}}(\mathrm{T}^n, \mathbb{C}^*) = \mathbb{Z}^n$$

Proof. First of all, note that

$$\mathrm{Hom}_{\text{alg-gp}}(\mathrm{T}^n, \mathbb{C}^*) = (\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*))^n$$

Thus, without loss of generality $n = 1$. Now we need a $\mathbb{C}$-algebra map

$$\chi^\sharp : \mathbb{C}[x^{\pm 1}] \to \mathbb{C}[y^{\pm 1}]$$

This is determined by $\chi^\sharp(x) = p(y, y^{-1})$. Note that x is a unit in $\mathbb{C}[x^{\pm 1}]$, and so $\chi^\sharp(x)$ is a unit. Thus, $p(y, y^{-1}) = ay^k$, for some $a \in \mathbb{C}, k \in \mathbb{Z}$.

Next, we need that $e^\sharp \circ \chi^\sharp = \chi^\sharp \circ e^\sharp$. That is,

$$a = e^\sharp(\chi^\sharp(x)) = \chi^\sharp(e^\sharp(x)) = \chi^\sharp(1) = 1$$

Hence $\chi^\sharp(x) = x^k$, for some $k \in \mathbb{Z}$. Note that $\chi^\sharp(x \otimes x) = \chi^\sharp(x) \otimes \chi^\sharp(x)$ automatically, so we are done. ☐

The characters of T^n are of the form

$$\chi(t_1, \dots, t_n) = t_1^{a_1} \cdots t_n^{a_n}$$

for some $(a_1, \dots, a_n) \in \mathbb{Z}^n$. Expressions of the form

$$t_1^{a_1} \cdots t_n^{a_n}$$

are called *Laurent monomials*. We will write

$$\chi^{\mathbf{a}}(\mathbf{t}) = \mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_n^{a_n}$$

for this character. The correspondence is then

$$\begin{aligned} \mathrm{Hom}_{\text{alg-gp}}(\mathrm{T}^n, \mathbb{C}^*) &\to \mathbb{Z}^n \\ \chi^{\mathbf{a}} &\mapsto \mathbf{a} \end{aligned}$$

The characters form a group. Given characters $\chi, \chi' : \mathrm{T}^n \to \mathbb{C}^*$,

$$\chi \cdot \chi'(\mathbf{t}) = \chi(\mathbf{t}) \cdot \chi'(\mathbf{t})$$

Thus, we have taht

$$\chi^{\mathbf{a}} \cdot \chi^{\mathbf{a}'} = \chi^{\mathbf{a}+\mathbf{a}'}$$

Hence the identification $\mathrm{Hom}_{\text{alg-gp}}(\mathrm{T}^n, \mathbb{C}^*) \cong \mathbb{Z}^n$ is also a group isomorphism.

Definition 2.3 (cochacter)
A *cocharacter* of T^n is

$$\lambda \in \mathrm{Hom}_{\text{alg-gp}}(\mathbb{C}^*, \mathrm{T}^n)$$

We also call them *one-parameter subgroup*.

We can check that

$$N = \mathrm{Hom}_{\text{alg-gp}}(\mathbb{C}^*, \mathrm{T}^n) \cong \mathbb{Z}^n$$

by showing that any $\lambda \in N$ is of the form

$$\lambda(t) = (t^{u_1}, \dots, t^{u_n})$$

for $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}$. So we can write the above cocharacter as $\lambda_{\mathbf{u}}$ as before. N is a group, and the identification is a group isomorphism. That is,

$$\lambda_{\mathbf{u}} \cdot \lambda_{\mathbf{u}'} = \lambda_{\mathbf{u}+\mathbf{u}'}$$

Example 2.4
For $n = 2$, we can identify

$$\mathrm{T}^2 = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \,\middle|\, t_1 t_2 \neq 0 \right\}$$

Under the identification $M \cong N \cong \mathbb{Z}^2$,

$$\chi^{(a,b)} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = t_1^a \cdot t_2^b$$

and

$$\lambda_{(u,v)}(t) = \begin{pmatrix} t^u & 0 \\ 0 & t^v \end{pmatrix}$$

Definition 2.5 (lattice)
A *lattice* is a free abelian group of finite rank.

Note that with this definition, any lattice is isomorphic to $\mathbb{Z}^n$ when we choose a basis.

Example 2.6

$\mathbb{Z}, 2\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}, \mathbb{Z}\oplus 3\mathbb{Z}, \ldots$ are all lattices.

There is a *perfect pairing*[1], given by

$$\begin{aligned} \langle\cdot,\cdot\rangle : M\times N &\to \mathbb{Z} \\ \langle \mathbf{a},\mathbf{u}\rangle = \mathbf{a}\cdot\mathbf{u} &= a_1u_1+\cdots+a_nu_n \end{aligned}$$

In a coordinate-free manner,

$$\begin{aligned} M\times N = \mathrm{Hom}(\mathbb{C}^*, \mathrm{T}^n)\times \mathrm{Hom}(\mathrm{T}^n,\mathbb{C}^*) &\to \mathrm{Hom}(\mathbb{C}^*,\mathbb{C}^*) \\ (\lambda,\chi) &\mapsto \chi\circ\lambda \end{aligned}$$

Using the identification $\mathrm{Hom}(\mathbb{C}^*,\mathbb{C}^*)\cong\mathbb{Z}$ as before gives the pairing.

We can describe a torus T_N starting from a lattice N, through

$$\mathrm{T}_N = N\otimes_{\mathbb{Z}}\mathbb{C}^*$$

with this,

$$\mathrm{Hom}(\mathbb{C}^*,\mathrm{T}_N) = N$$

Formally,

$$\mathrm{T}_N = \mathrm{Spec}(\mathbb{Z}[N]\otimes_{\mathbb{Z}}\mathbb{C}[x,x^{-1}])$$

With this, give a torus T, we get two dual lattices N, M. Conversely, given a lattice N, we get a torus T_N and a dual lattice M. Choosing a basis on one gives a basis on the other two.

3 Cocharacters and limit points - Part I

Back to toric varieties. Let X be a toric variety, with torus T. We can think of X as a "partial compactification" of T. That is, we can construct X by adding limit points[2] to T.

Given a one-parameter subgroup $\mathbf{u}\in N$, we have

$$\lambda_{\mathbf{u}} : \mathbb{C}^*\to \mathrm{T}\subseteq X$$

We can ask when does

$$\lim_{t\to 0}\lambda_{\mathbf{u}}(t)$$

exist in X. The idea will be that we can determine X by saying which $\lambda_{\mathbf{u}}$ have a limit. See later.

Example 3.1

Consider $\mathrm{T}^2\subseteq\mathbb{P}^2_{z_0:z_1:z_2}$. The embedding is given by $(t_1,t_2)\mapsto(1:t_1:t_2)$. Choose $\mathbf{u}\in\mathbb{Z}^2$, so $\mathbf{u}=(u_1,u_2)$.

$$\lim_{t\to 0}\lambda_{\mathbf{u}}(t) = \lim_{t\to 0}(1:t^{u_1}:t^{u_2})$$

All these limit points exist, since $\mathbb{P}^2$ is complete[a]. In particular, if $u_1,u_2>0$, the limit is $(1:0:0)$. If $u_1<0, u_2>u_1$, the limit is $(0:1:0)$. Similarly if $u_2<0, u_1>u_2$ then the limit is $(0:0:1)$.

The upshot is that we have partitioned $N_{\mathbb{R}} = N\otimes\mathbb{R} = \mathbb{R}^2$ into cones, so that if $\mathbf{u}_1,\mathbf{u}_2$ are in the same cone if and only if they have the same limit. In fact, we can construct $\mathbb{P}^2$ from this.

[a] Or in algebraic geometry, it satisfies the valuative criterion for properness for some choice of DVR.

Lecture 3

4 Affine toric varieties and cones

[1] That is, it induces isomorphism $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) = M^\vee\cong N$ and vice versa.

[2] In the Euclidean topology.

Definition 4.1 (cone)
A *(rational polyhedral) cone* is

$$\sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^n$$

where

$$\sigma = \sum_{i \in I} \lambda_i u_i \mid \lambda_i \geq 0, u_1, \ldots, u_n \in \mathbb{N}$$

We will also write

$$\sigma = \text{Cone}(u_1, \ldots, u_r)$$

and the set $\{u_i\}_{i=1}^r$ is the set of generators of the cone.

Example 4.2
If $N_{\mathbb{R}} \cong \mathbb{R}e$, then we have the cone $\sigma_1 = \text{Cone}(e)$,
Next, $\sigma_2 = \text{Cone}((1,0))$ is
and $\sigma_3 = \text{Cone}((1,0),(0,1))$ is
and $\sigma_4 = \text{Cone}((1,0),(1,2))$ is

Definition 4.3 (dimension)
The *dimension* of a cone σ is $\dim(\sigma) = \dim(V)$, where $V = \text{span}\{\sigma\} \subseteq N_{\mathbb{R}}$. A cone is *top-dimensional* if $V = N_{\mathbb{R}}$.

Definition 4.4 (strongly convex)
σ is *strongly convex* if it does not contain any lines through the origin. Equivalently, $\sigma \cap -\sigma = \{0\}$.

We would like to associate a to a cone σ, an affine toric variety $U_\sigma = \text{Spec}(R_\sigma)$. That is, we would like to define

$$R_d \subseteq \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

a sub-$\mathbb{C}$-algebra.

We will do this as follows:

1. We will associate a *dual cone* $\sigma^\vee \subseteq M_{\mathbb{R}}$,
2. Then we will generate a commutative semigroup S_σ,
3. from this, we will define R_σ.

4.1 Dual cone

Definition 4.5 (dual cone)
The *dual cone* $\sigma^\vee$ of σ is

$$\check{\sigma} = \{m \in M_{\mathbb{R}} \mid \langle u, m \rangle \geq 0 \text{ for all } u \in \sigma\} \subseteq M_{\mathbb{R}}$$

where we extend the pairing to $N_{\mathbb{R}} \otimes M_{\mathbb{R}} \to \mathbb{R}$.

Example 4.6
Using the same examples as before, we have the dual cones $\check{\sigma}_1$
and $\check{\sigma}_2$,
Note here we see that the dual cone depends on the embedding (or lattice). Moreover, the dual of a

strongly convex cone does not have to be strongly convex.

Fact: $\check{\check{\sigma}} = \sigma$.

Lemma 4.7. The dual $\check{\sigma}$ of a cone σ is a cone.

Proof. We'll define an algorithm to find $m_1, \ldots, m_s \in M$, such that

$$\check{\sigma} = \text{Cone}(m_1, \ldots, m_s)$$

From linear algebra, for $m \in M = N^\vee$, we have a hypersurface

$$m^\perp = H_m = \{u \in N_\mathbb{R} \mid \langle u, m \rangle = 0\}$$

We also have half spaces

$$H_m^+ = \{u \in N_\mathbb{R} \mid \langle m, u \rangle \geq 0\}$$

Definition 4.8 (supporting hyperplane)
A *supporting hyperplane* of σ is H_m, such that $\sigma \subseteq H_m^+$. Equivalently, $m \in \check{\sigma}$.

Now note that

$$\check{\sigma} = \text{Cone}(m_1, \ldots, m_s) \iff \sigma = \bigcap_{i=1}^{s} H_{m_i}^+$$

So to construct $\check{\sigma}$, we need to express σ as an intersection of half spaces.

For example, $\check{\sigma}_3 = \text{Cone}((1,0),(0,1))$

and similarly, $\check{\sigma}_4 = \text{Cone}((2,-1),(0,1))$.

In general, we can do this by identifying the *facets* $\tau \preccurlyeq \sigma$, and taking the corresponding dual element $m_\tau \in M$, and

$$\check{\sigma} = \text{Cone}(m_\tau \mid \tau \preccurlyeq \sigma)$$

when σ is top dimensional.

Definition 4.9
A *face* $\tau \preccurlyeq \sigma$ is the intersection $\sigma \cap H_m$, for a supporting hyperplane H_m. Note that the faces are all cones, and we call τ a *facet* if $\dim(\tau) = \dim(\sigma) - 1$, and τ an *edge* if $\dim(\tau) = 1$.

Note by taking $m = 0$, $\sigma \preccurlyeq \sigma$.

