Algebraic Geometry

Shing Tak Lam

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Lecture 1

0.1 Plan

1. Basics of sheaves of topological spaces
2. Definitions of schemes and of morphisms
3. Properties of schemes (the algebraic geometry analogues of compactness)
4. Rapid introduction to the cohomology of sheaves

0.2 Resources

1. Course webpage - Google Dhruv
2. Books:
   - Hartshorne - Algebraic Geometry, Chapter 2 and $\varepsilon \times$Chapter 3
   - Vakil - Rising Sea, good online (free) alternative
   - Eisenbud-Harris - Geometry of schemes
3. Four examples sheets on webpage
4. SAGES - Look on Dhruv's webpage

0.3 Why scheme theory?

Moduli theory – It is better to study families of varieties than one at a time. Or better, all varieties of a given type at the same time.

Examples of moduli: The set of all lines in $\mathbb{P}^2$. A line in $\mathbb{P}^2$ is

$$\{aX_0 + bX_1 + cX_2 = 0\}$$

In particular, as lines are paramterised by the triple (a, b, c), we have a correspondence

$$\mathbb{P}^2_{\text{dual}} = \text{lines in } \mathbb{P}^2 \leftrightarrow \mathbb{P}^2$$

The same logic applies for degree d hypersurface in $\mathbb{P}^n$, and we get

$$\text{degree } d \text{ hypersufaces in } \mathbb{P}^n \leftrightarrow \mathbb{P}^N$$

where $N = \binom{n+d}{d} - 1$. However, there is something wrong with this picture. Some polynomials are of the form

$$f = f_1^2 f_2$$

But in this case,

$$\mathbb{V}(f) = \mathbb{V}(f_1 f_2)$$

and $f_1 f_2$ is (in general) not of degree d.

A solution Take $U_d \subseteq \mathbb{P}^N$, where

$$[f] \in U_d \iff f \text{ has no repeated factors}$$

But U_d is not compact.

Output of scheme theory for a fixed projective space $\mathbb{P}^n$, we obtain a space

$$\text{subvarieties of } \mathbb{P}^n \leftrightarrow \text{Var}(\mathbb{P}^n) \subsetneq \text{Hilb}(\mathbb{P}^n) \leftrightarrow \text{ subschemes of } \mathbb{P}^n$$

where $\text{Hilb}(\mathbb{P}^n)$ is the Hilbert scheme of $\mathbb{P}^n$, which is compact in the Euclidean topology. That is, the limit of varieties need not be a variety, but limits of schemes are always schemes.

In scheme theory,

$$\mathbb{V}(X_0 + X_1 + X_2) \text{ and } \mathbb{V}((X_0 + X_1 + X_2)^2)$$

are not isomorphic as schemes.

Weil conjectures Fix $f \in \mathbb{Z}[X_0, \dots, X_{n+1}]$ homogeneous. We have

$$X = \mathbb{V}(f) \subseteq \mathbb{P}^{n+1}_{\mathbb{C}}$$

We will assume that X is smooth. That is, X is a (complex) manifold. In particular, X is a compact topological space, and so we have numbers $b_0(X), \dots, b_{2n}(X)$ called the Betti numbers, where

$$b_i(X) = \text{rank}(H_i(X, \mathbb{Z}))$$

In particular, we have the Euler characteristic $\chi(X)$.

On the other hand, fix a prime number p, and let

$$N_m = \text{number of solutions to } f \text{ in } \mathbb{F}_{p^m} = \left|X(\mathbb{F}_{p^m})\right|$$

Define the Weil ζ function

$$\zeta(X; t) = \exp\left(\sum_m \frac{N_m}{m} t^m\right)$$

Theorem 0.3.1 (Weil conjectures, Grothendieck).

1. $\zeta(X; t)$ is a rational function in t
2. Moreover,

$$\zeta(X; t) = \frac{P_0(t)P_2(t)\cdots P_{2n}(t)}{P_1(t)P_3(t)\cdots P_{2n-1}(t)}$$

   where $\deg(P_i(t)) = b_i(X)$.

Upshot: The proof is (fundamentally) via scheme theory. We need a space $\mathfrak{X}$ which interpolates geometry over $\mathbb{C}$ and geometry over finite fields.

Lecture 2

1 Beyond varieties

1.1 Summary of classical algebraic geometry

Let $k = \overline{k}$ be an algebraically closed field. Define the *affine space*

$$\mathbb{A}^n_k = k^n$$

as a set.

An *affine variety* is a subset

$$V = \mathbb{V}(S) \subseteq \mathbb{A}^n_k$$

for some $S \subseteq k[x_1, \dots, x_n]$. Note that $\mathbb{V}(S) = \mathbb{V}(\langle S \rangle)$, where $\langle S \rangle$ is the ideal generated by S. By the Hilbert basis theorem, or equivalently, $k[x_1, \dots, x_n]$ is Noetherian, $\langle S \rangle$ is generated by finitely many elements. Moreover,

$$\mathbb{V}(I) = \mathbb{V}\left(\