When σ is not top dimensional, then $\text{span}_\mathbb{R}(\sigma) = W \neq N_\mathbb{R}$. Write $\overline{\sigma}$ for σ considered as a subspace of W. Now $\overline{\sigma}$ is top-dimensional, and so we can the dual

$$\check{\overline{\sigma}} \subseteq \frac{M_\mathbb{R}}{W^\perp}$$

Say

$$\check{\overline{\sigma}} = \text{Cone}(\overline{m_1}, \ldots, \overline{m_s})$$

where

$$\overline{m_i} \in \frac{M_\mathbb{R}}{W^\perp}$$

Choose representatives $m_i \in M_\mathbb{R}$, and choose a basis $m_{s+1}, \ldots, m_k$ of $W^\perp$ in $M_\mathbb{R}$. With this,

$$\check{\sigma} = \text{Cone}(m_1, \ldots, m_s, \pm m_{s+1}, \ldots, \pm m_k)$$

□

4.2 Commutative semigroup

Definition 4.10 (commutative semigroup)
A *commutative semigroup* is $(S, +)$, such that $+$ is an associative commutative binary operation on S.

In particular, a semigroup with identity is called a *monoid*. S is *finitely generated* if there exists $A \subseteq S$ finite, such that

$$S = \left\{ \sum_n \lambda_n a_n \,\middle|\, \lambda_n \in \mathbb{Z}_{>0}, a_n \in A \right\}$$

S is *affine* if it is finitely generated, and $(S, +)$ is a subsemigroup of a lattice $(\mathbb{Z}^n, +)$.

Example 4.11
Any group is a semigroup. Given a (not necessarily unital) ring R, R under multiplication is a semigroup. For example, $\mathbb{Z}_{>0}$ is a semigroup under multiplication, but this is not finitely generated.

Lecture 4

Definition 4.12
Let σ be a cone. Define

$$S_\sigma = \check{\sigma} \cap M \subseteq M_{\mathbb{R}}$$

for the semigroup associated to σ.

This is a semigroup as cones are closed under addition. Now we want to show that this is affine. This does not follow immediately from being a subgroup of $(\mathbb{Z}^n, +)$. For example, consider

$$S = \{(x, y) \mid x \leq \sqrt{2}y\}$$

This is a semigroup, but it is not finitely generated.

Lemma 4.13 (Gordan's lemma). If σ is a cone, then S_σ is a finitely generated semigroup.

Proof. We know that $\check{\sigma} = \mathrm{Cone}(m_1, \dots, m_r)$, for some $m_i \in M$. Say $x \in S_\sigma$. By definition,

$$x = \sum_{i=1}^r \alpha_i m_i$$

where $\alpha_i \in \mathbb{R}_{\geq 0}$. Write

$$\alpha_i = a_i + b_i$$

where $a_i = \lfloor \alpha_i \rfloor$, $b_i = \{\alpha_i\} \in [0, 1)$. Then

$$x = \sum_i a_i m_i + \sum_i b_i m_i$$

Now $x \in M$, and $\sum_i a_i m_i \in M$, and M is a group, so

$$\sum_i b_i m_i \in M$$

On the other hand, $\sum b_i m_i$ is contained in a compact set

$$K = \left\{ \sum c_i m_i \mid 0 \leq c_i \leq 1 \right\}$$

Since M is discrete, K is compact, $K \cap M$ is finitely many points, say $K \cap M = \{m_{r+1}, \dots, m_\ell\}$. Then S_σ is generated by

$$m_1, \dots, m_r, m_{r+1}, \dots, m_\ell$$

so it is finitely generated. □

Example 4.14

When $\sigma = \text{Cone}(1)$, then $\check{\sigma} = \text{Cone}(1)$, and so S_σ is generated by 1.

Now if $\sigma = \text{Cone}(e_1) \subseteq \mathbb{R}^2$, then $\check{\sigma} = \text{Cone}(e_1, e_2, -e_2)$, and these are the generators of S_σ.

If $\sigma = \text{Cone}(e_1, e_2) \subseteq \mathbb{R}^2$, then $\check{\sigma} = \text{Cone}(e_1, e_2)$, which are the generators of S_σ.

Finally, if $\sigma = \text{Cone}(e_1, e_1 + 2e_2)$, then $\check{\sigma} = \text{Cone}(e_2, 2e_1 - e_2)$. Here, Gordan's lemma shows that S_σ is generated by

$$e_2, 2e_1 - e_2, e_1$$

4.3 Coordinate ring

Definition 4.15

The algebra $\mathbb{C}[S]$ of a semigroup $(S, +)$ is the $\mathbb{C}$-algebra of finite sums

$$\mathbb{C}[S] = \left\{ \sum_i a_i x^{s_i} \mid a_i \in \mathbb{C}, s_i \in S \right\}$$

with $x^s x^{s'} = x^{s+s'}$.

Example 4.16

The semigroup $\mathbb{Z}^n$ is generated by $e_1, \ldots, e_n, -e_1, \ldots, -e_n$, so

$$\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

where $x_i^{\pm 1} = x^{\pm e_i}$.

Lemma 4.17. If $(S, +)$ is a finitely generated semigroup, $\mathbb{C}[S]$ is a finitely generated $\mathbb{C}$-algebra. If $(S, +)$ is affine, then $\mathbb{C}[S]$ is the algebra of an affine variety.

Proof. If S is generated by $A = \{s_1, \ldots, s_\ell\}$, then $\{x^{s_1}, \ldots, x^{s_\ell}\}$ generates $\mathbb{C}[S]$ as a $\mathbb{C}$-algebra.

Now suppose S is affine. Then $\mathbb{C}[S]$ is finitely generated by the above, and we have that $(S, +) \subseteq (\mathbb{Z}^n, +)$, so we have

$$\mathbb{C}[S] \subseteq \mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

So $\mathbb{C}[S]$ is a subring of an integral domain, hence it is an integral domain. To conclude, every finitely generated integral domain is the coordinate ring of an affine variety. □

Define $R_\sigma = \mathbb{C}[S_\sigma]$. From this, we know that $U_\sigma = \text{Spec}(R_\sigma)$ is an affine variety.

Example 4.18

For $\sigma = \text{Cone}(1)$, let $x = x^{e_1}$, then $\mathbb{C}[S_\sigma] = \mathbb{C}[x]$, and so $U_\sigma = \mathbb{A}^1$. The inclusion $\mathbb{C}[x] \subseteq \mathbb{C}[x^{\pm 1}]$ defines the torus.

When $\sigma = \text{Cone}(e_1) \subseteq \mathbb{R}^2$, let $x = x^{e_1}, y = x^{e_2}$, then $\mathbb{C}[S_\sigma] = \mathbb{C}[x, y^{\pm 1}]$, and $U_\sigma = \mathbb{A}^1 \times \mathbb{C}^*$. The torus is given by the inclusion

$$\mathbb{C}[x, y^{\pm 1}] \subseteq \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$$

When $\sigma = \text{Cone}(e_1, e_2) \subseteq \mathbb{R}^2$, set $x = x^{e_1}, y = x^{e_2}$, then $U_\sigma = \text{Spec}(\mathbb{C}[x, y]) = \mathbb{A}^2$.

Finally, when $\sigma = \text{Cone}(e_1, e_1 + 2e_2)$. Let $A = e_1, B = 2e_1 - e_2, C = e_2$. Then S_σ is generated by A, B, C. We have a relation $B + A = 2C$. Define $x = x^A, y = x^B, z = x^C$, then

$$\mathbb{C}[S_\sigma] = \frac{\mathbb{C}[x, y, z]}{\langle xy = z^2 \rangle}$$

With this, $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma]) = \mathbb{V}(xy - z^2) \subseteq \mathbb{A}^3$. In particular, this is a *singular* toric variety. In this case, the torus is given by

$$\frac{\mathbb{C}[x, y, z]}{\langle xy - z^2 \rangle} \subseteq \mathbb{C}[x^{\pm 1}, z^{\pm 1}] = \mathbb{C}[M]$$

To see this, we have an inclusion

$$\frac{\mathbb{C}[x, y, z]}{\langle xy - z^2 \rangle} \subseteq \frac{\mathbb{C}[x^{\pm 1}, y, z^{\pm 1}]}{\langle xy - z^2 \rangle}$$

Remark 4.19. We know that $S_\sigma \subseteq M$, and so we have an inclusion of algebras $\mathbb{C}[S_\sigma] \subseteq \mathbb{C}[M]$. A fact from commutative algebra is that, the image of the map

$$\mathrm{T}^n = \mathrm{Spec}(\mathbb{C}[M]) \to U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$$

is dense, as these are integral domains.

Remark 4.20. T^n may map to some smaller torus $\mathrm{T}^{n'} \subseteq U_\sigma$, with $n' \leq n$. We claim that if σ is strongly convex, then $\mathrm{T}^n \subseteq U_\sigma$ as a dense open subset. See examples sheet 1, questions 1 and 3. For this, note that σ is strongly convex if and only if $\check{\sigma}$ is top dimensional.

From now on, restrict to strongly convex σ.

Lecture 5

What remains is to construct the torus action. Recall $T_N = \mathrm{Spec}(\mathbb{C}[M])$. We will define this by

$$\begin{aligned} a^\sharp : \mathbb{C}[S_\sigma] &\to \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M] \\ \chi^{\mathbf{m}} &\mapsto \chi^{\mathbf{m}} \otimes \chi^{\mathbf{m}} \end{aligned}$$

This extends

$$\begin{aligned} m^\sharp : \mathbb{C}[M] &\to \mathbb{C}[M] \otimes \mathbb{C}[M] \\ \chi^{\mathbf{m}} &\mapsto \chi^{\mathbf{m}} \otimes \chi^{\mathbf{m}} \end{aligned}$$

Remark 4.21. The reason this works is the fact that $S_\sigma \subseteq M$ tells us which functions on U_σ are monomials.

With this all in mind, given a strongly convex cone σ, we have an affine toric variety U_σ, with torus T_N. We will see that

$$\begin{array}{ccc} \{\text{affine toric varieties}\} & \text{———} & \{\text{affine semigroups}\} \\ \uparrow & & \uparrow \\ \{\text{affine } \textit{normal} \text{ toric varieties}\} & \text{———} & \{\text{strongly convex cones}\} \end{array}$$

5 Opens in U_σ

We will see that $\tau \preccurlyeq \sigma$, correspond to principal opens $U_\tau \subseteq U_\sigma$. Recall that a face is $\tau = \sigma \cap H_m$ for some $m \in \check{\sigma} \cap M = S_\sigma$.

Theorem 5.1. If $\tau = \sigma \cap H_m$, then

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^{\mathbf{m}}} = \mathbb{C}[S_\sigma][\chi^{\pm \mathbf{m}}]$$

Proof. First, since $\tau \in H_m$, $\langle u, m \rangle = 0$ for all $u \in \tau$. So m and $-m$ are in $\check{\tau} \cap M = S_\tau$. Also, if $m' \in S_\sigma$, then $\langle m', u \rangle \geq 0$ for all $u \in \tau$. Putting these together,

$$\mathbb{C}[S_\sigma]_{\chi^{\mathbf{m}}} \subseteq \mathbb{C}[S_\tau]$$

For the reverse inclusion, take $m' \in S_\tau \setminus S_\sigma$. We want to show $m' + km \in S_\sigma$ for some $k \in \mathbb{Z}$. For $u \in \sigma \setminus \tau$, $\langle m', u \rangle < 0$. But we also have that $\langle m, u \rangle > 0$, so we can pick k large enough such that $\langle m' + km, u \rangle > 0$. This tells us that $m' + km \in S_\sigma$[3]. □

Example 5.2

Let $\sigma = \text{Cone}(e_1, e_2)$. Let $\star = (0,0)$. This is a face, and $S_\star = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Hence $U_\star = \mathrm{T} \subseteq \mathbb{A}^2$.

Now if τ_1 is the face $y = 0$, then we get

$$\mathbb{C}[S_{\tau_1}] = \mathbb{C}[x, y]_{\chi^{e_2} = y} = \mathbb{C}[x, y^{\pm 1}]$$

So

$$U_{\tau_1} = \mathbb{A}^1 \times \mathbb{C}^*$$

6 Fans and toric varieties

Definition 6.1 (fan)

A *fan* Σ is a finite collection of strongly convex cones $\sigma \subseteq N_\mathbb{R}$, such that

1. for all $\sigma \in \Sigma$, $\tau \preccurlyeq \sigma$, then $\tau \in \Sigma$.
2. for any $\sigma_1, \sigma_2 \in \Sigma$, $\tau = \sigma_1 \cap \sigma_2$ is a face of both.

Notation 6.2. The *support* of Σ is

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_\mathbb{R}$$

and we will write

$$\Sigma(r) \subseteq \Sigma$$

for the subset of r-dimensional cones.

Example 6.3

First, let $\sigma = \text{Cone}(e) \subseteq \mathbb{R}$, then $\Sigma_1 = \{\sigma, -\sigma, \star\}$ is a fan.

Next, we have seen the fan Σ for $\mathbb{P}^2$. Here, $\Sigma(0) = \{\star\}$, $\Sigma(1) = \{\text{Cone}(e_1), \text{Cone}(e_2), \text{Cone}(-e_1 - e_2)\}$ and so on.

Next, consider the cone given by $\sigma_0 = \text{Cone}(e_1, e_1 + e_2)$, $\sigma_1 = \text{Cone}(e_1 + e_2, e_2)$ and their faces.

Finally, consider the rays passing through $(0,1), (1,0), (0,-1), (-1,a)$, for soem $a \in \mathbb{Z}_{>0}$.

By the above construction, we can associate to each $\sigma \in \Sigma$ an affine toric variety, and if $\tau \preccurlyeq \sigma$ is a face, then $U_\tau \subseteq U_\sigma$ is an open. Thus, we can glue the U_σs together, using the explicit open embeddings. This gives an (abstract) variety X_Σ.

Remark 6.4. The second condition will imply separatedness.

Example 6.5

For the first example, $U_\sigma = \text{Spec}(\mathbb{C}[x])$, and $U_{-\sigma} = \text{Spec}(\mathbb{C}[x^{-1}])$. Finally, the origin corresponds to $\text{Spec}(\mathbb{C}[x, x^{-1}])$. So we get $\mathbb{A}^1$ and $\mathbb{A}^1$ glued along $\mathbb{C}^*$, which is $\mathbb{P}^1$.

Lecture 6

[3]Since we only need to show positivity on the finitely many generators of the cone.

Example 6.6
For the fan with for $\mathbb{P}^2$ as above, the dual fan is

diagram

Let

$$x = \chi^{(1,0)}$$
$$y = \chi^{(0,1)}$$

In this case,

$$U_{\sigma_0} = \mathrm{Spec}(\mathbb{C}[x, y])$$
$$U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y])$$
$$U_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[y^{-1}, xy^{-1}])$$

We would like to identify X_Σ with $\mathbb{P}^2$, with homogeneous coordinates $[z_0 : z_1 : z_2]$. If

$$x = \frac{z_1}{z_0} \quad \text{and} \quad y = \frac{z_2}{z_0}$$

This identifies U_{σ_0} with the open $\{z_0 \neq 0\}$. Similarly, we can identify the other opens.

Example 6.7
Consider now the fan given by

$$\sigma_1 = \mathrm{Cone}(e_1, e_1 + e_2)$$
$$\sigma_2 = \mathrm{Cone}(e_1 + e_2, e_2)$$

The dual cones are

$$\sigma_1^\vee = \mathrm{Cone}(e_2, e_1 - e_2)$$
$$\sigma_2^\vee = \mathrm{Cone}(e_1, -e_1 + e_2)$$

Note that the dual of a fan is not a fan. In this case,

$$U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[y, xy^{-1}])$$
$$U_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[x, x^{-1}y])$$

Set $t = xy^{-1}$. Then we have a projection map to $\mathbb{P}^1$. In particular, this is the total space of $\mathcal{O}(-1)$[a], which is

$$\{xz_1 = yz_0\} \subseteq \mathbb{C}^2_{x,y} \times \mathbb{P}^1_{[z_0, z_1]}$$

Setting $t = z_0/z_1$ gives the identification. This is also the blowup of $\mathbb{A}^2$ at 0.

[a]which is the tautological bundle. Using relative spec, we can also write this as

$$\mathrm{Spec}_{\mathbb{P}^1}\left(\mathrm{Sym}^\bullet_{\mathbb{P}^1}\mathcal{O}(1)\right)$$

More generally, taking the spec of the symmetric algebra associates a vector bundle to a locally free sheaf. The vector bundle has a trivialising open affine cover.

Conversely, for a vector bundle, take the sheaf of sections.

Fact: If Σ_1, Σ_2 are fans, then so is $\Sigma_1 \times \Sigma_2$, and $X_{\Sigma_1 \times \Sigma_2} = X_{\Sigma_1} \times X_{\Sigma_2}$.

7 Affine embeddings and toric ideals

Lecture 7

We'll see two alternative ways of describing toric varieties:

1. toric ideals, $V = \mathbb{V}(I_L) \subseteq \mathbb{A}^s$, where I_L is a toric ideal,
2. torus embedding, $V = \overline{\Phi_{\mathscr{A}}(T)}$, where $\Phi_{\mathscr{A}} : T \to \mathbb{A}^s$ given by monomials.

Recall for $\sigma = \text{Cone}((1,0),(1,2))$, we have $\sigma^\vee = \text{Cone}((0,1),(2,-1))$, and

$$S_\sigma = \mathbb{Z}_{\geq 0}\mathscr{A}$$

where $\mathscr{A} = \{A, B, C\}$, where $A = (0,1)$, $B = (1,0)$, $C = (2,-1)$. We have a map

$$\begin{aligned}\pi : \mathbb{Z}^3_{A,B,C} &\to M \cong \mathbb{Z}^2\\ A &\mapsto (0,1)\\ B &\mapsto (1,0)\\ C &\mapsto (2,-1)\end{aligned}$$

As a matrix,

$$\begin{pmatrix} 0 & 1 & 2\\ 1 & 0 & -1\end{pmatrix}$$

Then

$$\ker(\pi) = \mathbb{Z}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

If we restrict to $\mathbb{N}^3_{A,B,C}$ and S_σ, then we have a surjection of semigroups. In particular, $S_\sigma = \mathbb{N}^3_{A,B,C}/\sim$, where $\sim$ is the relation $A + C = 2B$. On algebras,

$$\mathbb{C}[S_\sigma] = \frac{\mathbb{C}[\mathbb{N}_{A,B,C}]}{I_L}$$

Let $x = \chi^A, y = \chi^B, z = \chi^C$. Then $A + C = 2B$ corresponds to $xz = y^2$. Then $I_L = \langle xz - y^2\rangle$. In particular, this shows that we have an affine embedding $U_\sigma = \mathbb{V}(I_L) \subseteq \mathbb{A}^3$.

On the other hand, χ^A, χ^B, χ^C are maps $T \to \mathbb{C}^*$. We can define a map

$$\begin{aligned}\Phi_{\mathscr{A}} &: T \to \mathbb{A}^3\\ (s,t) &\mapsto \left(\chi^A(s,t), \chi^B(s,t), \chi^C(s,t)\right) = (t, s, s^2t^{-1})\end{aligned}$$

In this case,

$$U_\sigma = \overline{\Phi_{\mathscr{A}}(T)}$$

Moreover, $\Phi_{\mathscr{A}}$ with coordinates s, t, acts on U_σ by

$$(s,t)(x,y,z) = (tx, sy, s^2t^{-1}z)$$

More generally, if S is an affine semigroup, generated by $\mathscr{A} = \{m_1, \dots, m_s\}$, where $m_i \in M$, a lattice. So $S = \mathbb{Z}_{\geq 0}\mathscr{A}$. We can associate an ideal and a torus embedding as above.

Definition 7.1

A *lattice ideal* is an ideal I_L in $\mathbb{C}[x_1, \dots, x_s]$, of the form

$$I_L = \left\langle x^\alpha - x^\beta \mid \alpha - \beta \in L\right\rangle$$

for some lattice L. Note here we use the multi-index notation.

I_L is *toric* if it is prime.

Now given S as above, consider the map

$$\pi : \bigoplus_{i=1}^{s} \mathbb{Z}\cdot m_i \to M$$

and $L = \ker(\pi)$ is a sublattice. These relations give an ideal

$$I_L = \left\langle x^{\ell^+} - x^{\ell^-} \mid \ell^+ - \ell^- \in L; \ell^+, \ell^- \in \bigoplus \mathbb{Z}_{\geq 0} \cdot m_i \right\rangle$$

This is an ideal of

$$\mathbb{C}[x_1, \dots, x_s] = \mathbb{C}\left[\bigoplus_{i=1}^{s} \mathbb{Z}_{\geq 0} \cdot m_i\right]$$

where $x_i = \chi^{m_i}$. We can define a variety $V = \mathbb{V}(I_L) \subseteq \mathbb{A}^s$.

Now for $\Phi_{\mathscr{A}}$, we recall that $M = \mathbb{Z}S$, and $T_N = \operatorname{Spec}(\mathbb{C}[M])$. We can define

$$\begin{aligned} \Phi_{\mathscr{A}} : T_N &\to \mathbb{A}^s \\ \mathbf{t} &\mapsto (\chi^{m_1}(\mathbf{t}), \dots, \chi^{m_s}(\mathbf{t})) \end{aligned}$$

Theorem 7.2. If V is an affine variety, then the following are equivalent:

(i) V is an affine toric variety,

(ii) $V = \operatorname{Spec}(\mathbb{C}[S])$ for an affine semigroup S,

(iii) $V = \mathbb{V}(I_L)$ for a toric ideal I_L,

(iv) $V = \overline{\Phi_{\mathscr{A}}(T)}$ for some finite collection $\mathscr{A}$ generating an affine semigroup.

Proof. We have seen (ii) implies (iii) and that (iii) implies (iv). To see that (iv) implies (i), note that $\Phi_{\mathscr{A}}$ gives an action of T on $\mathbb{A}^s$, and if $V = \overline{\Phi_{\mathscr{A}}(T)}$, then $\Phi_{\mathscr{A}}(T) \subseteq V$ is a dense open torus, and we just need to check that for $t \in T$, $t \cdot V \subseteq V$. Now $t \cdot V$ contains $\Phi_{\mathscr{A}}(T)$, so it is contained in the closure of $\Phi_{\mathscr{A}}(T)$, which is V.

For (iv) implies (i), we need a lemma from representation theory:

Lemma 7.3. If $A \subseteq \mathbb{C}[M]$ is a subrepresentation of T_N, then A splits into

$$A = \bigoplus_{m \in S} \mathbb{C} \cdot \chi^m$$

for some subset[a] $S \subseteq M$.

[a] Since the multiplicity in $\mathbb{C}[M]$ is one for each eigenspace, each eigenspace appears either zero or one times, so we can consider a *subset* without thinking about multiplicity.

Note that here, multiplication by t defines a map $\mathrm{T}_N \to \mathrm{T}_N$, which in turn is a map on coordinate rings $\mathbb{C}[M] \to \mathbb{C}[M]$.

Applying the lemma to $\mathbb{C}[V] \to \mathbb{C}[M]$. The collection S of monomial appearing in $\mathbb{C}[V]$ needs to form a semigroup S, as $\mathbb{C}[V]$ is a ring. This is finitely generated as $\mathbb{C}[V]$ is finitely generated, as it is the coordinate ring of an affine variety. ☐

Under the correspondence

$$\{\text{affine toric variety}\} \leftrightarrow \{\text{affine semigroups}\}$$

Restricting, we get

$$\{\text{normal affine toric varieties}\} \leftrightarrow \{\text{saturated semigroups}\}$$

Example 7.4

Let $M = \mathbb{Z}$, and $S = \langle 2, 3 \rangle = \{0, 2, 3, \dots\}$. But we can still construct a toric variety

$$V = \operatorname{Spec}(\mathbb{C}[S])$$

So we have a map

$$\begin{aligned}\mathbb{Z}^2 &\to \mathbb{Z}\\ e_1 &\mapsto 2\\ e_2 &\mapsto 3\end{aligned}$$

and the kernel is generated by $2e_2 - 3e_1$. Now take $x = \chi^{e_1}, y = \chi^{e_2} \in \mathbb{C}[\mathbb{Z}^2]$. We have an associated ideal, $I_L = \langle y^2 = x^3 \rangle$. Then $C = \operatorname{Spec}(\mathbb{C}[x, y]/I_L) = \mathbb{V}(I_L)$ is the affine toric variety associated to S. In particular, this is a cuspidal cubic.

For a parametrisation, we have

$$\begin{aligned}\Phi_{\mathscr{A}} : \mathbb{C}^* &\to \mathbb{A}^2\\ t &\mapsto (t^2, t^3)\end{aligned}$$

and $C = \overline{\Phi_{\mathscr{A}}(\mathbb{C}^*)}$.

On the other hand, this is not a normal variety.

Lecture 8

Definition 7.5 (normal)
An integral domain R with $K = \operatorname{Frac}(R)$ is *normal* (i.e. integrally closed) if for all $x \in K$, with x being a root of a monic polynomial with coefficients in R, $x \in R$.

Equivalently, if $x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with $a_i \in R$, then $x \in R$.

For an affine variety, it is normal if and only if its coordinate ring is normal. In general, we need to consider stalks, or an affine cover.

Example 7.6
We'll show that $R = \mathbb{C}[x, y]/\langle y^2 - x^3 \rangle$ is not normal. Note that

$$x = \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2 \in \operatorname{Frac}(R)$$

Now

$$z = \frac{y}{x} \notin R$$

but $z^2 - x^2 = 0$.

Definition 7.7 (saturated)
An affine semigroup $S \subseteq M$ is *saturated* if for every $m \in M, k \in \mathbb{N}$ with $km \in S$, then $m \in S$.

Lemma 7.8. The semigroup S_σ of a (strongly convex) cone is saturated.

Proof. We defined S_σ as the intersection of a cone (which is convex) and M. □

Theorem 7.9. If V is an affine toric variety, with torus T_N, then the following are equivalent:

(i) V is normal,

(ii) $V = \operatorname{Spec}(\mathbb{C}[S_\sigma])$, for some cone σ,

(iii) $V = \mathrm{Spec}(\mathbb{C}[S])$, for some saturated affine semigroup S,

Proof. For (i) implies (iii), if V is affine toric, then $V = \mathrm{Spec}(\mathbb{C}[S])$, for some affine semigroup S. Say $m \in M$, and $km \in S$, for some $k \in \mathbb{Z}_{>0}$. This means that $\chi^{km} \in \mathbb{C}[S]$, and $\chi^m \in \mathbb{C}(S) = \mathrm{Frac}(\mathbb{C}[S])$. But χ^m is the root of the monic integral polynomial

$$y^k - \chi^k m = 0$$

so $\chi^m \in \mathbb{C}[S]$ as $\mathbb{C}[S]$ is normal.

For (iii) implies (ii), let S be saturated and affine, and let $\mathscr{A}$ be a finite generating set. Define $\sigma^\vee = \mathrm{Cone}(\mathscr{A})$. Then $S = M \cap \sigma^\vee$. Taking the dual gives σ.

For (ii) implies (i), take $\sigma = \mathrm{Cone}(v_1, \ldots, v_r)$. Let ρ_i be the ray generated by v_i. Without loss of generality, we can assume that v_i is a minimal generator of ρ_i. Now we have that

$$\sigma^\vee = \bigcap_{i=1}^{r} H_{v_i}^+ = \bigcap_{i=1}^{r} \rho_i^\vee$$

Intersecting with the lattice,

$$S_\sigma = \bigcap_{i=1}^{r} S_{\rho_i}$$

and so

$$\mathbb{C}[S_\sigma] = \bigcap_{i=1}^{r} \mathbb{C}[\rho_i]$$

We just have to check the varieties $U_i = \mathrm{Spec}(\mathbb{C}[S_{\rho_i}])$ are normal. But if we only have one ray ρ, say generated by x_1, then

$$\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

So $U_\rho = \mathbb{A}^1 \times (\mathbb{C}^*)^{n-1}$ which is normal. $\square$

With this, we have a correspondence

$$\{\text{normal affine toric varieties}\} \leftrightarrow \{\text{strongly convex cones}\}$$

8 Points of U_σ and limit points - Part II

We want $u \in \sigma$ to correspond to one-parameter subgroups $\lambda_u : \mathbb{C}^* \to T \subseteq U_\sigma$, such that

$$\lim_{t \to 0} \lambda_u(t)$$

exists in U_σ (with the Euclidean topology). Equivalently, λ_u extends to a map $\mathbb{C} \to U_\sigma$.

Theorem 8.1. Let σ be a strongly convex cone. Let $u \in N$. Then $u \in \sigma$ if and only if

$$\lim_{t \to 0} \lambda_u(t)$$

exists in U_σ.

Proof. For any $u \in N$, $m \in S_\sigma$, say $\langle u, m \rangle = d \in \mathbb{Z}$. This means that the composition

$$\mathbb{C}^* \xrightarrow{\lambda_u} T \xrightarrow{\chi^m} \mathbb{C}^*$$

is $x \mapsto x^d$. Moreover,

$$u \in \sigma \iff \langle u, m \rangle \geq 0 \text{ for all } m \in S_\sigma \iff \lim_{t \to 0} (\chi^m \circ \lambda_u)(t) \text{ exists for all } m \in S_\sigma$$

So we need to show that

$$\lim_{t \to 0} (\chi^m \circ \lambda_u)(t) \text{ exists for all } m \in S_\sigma \iff \lim_{t \to 0} \lambda_u(t) \text{ exists in } U_\sigma$$

$\Longleftarrow$ is clear. Conversely, recall (a closed point) $x \in U_\sigma$ is a $\mathbb{C}$-algebra homomorphism

$$x^\sharp : \mathbb{C}[S_\sigma] \to \mathbb{C}$$

Define

$$x_0(\chi^m) = \lim_{t\to 0} \chi^m(\lambda_u(t)) = \begin{cases} 0 & \text{if } \langle u, m\rangle > 0 \\ 1 & \text{if } \langle u, m\rangle = 0 \end{cases}$$

Then

$$\lim_{t\to 0} \lambda_u(t) = x_0$$

□

Remark 8.2.

$$\lim_{t\to 0} \chi^m(\lambda_u(t)) = \begin{cases} 0 & \text{if } \langle u, m\rangle > 0 \\ 1 & \text{if } \langle u, m\rangle = 0 \end{cases}$$

and the limit does not exist if $\langle u, m\rangle < 0$.

Definition 8.3 (distinguished point)

The *distinguished point of σ* is

$$x_\sigma : \mathbb{C}[S_\sigma] \to \mathbb{C}$$
$$\chi^m \mapsto \begin{cases} 0 & \text{if } m \in S_\sigma \setminus \sigma^\perp \\ 1 & \text{otherwise} \end{cases}$$

Note $m \in S_\sigma \setminus \sigma^\perp$ if and only if $\langle u, m\rangle > 0$ for all m. Lecture 9

Lemma 8.4. We have a one-to-one correspondence between points $x \in U_\sigma$ and semigroup homomorphisms

$$\gamma_x : (S_\sigma, +) \to (\mathbb{C}, \cdot)$$

Proof. For $x \in U_\sigma$, we have a corresponding map

$$x^\sharp : \mathbb{C}[S_\sigma] \to \mathbb{C}$$
$$x^\sharp(\chi^m) = \chi^m(x)$$

and we can take $\gamma_x(m) = \chi^m(x)$. Conversely, given $\gamma : S_\sigma \to \mathbb{C}$, define

$$x_\gamma^\sharp : \mathbb{C}[S_\sigma] \to \mathbb{C}$$
$$x_\gamma^\sharp(\chi^m) = \gamma(m)$$

□

Example 8.5

The distinguished point $x_\sigma \in U_\sigma$ corresponds to the semigroup homomorphism

$$\gamma_\sigma : S_\sigma \to \mathbb{C}$$
$$\gamma_\sigma(m) = \begin{cases} 1 & m \in \sigma^\perp \cap S_\sigma \\ 0 & m \in S_\sigma \setminus \sigma^\perp \end{cases}$$

Corollary 8.6. For all $u \in \text{Int}(\sigma) \cap N$,

$$\lim_{t \to 0} \lambda_u(t) = x_\sigma$$

Example 8.7

Consider the cone $\sigma = \text{Cone}(e_1, e_2) \subseteq \mathbb{R}^2$. In this case, $\sigma^\perp = \{0\}$. Now $U_\sigma = \mathbb{A}^2$, and we would like to find x_σ, x_{τ_1}, where $\tau_1 = \text{Cone}(e_1)$.

First of all, in this case, we have that $\gamma_\sigma(0) = 1$, and $\gamma_\sigma(m) = 0$ otherwise. The corresponding point x_σ is given by

$$\begin{aligned} x_\sigma : \mathbb{C}[S_\sigma] &\to \mathbb{C} \\ \chi^m &\mapsto \begin{cases} 1 & m = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

This is just evaluation at zero of the polynomial ring. So $x_\sigma = (0, 0) \in \mathbb{A}^2$. Next, we want to show that this is the limit point for all one-parameter subgroups λ_u, for $u \in \text{Int}(\sigma) \cap N$. That is, $u = (a, b)$, with $a, b > 0$. Now

$$\lambda_{(a,b)}(t) = (t^a, t^b)$$

Since $a, b > 0$,

$$\lim_{t \to 0} \lambda_{(a,b)}(t) = (0, 0) = x_\sigma$$

For τ_1, we have that $\tau_1^\vee = \text{Cone}(e_1, e_2, e_2^{-1})$. In this case,

$$U_{\tau_1} = \mathbb{A}^1 \times \mathbb{C}^*$$

and $S_{\tau_1} = \mathbb{Z}_{\geq 0} e_1 \oplus \mathbb{Z} e_2$. First of all, $(a, b) \in \text{Int}(\tau_1) \cap N$ if and only if $a > 0, b = 0$. So we have

$$\lambda_{(a,0)}(t) = (t^a, 1)$$

and so

$$\lim_{t \to 0} \lambda_{(a,0)}(t) = (0, 1)$$

On the other hand,

$$\begin{aligned} \gamma_{\tau_1} : S_{\tau_1} &\to \mathbb{C} \\ e_1 &\mapsto 0 \\ e_2 &\mapsto 1 \end{aligned}$$

and so the map is

$$\begin{aligned} \mathbb{C}[x, y^{\pm 1}] &\to \mathbb{C} \\ x &\mapsto 0 \\ y &\mapsto 1 \end{aligned}$$

and so $x_{\tau_1} = (0, 1)$.

Example 8.8

For $\sigma = \text{Cone}(\varnothing) = \{0\} \subseteq N_\mathbb{R}$. Then $\sigma^\vee = M_\mathbb{R}$, and $S_\sigma = \mathbb{Z} e_1 \oplus \cdots \oplus \mathbb{Z} e_n$. Then

$$\mathbb{C}[S_\sigma] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

The distinguished point is given by $\gamma_\sigma(m) = 1$ for all m. So we have the point $(1, \ldots, 1)$, which is the identity of the group $T_N = U_\sigma$. This is the limit point for all $u \in \text{Int}(\sigma)$.

9 Smoothness

Lemma 9.1. If $\sigma = \text{Cone}(v_1, \dots, v_k)$ for $v_i \in N$, and suppose $v_1, \dots, v_k$ can be completed to a $\mathbb{Z}$-basis $v_1, \dots, v_n$ of N. Then $U_\sigma \cong \mathbb{A}^k \times (\mathbb{C}^*)^{n-k}$.

Moreover, we claim that these are the only non-singular affine toric varieties.

Proof. Complete $v_1, \dots, v_k$ to a basis $v_1, \dots, v_n$. Then S_σ is generated by $w_1^*, \dots, w_k^*, \pm w_{k+1}^*, \dots, \pm w_n^*$. So

$$\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, \dots, x_k] \otimes \mathbb{C}[x_{k+1}^{\pm 1}, \dots, x_n^{\pm 1}]$$

□

Recall if $v_1, \dots, v_n \in \mathbb{Z}^n$ forms a basis of $\mathbb{R}^n$, it does not have to form a basis of $\mathbb{Z}^n$.

Example 9.2

Take $v_1 = (1, 0), v_2 = (1, 2)$. These form a $\mathbb{R}$-basis, but not a $\mathbb{Z}$-basis. In this case, U_σ is a singular variety.

In general, if $v_1, \dots, v_n \in \mathbb{Z}^n$ are $\mathbb{R}$-linearly independent, then they form a basis of $\mathbb{Z}^n$ if and only if

$$V = \begin{pmatrix} \vdots & & \vdots \\ v_1 & \cdots & v_n \\ \vdots & & \vdots \end{pmatrix} \in GL(n, \mathbb{Z})$$

Equivalently, $\det(V) = \pm 1$.

Definition 9.3 (regular point)

Let X be a variety of dimension n. Then $x \in X$ is called a *regular* or *nonsingular* point if $\dim_{\mathbb{C}}(\Omega_{X,x}) = n$.

Otherwise, we say that x is a singular point.

Here, $\Omega_{X,x}$ is the Zariski cotangent space to x at X. Recall that a closed point $x \in \text{Spec}(R)$ corresponds to $x^\sharp : R \to \mathbb{C}$, and defines a maximal ideal $\mathfrak{m}_x = \ker(x^\sharp)$.

Then

$$\Omega_{X,x} = \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$$

This is a $\mathbb{C}$-vector space.

Definition 9.4

We say that X is *smooth* or *nonsingular*, if X has no singular points.

Note that over some fields, smooth implies non-singular, but the converse is false. An alternative definition of smoothness:

Proposition 9.5 (Jacobian criterion). Let $V = \mathbb{V}(f_1, \dots, f_r) \subseteq \mathbb{A}^s$ be an affine variety of dimension d. Then V is *smooth* at $x \in V$ if and only if the matrix

$$J_{(f_1,\dots,f_r)}(x) = \left(\frac{\partial f_i}{\partial x_j}\right)_{i=1,\dots,r; j=1,\dots,s}$$

has rank $s - d$ for all $x \in V$.

We can identify $\ker(J(x)) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. In terms of differential geometry, we have a map $F : \mathbb{A}^s \to \mathbb{A}^r$, and $\ker(dF) = TV$.

Lecture 10

Theorem 9.6. If U is an affine toric variety, then the following are equivalent:

(i) $U = U_\sigma$ for some σ is generated by $e_1, \ldots, e_k$, which is part of a basis for N,

(ii) $U \cong \mathbb{A}^k \times (\mathbb{C}^*)^{n-k}$,

(iii) U is smooth,

(iv) $x_\sigma \in U$ is smooth.

Proof. So far we have shown (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv).

Lemma 9.7. If σ is a strongly convex top dimensional cone, then the set of indecomposable elements

$$\mathcal{H} = \{m \in S_\sigma \setminus \{0\} \mid m \neq m' + m'' \text{ for all } m', m'' \in S_\sigma \setminus \{0\}\}$$

generates S_σ as a semigroup.

$\mathcal{H}$ is the *Hilbert basis* of S_σ.

Proof. Let $m \in S_\sigma \setminus 0$. Suppose $m \notin \mathcal{H}$. Then $m = m' + m''$, for some $m', m'' \in S_\sigma \setminus \{0\}$. If $m', m'' \in \mathcal{H}$, then we are done. Otherwise, we can keep splitting. We need to show that the process terminates.

Since σ is top dimensional and strongly convex, $\sigma^\vee$ is top dimensional and strongly convex. Then there exists $u \in \sigma$ such that

$$\langle m, u \rangle \in \mathbb{Z}_{>0}$$

for all $m \in S_\sigma \setminus \{0\}$. So if $m = m' + m''$ as above, then

$$\langle m, u \rangle = \left\langle m', u \right\rangle + \left\langle m'', u \right\rangle$$

Each of the terms is a positive integer, and so the process must terminate. □

We will now show (iv) implies (i). First assume σ is top dimensional. Then by assumption,

$$\Omega_{X, x_\sigma} = \frac{\mathfrak{m}_{x_\sigma}}{\mathfrak{m}_{x_\sigma}^2}$$

has rank n. In this case,

$$\begin{aligned} \gamma_\sigma : S_\sigma &\to \mathbb{C} \\ m &\mapsto 0 \text{ if } m \in S_\sigma \setminus S_\sigma^\perp = S_\sigma \setminus \{0\} \\ 0 &\mapsto 1 \end{aligned}$$

So the associated map of rings is

$$\begin{aligned} \mathbb{C}[S_\sigma] &\to \mathbb{C} \\ \chi^m &\mapsto 0 \text{ for } m \neq 0 \\ \chi^0 &\mapsto 1 \end{aligned}$$

In this case,

$$\mathfrak{m}_{x_\sigma} = \langle \chi^m \mid m \in S_\sigma \setminus \{0\} \rangle$$

and so

$$\mathfrak{m}_{x_\sigma}^2 = \left\langle \chi^{m'+m''} \mid m', m'' \in S_\sigma \setminus \{0\} \right\rangle$$

Hence

$$\frac{\mathfrak{m}_{x_\sigma}}{\mathfrak{m}_{x_\sigma}^2} = \langle \chi^m \mid m \in S_\sigma \setminus \{0\} \text{ indecomposable} \rangle$$

We know that this has rank n, so we must have that

$$\frac{\mathfrak{m}_{x_\sigma}}{\mathfrak{m}_{x_\sigma}^2} = \langle \chi^m \mid m \in \mathcal{H} \rangle$$

Since $\mathcal{H}$ is a basis of S_σ, and $\mathbb{Z}S_\sigma = N$, σ is generated by an integral basis of N. This shows (i) for this case.

Now suppose $\dim(\sigma) = k < n$. We will reduce to the previous case.

Remark 9.8 (interlude on non-top-dimensional affine toric varieties). Suppose $\dim(\sigma) = k < \dim(N_{\mathbb{R}})$. Then $N_\sigma = \mathrm{span}(\sigma) \cap N$ is a k-dimensional sublattice of N. Then we have an exact sequence

$$0 \longrightarrow N_\sigma \longrightarrow N \longrightarrow N(\sigma) = N/N_\sigma \longrightarrow 0$$

Note everything is torsion free. Dualising, we get

$$0 \longrightarrow N_\sigma^\perp \longrightarrow M \longrightarrow M/N_\sigma^\perp \longrightarrow 0$$

Consider $\overline{\sigma} \subseteq N_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$. We define this as the same cone as σ, but viewed as a top dimensional cone in $\mathrm{span}_{\mathbb{R}}(\sigma) = N_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$. Now

$$\overline{\sigma}^\vee \subseteq \frac{M}{N_\sigma^\perp} \otimes_{\mathbb{Z}} \mathbb{R}$$

Say $\overline{\sigma}^\vee = \mathrm{Cone}(\overline{v_1}, \dots, \overline{v_r})$. Choosing a splitting of the second exact sequence, we find $v_i \in M$ mapping to $\overline{v_i}$. Then

$$\sigma^\vee = \mathrm{Cone}(v_1, \dots, v_r) \times \mathrm{Cone}(\pm e_1, \dots, \pm e_{n-k})$$

where $e_1, \dots, e_{n-k}$ is a basis of $N_\sigma^\perp$.

So $U_\sigma = U_{\overline{\sigma}} \times T_{N(\sigma)} = U_{\overline{\sigma}} \times (\mathbb{C}^*)^{n-k}$. The torus action on this splits as $T_N = T_{N_\sigma} \times T_{N(\sigma)}$. In particular, $x_\sigma = (x_{\overline{\sigma}}, 1_{T_{N(\sigma)}})$, where $1_{T_{N(\sigma)}}$ is the identity of $T_{N(\sigma)}$.

If x_σ is smooth, then $x_{\overline{\sigma}}$ is smooth, and $\overline{\sigma}$ is a top dimensional cone in N_σ. So by the above, $\overline{\sigma} = \mathrm{Cone}(e^*_{n-k+1}, \dots, e^*_n)$, where the $e^*_{n-k+1}, \dots, e^*_n$ is a basis of N_σ. With this,

$$\sigma = \mathrm{Cone}(e^*_{n+k+1}, \dots, e^*_n)$$

and

$$\sigma^\vee = \mathrm{Cone}(\pm e_1, \dots, \pm e_{n-k}, e_{n-k+1}, \dots, e_n)$$

and so

$$U_\sigma = \mathbb{A}^k \times (\mathbb{C}^*)^{n-k}$$

□

10 Toric morphisms

Definition 10.1 (morphism of toric varieties)

Let X, Y be toric varieties, with torus T_X, T_Y respectively. Then a *morphism of toric varieties* $X \to Y$ is *toric* if $f(T_X) \subseteq T_Y$, and $f : T_X \to T_Y$ is a group homomorphism.

Note that if $T_N, T_{N'}$ are tori, then

$$\mathrm{Hom}_{\text{alg-gp}}(T_N, T_{N'}) \cong \mathrm{Hom}_{\mathbb{Z}}(N, N') \cong \mathrm{Mat}(n' \times n, \mathbb{Z})$$

For a morphism $\varphi \in \mathrm{Hom}_{\text{alg-gp}}(T_N, T_{N'})$, we will write $\widehat{\varphi} = J_\varphi(1)$ for the associated element of $\mathrm{Hom}_{\mathbb{Z}}(N, N')$.

We want to start with φ and understand from pictures of Σ, Σ', whether φ extends to a toric morphism $f : X_\Sigma \to X_{\Sigma'}$.

Example 10.2

Consider the map $\mathbb{A}^1 \to C = \mathrm{Spec}(\mathbb{C}[x, y]/\langle y^2 - x^3 \rangle) \subseteq \mathbb{A}^2$, given by $t \mapsto (t^2, t^3)$. This clearly restricts to a map $\mathbb{C}^* \to (\mathbb{C}^*)^2$ given by $t \mapsto (t^2, t^3)$. This is a group homomorphism, so the above is a morphism of toric varieties.

On semigroups, this is the inclusion

$$2\mathbb{Z}_{\geq 0} + 3\mathbb{Z}_{\geq 0} \hookrightarrow \mathbb{Z}_{\geq 0}$$

This is called saturation, and the map $\mathbb{A}^1 \to C$ is the normalisation.

Example 10.3 (non-example)
The map

$$\mathbb{A}^2 \to \mathbb{A}^1$$
$$(x, y) \mapsto x + y$$

is not a toric morphism, since it does not restrict to a map on tori.

Lecture 11

Lemma 10.4. A toric morphism is equivariant, with respect to the torus actions. That is,

$$f(tx) = f(t)f(x)$$

for all $t \in T_N, x \in X$.

Proof. It is true for $x \in T_N$, and we can extend by continuity to all of X, since T_N is dense in X. □

To start off, given $\varphi : T_N \to T_{N'}$, or equivalently $\widehat{\varphi} : N \to N'$, when does it extend to $f : X \to Y$? Note that if the extension exists, then it is unique. In addition,

$$\mathrm{Aut}_{\mathrm{toric}}(X) \leftrightarrow \mathrm{Aut}_{\text{alg-gp}}(T_N) \leftrightarrow \mathrm{GL}_n(\mathbb{Z})$$

Definition 10.5 (morphism of fans)
Let $\Sigma \subseteq N_\mathbb{R}$ and $\Sigma' \subseteq N'_\mathbb{R}$ be fans, and fix a morphism $\widehat{\varphi} : N \to N'$. Then $\widehat{\varphi}_\mathbb{R}$ is a morphism of fans $\Sigma \to \Sigma'$ if for any cone $\sigma \in \Sigma$,

$$\widehat{\varphi}_\mathbb{R}(\sigma) \subseteq \sigma'$$

for some $\sigma' \in \Sigma'$.

In particular, a morphism of cones $\sigma \to \sigma'$ is $\widehat{\varphi}$ such that $\widehat{\varphi}_\mathbb{R}(\sigma) \subseteq \sigma'$.

Example 10.6
diagram
This is a morphism of fans, given by the projection $\mathbb{Z}^2 \to \mathbb{Z}$.

Theorem 10.7. Let $\varphi : T_N \to T_{N'}$ be a homomorphism of tori. Then this extends to $f : X_\Sigma \to X_{\Sigma'}$ if and only if $\widehat{\varphi} : N \to N'$ is a morphism of fans from Σ to Σ'.

Proof. Omitted. □

Corollary 10.8. There is an equivalence of categories

$$\{\text{normal toric varieties with toric morphisms}\} \leftrightarrow \{\text{fans with morphisms of fans}\}$$

Example 10.9
Consider the previous example, the morphism between tori is $(x, y) \mapsto x$. This is also the map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Example 10.10

Consider the map

diagram

The domain is the total space of the tautological bundle, equivalently

$$\mathbb{V}(xY - yX) \subseteq \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{[X:Y]}$$

and equivalently again, this is the blowup $\mathrm{Bl}_0 \mathbb{A}^2$. The map is to $\mathbb{A}^2$.

We are replacing the point $x_\sigma = (0,0) \in \mathbb{A}^2$ by some other cones. In fact,

$$\pi^{-1}(0,0) = \mathbb{P}^1_{[X:Y]}$$

Example 10.11

Consider

diagram

This is not a morphism of fans. This is because the image of the red cone does not fall inside any cone. In fact, there is no morphism $\mathbb{P}^2 \to \mathbb{P}^1$. This is just a rational map (so it is defined on a dense open). If we compute what the map is, in homogeneous coordinates

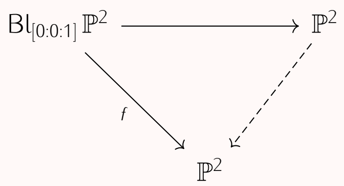
$$[z_0 : z_1 : z_2] \mapsto [z_0 : z_1]$$

This is undefined at $[0:0:1]$. Note that the left fan corresponds to $[0:0:1]$.

(maybe all up to rotation?)

If we add a ray generated by $(-1,0)$, then f turns into a toric morphism. In fact, we have added $(-1,-1)+(0,1)$. But now the red cone is isomorphic to $\mathbb{A}^2$. So what we have done is a blowup. in particular, we get

$$X = \mathrm{Bl}_{[0:0:1]} \mathbb{P}^2 \to \mathbb{P}^2$$



Remark 10.12. In general, we can blow up a variety X at any subvariety $Z \subseteq X$,. Let $\mathcal{I}_Z$ be the corresponding ideal sheaf. Then

$$\bigoplus_{d=0}^{\infty} \mathcal{I}_Z^d$$

is a sheaf of graded algebras, where we assume the generators of $\mathcal{I}_Z$ live in degree 1. Then

$$\pi : \underline{\mathrm{Proj}}_X \left(\bigoplus_{d=0}^{\infty} \mathcal{I}_Z^d \right) \to X$$

is the blowup. If $Z \subseteq X$ is regular, then

$$\bigoplus_{d=1}^{\infty} \mathcal{I}_Z^d = \mathrm{Sym}^\bullet_{\mathcal{O}_Z}(\mathcal{I}_Z)$$

We can think of this as the conormal bundle of Z in X.

Some facts:

- π is a proper birational map,
- π is an isomorphism away from Z, and $\pi^{-1}(Z)$ is the projectivised conormal bundle,
- π "enlarges" Z to a divisor.

We can use blowups to:

- resolve the indeterminancy locus of a rational map,
- resolving singularities

The only toric blowups we'll see are when Z is a point.

Theorem 10.13 (Hironaka). If X is a singular variety over a field k of characteristic 0, there exists a resolution of singularities

$$f : \widetilde{X} \to X$$

which is a finite composition of blowups.

Definition 10.14

A *resolution of singularities* for a singular variety X is a morphism $f : \widetilde{X} \to X$, such that

1. $\widetilde{X}$ is smooth,
2. f is proper and birational,

Remark 10.15. Blowups are proper and birational.

Theorem 10.16. If X_Σ is a singular toric surface, then we can find a toric resolution of singularities. That is, we have

$$X_{\widetilde{\Sigma}_n} \longrightarrow \cdots \longrightarrow X_{\widetilde{\Sigma}_1} \longrightarrow X_\Sigma$$

where $X_{\widetilde{\Sigma}_n}$ is smooth, and all morphisms are toric, proper and birational.

Lecture 12

Proposition 10.17. A morphism of toric varieties $X_{\widetilde{\Sigma}} \to X_\Sigma$ is *proper* if $\widehat{\varphi} : \widetilde{N} \to N$ has $\varphi^{-1}|\Sigma| = \left|\widetilde{\Sigma}\right|$. In particular, X_Σ is *proper* (or complete) if $|\Sigma| = N_\mathbb{R}$.

Proof. Fulton §2.4. □

So we can obtain proper morphisms by refinement of fans. That is, if we keep N fixed, and subdivide the fans, then we get a proper morphism, and birational. To see this, $f|_{T_N} = \mathrm{id}$, and T_N is a dense open, so f is a birational map.

Proof of theorem 10.16. Since X_Σ is normal, it is regular in codimension 1, and so it only has points as singularities. Now we can work locally, and assume $X_\Sigma = U_\sigma$, where $\sigma \subseteq N_\mathbb{R} = \mathbb{R}^2$ is a top-dimensional cone.

Note if U_σ did not come from a top-dimensional cone, then U_σ is isomorphic to $U_{\overline{\sigma}} \times \mathbb{C}^*$, where $U_{\overline{\sigma}}$ is a normal 1-dimensional toric variety, and so it is smooth.

So we have that the only singular point is x_σ. Say $U_\sigma = \mathrm{Cone}(v_1, v_2)$, where v_1, v_2 are minimal.

Claim 10.18. There is a toric automorphism of U_σ, such that v_1 is mapped to $(0,1)$ and v_2 is mapped to $(m, -k)$, $0 < k < m$ and $\gcd(m,k) = 1$.

Proof. This is just linear algebra. We can send $v_1 \mapsto (0,1)$ and $v_2 \mapsto (m, x)$, for $m \in \mathbb{Z}_{>0}$, with a matrix in $\mathrm{GL}_2(\mathbb{Z})$. Then change basis by

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

We get

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 0 & m \\ 1 & x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ m & x + mc \end{pmatrix}$$

For an appropriate choice of c, we can make it so that $0 \leq k < m$. Note that if $k = 0$, then $\sigma = \text{Cone}((0,1),(m,0))$, so we can pick the minimal generator $(1,0)$ of the second ray and σ is smooth, since U_σ is $\mathbb{A}^2$. But we assumed minimality, so we can assume $k > 0$.

If $\gcd(m,k) > 1$, then we can just divide by the common factor, which again contradicts minimality. ☐

Thus, we can just consider U_σ, where $\sigma = \text{Cone}((0,1),(m,-k))$. Now insert a new ray generated by $(1,0)$. **diagram**

Let $\sigma' = \text{Cone}((0,1),(1,0))$ and $\sigma_1 = \text{Cone}((m,-k))$. Then σ' is smooth, and σ_1 is "less singular" than σ. Let $\widetilde{\Sigma}_1$ be this fan. Then $X_{\widetilde{\Sigma}_1}$ is still possibly singular, but we only have to worry about σ_1. Now apply the claim and make σ_1 into the form above.

1. Apply a rotation to get
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & -k \end{pmatrix} = \begin{pmatrix} 0 & k \\ 1 & m \end{pmatrix}$$
2. apply a shear to get
$$\begin{pmatrix} 0 & k \\ 1 & -k_1 \end{pmatrix}$$
with $0 \leq k_1 < k$ and $\gcd(k, k_1) = 1$. Let $m_1 = k$.

With this, we get a new cone $\sigma_1 = \text{Cone}((0,1),(m_1,-k_1))$. Now $m_1 < m$, and $k_1 < m_1$. If $k_1 = 0$, then we are done, since $U_{\sigma_1} = \mathbb{A}^2$, and $X_{\widetilde{\Sigma}_1}$ is a toric resolution of singularities. Otherwise, we repeat the process. ☐

10.1 Quotient maps

Let $X = \text{Spec}(S)$, let G be a finite group acting on S by ring homomorphisms. Then the *ring of invariants* is

$$S^G = \{s \in S \mid gs = s \text{ for all } g \in G\}$$

The inclusion map $S^G \hookrightarrow S$ gives a morphism

$$\pi : X = \text{Spec}(S) \to \text{Spec}(S^G)$$

But we claim that the orbit space X/G is just $\text{Spec}(S^G)$. Note that this only works for G finite. To generalise, we need GIT.

Example 10.19

Let $G = C_d = \{\zeta = e^{2\pi i/d}\} \subseteq \mathbb{C}^*$ act on $\mathbb{C}[x,y]$ with weights $(1,1)$. So

$$(\zeta \cdot f)(x,y) = f(\zeta x, \zeta y)$$

Now $\mathbb{C}[x,y]^G = \mathbb{C}[x^d, x^{d-1}y, \dots, xy^{d-1}, y^d]$. So we can identify

$$\text{Spec}(\mathbb{C}[x^d, x^{d-1}y, \dots, xy^{d-1}, y^d]) = \mathbb{A}^2/G$$

This has a *cyclic quotient singularity*. Note also that

$$\mathbb{C}[x^d, x^{d-1}y, \cdots, xy^{d-1}, y^d] = \mathbb{C}[u, uv, \dots, uv^d]$$

where $u = x^d, v = y/x$. The right hand side is the ring of invariants

$$\mathbb{C}[u^{1/d}, u^{1/d}v]^G$$

where G acts with weights $(1,1)$, i.e. $(\xi \cdot f)(u^{1/d}, u^{1/d}v) = f(\xi u^{1/d}, \xi u^{1/d}v)$. So

$$(\zeta \cdot f)(u,v) = f(u, \zeta v)$$

Note this does not work for the action of $\mathbb{C}^*$ on $\mathbb{A}^2$, since

$$\mathbb{C}[x,y]^{\mathbb{C}^*} = \mathbb{C}$$

But $\mathbb{A}^2/\mathbb{C}^* \neq \mathrm{Spec}(\mathbb{C})$.

Let $\widehat{\varphi} : N' \to N$ be a morphism of lattices of finite index. That is, we have an exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N/N' \longrightarrow 0$$

where N/N' is finite. Note by the structure theorem, N/N' is a product of cyclic groups.

Theorem 10.20. Say $N' \hookrightarrow N$ is a morphism of lattices with finite index, and say $\sigma \subseteq N_{\mathbb{R}} \cong N'_{\mathbb{R}}$ is a cone. Then we have a morphism

$$f : U_{\sigma,N'} \to U_{\sigma,N}$$

and

(i) $G = N/N' \cong M'/M \cong \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*)$,

(ii) G acts on $\mathbb{C}[U_{\sigma,N'}] = \mathbb{C}[\sigma^\vee \cap M']$, and

$$\mathbb{C}[U_{\sigma,N'}]^G = \mathbb{C}[U_{\sigma,M}] = \mathbb{C}[\sigma^\vee \cap M]$$

(iii) $U_{\sigma,N} = U_{\sigma,N'}/G$.

Lecture 13

Proof. For simplicity, say we have

$$0 \longrightarrow N' = d\mathbb{Z} \longrightarrow N = \mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0$$

Dualising, we get

$$0 \longrightarrow M = \mathbb{Z} \longrightarrow M' = \tfrac{1}{d}\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0$$

Say $G = \mu_d = \mathbb{Z}/d\mathbb{Z}$. Then this acts on M' by multiplication. Taking $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^*)$, gives (as groups)

$$0 \longrightarrow \mu_d \longrightarrow \mathrm{Hom}(M', \mathbb{C}^*) = T_{N'} \longrightarrow T_N \longrightarrow 0$$

Now we can see that $N/N' \cong M'/M \cong \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*)$. In particular, given $[v] \in N/N'$, the corresponding homomorphism sends $[m] \in M'/M$ by

$$[m] \mapsto e^{2\pi i \langle v,m\rangle}$$

Now μ_d is a subgroup of $T'_{N'}$, which acts on $U_{\sigma,N'}$, and so it acts on the ring $\mathbb{C}[\sigma^\vee \cap M']$, and $\mathbb{C}[\sigma^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M]$. □

Example 10.21

Consider

$$N' = 2\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow N = \mathbb{Z}^2$$

Say we have the cone $\mathrm{Cone}((0,1),(1,-1)$ in N'. In N, this will be the cone $\mathrm{Cone}(((0,1),(2,-1)))$. Dualising,

$$0 \longrightarrow M \longrightarrow M' = \tfrac{1}{2}\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mu_2 \longrightarrow 0$$

In M, we have the dual cone $\mathrm{Cone}((1,0),(1,2))$, and so $U_{\sigma,N} = \mathrm{Spec}(\mathbb{C}[u, uv, uv^2])$. On the other hand, in M', we have

$$U_{\sigma,N'} = \mathrm{Spec}(\mathbb{C}[u, uv]) = \mathbb{A}^2$$

and we have a corresponding toric morphism.

More generally, $\widehat{\varphi} : d\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}^2$ and $\sigma = \mathrm{Cone}((0,1),(1,-1))$ gives a morphism $\mathbb{A}^2 \to \mathbb{A}^2/\mu_d$. Singularities arising in this way are called *cyclic quotient singularities*. In particular, this is the A_{d-1} singularity.

Definition 10.22

Suppose $\sigma = \mathrm{Cone}(v_1, \ldots, v_r)$, with $v_1, \ldots, v_r \in N$ and $v_1, \ldots, v_r$ being $\mathbb{R}$-linearly independent. Then we say σ is a *simplicial* cone,

In particular, from the theorem,

$$U_\sigma = \frac{\mathbb{A}^r \times (\mathbb{C}^*)^{n-r}}{G} = \frac{\mathbb{A}^r}{G} \times (\mathbb{C}*)^{n-r}$$

So U_σ has a cyclic quotient singularity.

In two dimensions, every cone is simplicial. But this is not true in higher dimensions, for example, we can have a cone in $\mathbb{R}^3$ which is generated by (at least) four rays.

11 Torus orbits and orbit-cone correspondence

Let $x \in X_\Sigma$. The *orbit* of x is

$$T \cdot x = \{t \cdot x \mid t \in T\} \subseteq X_\Sigma$$

Remark 11.1. $T \cdot x = a(T \times \{x\})$.

As usual, as a set, X_Σ is a disjoint union of torus orbits.

Example 11.2

Consider the action $(t,s)(x,y) = (tx, sy)$ of $(\mathbb{C}^*)^2$ on $\mathbb{A}^2$. Orbits are:

- $(\mathbb{C}^*)^2 \subseteq \mathbb{A}^2$, which is the orbit of $(1,1)$, the distinguished point of the origin.
- $\mathbb{C}^* \times \{0\}$, which is the orbit of $(1,0)$. This is the distinguished point of τ_2.
- $\{0\} \times \mathbb{C}^*$, which is the orbit of $(0,1)$, the distinguished point of τ_1,
- $\{(0,0)\}$, which is the distinguished point of σ, the top dimensional cone.

We can see that a smaller cone corresponds to a larger orbit, and so on.

Theorem 11.3 (orbit-cone correspondence)**.** There is a bijection

$$\{\text{cones in } \Sigma\} \leftrightarrow \{\text{torus orbits in } X_\Sigma\}$$

given by sending σ to $\mathrm{Orb}(\sigma) = T \cdot x_\sigma$, such that $\dim(\mathrm{Orb}(\sigma)) = n - \dim(\sigma)$, where $n = \mathrm{rank}(N) = \dim(X_\Sigma)$. Moreover,

$$\mathrm{Orb}(\sigma) \cong (\mathbb{C}^*)^{n-\dim(\sigma)}$$

Lemma 11.4. If $\sigma \subseteq N_\mathbb{R} \cong \mathbb{R}^n$ is a strongly convex cone of dimension k, then $\mathrm{Orb}(\sigma) = T_{N(\sigma)} \cong (\mathbb{C}^*)^{n-k}$, where $N_\sigma = \mathrm{span}_\mathbb{R}(\sigma) \cap N$ and $N(\sigma) = N/N_\sigma$.

Proof. Consider $\sigma = \overline{\sigma} \times \{0\}$, where $\overline{\sigma} \subseteq N_\sigma \otimes \mathbb{R}$ is top dimensional. Then $U_\sigma = U_{\overline{\sigma}, N_\sigma} \times U_{\{0\}, N(\sigma)}$. Then $x_\sigma = (x_{\overline{\sigma}}, x_{\{0\}}) = (0, \ldots, 0, 1, \ldots, 1)$. Hence $\mathrm{Orb}(x_\sigma) = T_{N(\sigma)} x_{\{0\}} = (\mathbb{C}^*)^{n-k}$. ◻

Corollary 11.5. x_σ is a torus fixed point if and only if σ is a top dimensional cone.

First of all, note that for all $\sigma \in \Sigma$, U_σ is fixed by T. Let $x \in U_\sigma$, or equivalently, a ring homomorphism $\mathbb{C}[S_\sigma] \to \mathbb{C}$, or a map of semigroups $S_\sigma \to \mathbb{C}$. For $t \in T_N = \mathrm{Hom}(M, \mathbb{C}^*)$, we have a corresponding $\gamma_t : M \to \mathbb{C}^*$ group homomorphism, sending m to $\chi^m(t)$.

Then $t \cdot x \in U_\sigma$ corresponds to

$$\gamma_{t\cdot x}(m) = \chi^m(t)\chi_x(m)$$

Lemma 11.6. $(\mathbb{C}^*)^{n-\dim(\sigma)} \cong \mathrm{Orb}(\sigma) = \mathrm{Hom}_{\mathrm{semigroup}}(\sigma^\perp \cap M, \mathbb{C}^*)$

Proof. Observe that $\sigma^\perp \subseteq \sigma^\vee$, and $\sigma^\perp \cap M \subseteq \sigma^\vee \cap M = S_\sigma$ is the largest lattice contained in S_σ, and has rank $n - \dim(S_\sigma)$.

Recall x_σ corresponds to the morphism $\gamma_\sigma : S_\sigma \to \mathbb{C}$, with

$$\gamma_\sigma(m) = \begin{cases} 1 & m \in \sigma^\perp \cap M \\ 0 & \text{otherwise} \end{cases}$$

Now $t \cdot x_\sigma$ corresponds to the morphism

$$\gamma_{t\cdot\sigma}(m) = \begin{cases} \chi^m(t) & m \in \sigma^\perp \cap M \\ 0 & \text{otherwise} \end{cases}$$

This shows that $\mathrm{Orb}(\sigma)$ is contained in $\mathrm{Hom}(\sigma^\perp \cap M, \mathbb{C}^*)$.

We will sketch the converse. For all morphisms of semigroups, $\gamma : \sigma^\perp \cap M \to C$, pick $t \in T_N$, such that γ sends all elements of $\sigma^\perp \cap M$ to 1, and extend by zero on points $m \in S_\sigma \setminus \sigma^\perp$. This will give γ_σ. □

To prove theorem 11.3, it remains to show that we have a bijection

$$\{\text{cones in } \Sigma\} \leftrightarrow \{\text{torus orbits in } X_\Sigma\}$$

Proof. The map sending a cone to its orbit is clear. For the reverse map, let O be an orbit. Then $O = T \cdot x$ for some $x \in X_\Sigma$. Now choose σ_x to be the smallest cone containing x. Since U_{σ_x} is closed under T-action, $O = T \cdot x \subseteq U_{\sigma_x}$. We claim that $O = \mathrm{Orb}(\sigma_x)$. Note that it suffices to show that $x \in \mathrm{Orb}(\sigma_x)$. □

Corollary 11.7.

$$U_\sigma = \bigcup_{\tau \preccurlyeq \sigma} \mathrm{Orb}(\tau)$$

Definition 11.8

The *orbit closure* of $\mathrm{Orb}(\sigma)$ in X_Σ is the Zariski closure

$$V(\sigma) = \overline{\mathrm{Orb}(\sigma)}$$

Lemma 11.9. The correspondence in theorem 11.3 is order reversing. That is, for $\tau, \sigma \in \Sigma$,

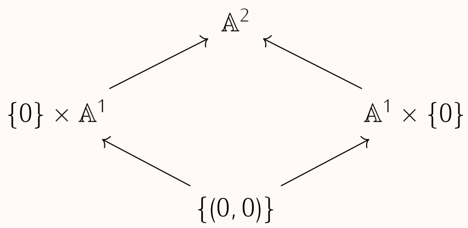
(i) $\tau \preccurlyeq \sigma$ if and only if $\mathrm{Orb}(\sigma) \subseteq V(\tau)$,

(ii) $V(\sigma) = \bigcup_{\sigma \preccurlyeq \tau} \mathrm{Orb}(\tau)$.

Example 11.10
For $\mathbb{A}^2$, the orbits are:

- $\{(0,0)\}$,
- $\{0\} \times \mathbb{C}^*$,
- $\mathbb{C}^* \times \{0\}$,
- $(\mathbb{C}^*)^2$,

and the orbit closures are



Example 11.11
Consider $\mathrm{Bl}_0(\mathbb{A}^2)$. Say τ_3 the new edge we add. $\mathrm{Orb}(\tau_3) = \mathbb{C}^*$, σ_1 and σ_2 lie above τ_3, and their orbits are a point. Thus,

$$V(\tau_3) = \mathbb{C}^* \cup \{\mathrm{pt}\} \cup \{\mathrm{pt}\} = \mathbb{P}^1$$

In particular, $V(\sigma)$ is a toric variety, and a toric subvariety of X_Σ. Let $N_\sigma = \mathrm{span}_{\mathbb{R}}\{\sigma\} \cap N$. Then we have the short exact sequence

$$0 \longrightarrow N_\sigma \longrightarrow N \longrightarrow N(\sigma) \longrightarrow 0$$

If τ is a cone which contains σ as a face, let $\overline{\tau}$ be the image of τ under the projection $\pi : N_{\mathbb{R}} \to N(\sigma) \otimes \mathbb{R}$. The collection

$$\mathrm{Star}(\sigma) = \{\overline{\tau} \mid \sigma \preccurlyeq \tau\}$$

defines a fan in $N(\sigma) \otimes \mathbb{R} \cong \mathbb{R}^{n-\dim(\sigma)}$.

Theorem 11.12. For any cone $\sigma \in \Sigma$,

$$V(\sigma) = X_{\mathrm{Star}(\sigma)}$$

is a toric subvariety of X_Σ.

Proof sketch. For $\sigma \preccurlyeq \tau$, we claim that $\overline{\tau}$ is a strongly convex cone in $N(\sigma)_{\mathbb{R}}$, which corresponds to the variety

$$U_{\overline{\sigma}} = \mathrm{Spec}(\mathbb{C}[\overline{\tau}^\vee \cap N(\sigma)^\vee]) = \mathrm{Spec}(\mathbb{C}[\tau^\vee \cap \sigma^\perp \cap M])$$

and the closed embedding is given by

$$\mathbb{C}[\tau^\vee \cap M] \twoheadrightarrow \mathbb{C}[\tau^\vee \cap \sigma^\perp \cap M]$$
$$\chi^m \mapsto \begin{cases} \chi^m & m \in \sigma^\perp \\ 0 & \text{otherwise} \end{cases}$$

We need to check that these local closed maps glue to a closed embedding $X_{\mathrm{Star}(\sigma)} \hookrightarrow X_\Sigma$, and that the iamge of $X_{\mathrm{Star}(\sigma)}$ is $V(\sigma)$. $\square$

Example 11.13
For $\mathbb{P}^2$, we have

$$V(\tau_2) = X_{\text{Star}(\tau_2)} \cong \mathbb{P}^1 \subseteq \mathbb{P}^2$$

as $\mathbb{V}(z_2)$. Recall $X_\Sigma \cong \mathbb{P}^2$, with

$$\begin{aligned} U_{\sigma_0} &= \text{Spec}(\mathbb{C}[x, y]) \\ U_{\sigma_1} &= \text{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]) \\ U_{\sigma_2} &= \text{Spec}(\mathbb{C}[y^{-1}, y^{-1}x]) \end{aligned}$$

The homogeneous coordinates on $\mathbb{P}^2$ are $[z_0 : z_1 : z_2]$, which corresponds to $x = z_1/z_0, y = z_2/z_0$. So $U_{\sigma_i} = \{z_i \neq 0\}$ in $\mathbb{P}^2$. Now $U_0 \cong \mathbb{A}^2 \subseteq \mathbb{P}^2$, by mapping $(x, y) \mapsto [1 : x : y]$, and so on.

Now we focus on $\tau_2 = \mathbb{R}_{\geq 0} \cdot e_2$. The short exact sequence is

$$0 \longrightarrow N_{\tau_2} = \mathbb{Z} \cdot e_2 \longrightarrow N = \mathbb{Z} \cdot e_1 \oplus \mathbb{Z} \cdot e_2 \longrightarrow N(\tau_2) = \mathbb{Z} \cdot e_1 \longrightarrow 0$$

Now $\text{Star}(\tau_2)$ is a fan in $\mathbb{Z} \cdot e_1 \otimes \mathbb{R}$, given by

$$\{\overline{\tau_2} = \{0\}, \overline{\sigma_0} = \text{Cone}(e_1), \overline{\sigma_1} = \text{Cone}(-e_1)\}$$

Lecture 15

Example 11.14 (example continued)
Thus, $X_{\text{Star}(\tau_2)} \cong \mathbb{P}^1$, and we have a morphism $N(\tau_2) = \mathbb{Z} \cdot e_2 \hookrightarrow \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, which gives the embedding $X_{\text{Star}(\tau_2)} \hookrightarrow X_\Sigma$. So we can check that $X_{\text{Star}(\tau_2)}$ is embedded as

$$[w_0 : w_1] \mapsto [w_0 : w_1 : 0]$$

On the other hand,

$$V(\tau_2) = \overline{T \cdot x_{\tau_2}}$$

but $x_{\tau_2} = [1 : 1 : 0]$, and by the orbit-cone correspondence

$$\text{Orb}(\tau_2) = (\mathbb{C}^*)^2 \subseteq \mathbb{P}^2(s, t) \qquad \mapsto [s : t : 0]$$

and so

$$\begin{aligned} V(\tau_2) = \mathbb{C}^* \times \mathbb{A}^1 &\subseteq \mathbb{P}^2 \\ (t, z) &\mapsto [1 : t : z] \end{aligned}$$

X_Σ is *stratified* by orbit closures $V(\tau)$ correspond to torus invariant[4] subvarieties. Moreover, $\text{codim}(V(\tau)) = \dim(\tau)$.

12 Divisors

Assume that X is separated, Noetherian, normal in codimension 1. In particular, for any toric variety which comes from a cone, $X = X_\Sigma$ satisfies the above requirements.

The group of *Weil divisors* is

$$\text{Div}(X) = \bigoplus_{V \subseteq X \text{ codimension 1 subvariety}} \mathbb{Z} \cdot V$$

The *principal divisors* is

$$\text{Div}_0(X) = \{\text{div}(f) \mid f \in \Gamma(\mathcal{K}_X^*)\}$$

[4] i.e. $TV \subseteq V$

where

$$\operatorname{div}(f) = \sum_{V \subseteq X} \operatorname{ord}_V(f) \cdot V$$

where $\operatorname{ord}_V(f)$ is the order of vanishing of f on V, and $\mathcal{K}_X^*$ is the sheaf of non-zero rational functions on X. Recall $\mathcal{O}_{X,V}$ is a discrete valuation ring, and we have a corresponding discrete valuation $\nu : \Gamma(\mathcal{K}^*) \to \mathbb{Z}$.

Example 12.1

For $X = \operatorname{Spec}(\mathbb{C}[X]) = \mathbb{A}^1$, the origin is a codimension 1 subvariety, and $\mathcal{O}_{\mathbb{A}^1,0} = \mathbb{C}[x]_{\langle x \rangle}$. This is a DVR, and it induces a valuation $\mathbb{C}(t) \to \mathbb{Z}$, which is the order of vanishing at 0.

The *Weil class group* is

$$\operatorname{Cl}(X) = \frac{\operatorname{Div}(X)}{\operatorname{Div}_0(X)} = A_{n-1}(X)$$

Note there is a higher dimensional generalisation, producting a group $A_\bullet(X)$ called the *Chow group*.

Fact: If $X = \operatorname{Spec}(R)$ where R is a UFD, then $\operatorname{Cl}(X) = 0$.

Thus, $\operatorname{Cl}(\mathbb{A}^n)$ and $\operatorname{Cl}((\mathbb{C}^*)^n)$ are zero.

If $X = X_\Sigma$, the group of *T-invariant Weil divisors* is

$$\operatorname{Div}_T(X) = \bigoplus_{V \subseteq X_\Sigma \text{ codim. 1 subvar., } T \cdot V \subseteq V} \mathbb{Z} \cdot V = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot V(\rho) = \mathbb{Z}^{\Sigma(1)}$$

Let $f \in \Gamma(\mathcal{K}_{X_\Sigma}^*)$, then $\operatorname{div}(f) \in \operatorname{Div}_T(X_\Sigma)$ if and only if $f|_{T_N} \in \Gamma(\mathcal{O}_{T_N}^*)$ has no zeroes or poles.

But this is the same as morphisms $T_N \to \mathbb{C}^*$, and so this is the same as $f = \chi^m$ for some $m \in M$. That is, we have a morphism $M \to \operatorname{Div}_0(X) \cap \operatorname{Div}_T(X)$, sending m to $\operatorname{div}(\chi^m)$. Recall

Theorem 12.2 (Excision)**.** Let $U \subseteq X$ be open, $Z = X \setminus U$. Write $Z = Z_1 \cup \cdots \cup Z_s \cup W$, where Z_i are the codimension 1 irreducible components, and W consists of the higher codimension components. Then we have an exact sequence

$$\mathbb{Z}^s \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0$$

Apply this to a toric variety $X = X_\Sigma$ and $U = T_N$. Then the codimension 1 irreducible components of $X_\Sigma \setminus T_N$, which are the $V(\rho)$ for $\rho \in \Sigma(1)$. We also know that $\operatorname{Cl}(T_N) = 0$, and so we get a surjection

$$\operatorname{Div}_T(X) = \mathbb{Z}^{\Sigma(1)} \twoheadrightarrow \operatorname{Cl}(X_\Sigma)$$

Thus, we have a sequence

$$M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \operatorname{Cl}(X_\Sigma) \longrightarrow 0 \tag{i}$$

Let us look at the morphism $M \to \operatorname{Div}_T(X_\Sigma)$ more closely.

Lemma 12.3.

$$\operatorname{ord}_{V(\rho)}(\chi^m) = \langle v_\rho, m \rangle$$

where v_ρ is the minimal generator of ρ.

Proof. We can work locally in $U_\rho \cong \mathbb{A}^1 \times (\mathbb{C}^*)^{n-1}$, by considering $V(\rho) \cap U_\rho$. Now consider the composite map

$$\mathbb{C} \xrightarrow{\lambda_{v_\rho}} U_\rho \xrightarrow{\chi^m} \mathbb{C}$$

This sends x to x^d, where $d = \langle v_\rho, m \rangle$. Now

$$\operatorname{ord}_{v_\rho}(\chi^m) = \operatorname{ord}_0(\chi^m \circ \lambda_{v_\rho}) = \langle v_\rho, m \rangle$$

∎

Then

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle v_\rho, m \rangle \cdot V(\rho) \in \operatorname{Div}_T(X)$$

If $\operatorname{span}_{\mathbb{R}}(\Sigma(1)) = N_{\mathbb{R}}$, or equivalently if X_Σ has no torus factor, then

$$\operatorname{div}(\chi^m) = 0 \iff \langle v_\rho, m \rangle = 0$$

for all $\rho \in \Sigma(1)$. In turn, this is true if and only if $m = 0$. In this case, the sequence (i) is exact on the left. So we get

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \operatorname{Cl}(X) \longrightarrow 0 \tag{ii}$$

Lecture 16

12.1 Examples

Let X be a toric variety, $T \subseteq X$ the dense torus. In this case, we have a supply of

- integral closed codimension 1 subschemes, from D_ρ, for $\rho \in \Sigma(1)$,
- for $m \in M$, we have $\chi^m : T \to \mathbb{C}^*$.

and from the above, we have

$$\operatorname{Cl}(X) = \frac{\bigoplus_\rho \mathbb{Z} \cdot \rho}{\operatorname{im}(M \to \bigoplus_\rho \mathbb{Z} \cdot \rho)}$$

where $m \mapsto \sum \langle v_\rho, m \rangle \rho$.

Example 12.4

Recall that we know $\operatorname{Cl}(\mathbb{P}^n) = \mathbb{Z}$, where 1 is sent to a hyperplane class. In this case, the rays in the fan of $\mathbb{P}^n$ are generated by $e_1, \dots, e_n, e_{n+1} = -e_1 - \cdots - e_n \in N \cong \mathbb{Z}^n$. By (ii), we have

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^{n+1} \mathbb{Z} \cdot \rho_i \longrightarrow \operatorname{Cl}(\mathbb{P}^n) \longrightarrow 0$$

In this case,

$$m \mapsto \sum_i \langle m, e_i \rangle \rho_i$$

By choosing the dual basis $e_i^\vee$ for M, the image of $e_i^\vee$ is $\rho_i - \rho_{n+1}$. Thus, the cokernel can be identified with $\mathbb{Z} \cdot \rho_{n+1}$ (or any $Z \cdot \rho_i$).

A similar computation shows thwt $\operatorname{Cl}(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathbb{Z}^2$.

For nice schemes, we know that the class group is a finitely generated abelian group. Can it have torsion? Yes, say if we take $U = \mathbb{P}^n \setminus X_d$, where X_d is a degree d irreducible hypersurface. In this case, $\operatorname{Cl}(U) = \mathbb{Z}/d\mathbb{Z}$. Is there a toric example?

Example 12.5

Let

$$\widehat{C}_2 = \operatorname{Spec}\left(\frac{\mathbb{C}[x, y, z]}{\langle x^2 - yz \rangle}\right)$$

$\widehat{C}_2$ is toric, and a possible fan is given by

$$\operatorname{Cone}((2, -1), (0, 1))$$

Choose basis e_1, e_2 of N, $v_1 = e_2, v_2 = 2e_1 - e_2$. The exact sequence (ii) is as follows

$$0 \longrightarrow \mathbb{Z} \cdot e_1^\vee \oplus \mathbb{Z} \cdot e_2^\vee \xrightarrow{A} \mathbb{Z} \cdot v_1 \oplus \mathbb{Z} \cdot v_2 \longrightarrow \operatorname{Cl}(\widehat{C}_2) \longrightarrow 0$$

where

$$A(e_1^\vee) = 2v_1$$
$$A(e_2^\vee) = v_1 - v_2$$

Thus, in the class group

$$[D_1] - [D_2] = [\mathrm{div}(\chi^{e_2^\vee})]$$
$$2[D_1] = [\mathrm{div}(\chi^{e_1^\vee})]$$

Thus, $\mathrm{Cl}(\widehat{C}_2) = \mathbb{Z}/2\mathbb{Z}$.

Warning: Calculating the cokernel of an integer matrix requires calculating Smith normal form.
End of examinable material

13 *Line bundles*

Let $\mathrm{Pic}(X)$ be the group of line bundles on X, up to isomorphism, with group operation $\otimes$. Given a line bundle $\mathcal{L}$, let

$$\mathcal{L}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$$

denote the *dual*, which is the inverse of $\mathcal{L}$, and $\mathcal{O}_X$ is the identity element. For toric varieties (or anything which is integral over a field),

$$\mathrm{Pic}(X) \cong \mathrm{CaCl}(X) = \frac{\mathrm{CaDiv}(X)}{\mathrm{Prin}(X)}$$

which is the Cartier class group. If X is normal, then the group of Cartier divisors $\mathrm{CaDiv}(X)$ is a subgroup of $\mathrm{Div}(X)$, where the image correspond to the Weil divisors which are locally principal. That is, divisors D such that there exists an open cover U_i, such that

$$D|_{U_i} = \mathrm{div}(f_i)$$

For X toric, $\mathrm{Pic}(X) = \mathrm{CaCl}(X)$, we have the following

$$\begin{array}{ccccccc} M & \longrightarrow & \mathrm{CaDiv}_T(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \\ M & \longrightarrow & \mathrm{Div}_T(X) & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0 \end{array}$$

How do we tell when $D \in \mathrm{Div}_T(X)$ is Cartier?

Remark 13.1. If X is smooth, then $\mathrm{CaDiv}_T(X) = \mathrm{Div}_T(X)$.

In general, to find $\mathrm{Pic}(X)$, we need to answer this question.

Lemma 13.2 (Fulton chapter 3, CLS chapter 4)**.** Let U_σ be affine toric, with cone σ. Then

$$\mathrm{CaDiv}_T(U_\sigma) = \{\mathrm{div}(\chi^m) \mid m \in S_\sigma\}$$

Thus, $D \sim 0$ for all $D \in \mathrm{CaDiv}_T(U_\sigma)$, and $\mathrm{CaCl}(U_\sigma) = \mathrm{Pic}(U_\sigma) = 0$.

Example 13.3
Say $X_\Sigma = \mathbb{V}(xy - zw) \subseteq \mathbb{A}^3 \subseteq \mathbb{P}^3$. In this case,

$$\mathrm{Div}_T(X_\Sigma) = \bigoplus_i \mathbb{Z} \cdot \rho_i$$

For $m \in M$, then

$$\operatorname{div}(\chi^m) = \sum \langle m, v_i \rangle D_{\rho_i}$$

Theorem 13.4. If X_Σ is toric, then

$$\operatorname{Pic}(X_\Sigma) = \frac{\{\varphi : |\Sigma| \to \mathbb{R} \mid \text{continuous and restricts to a linear function on each } \sigma \in \Sigma\}}{M = \text{globally linear function}}$$

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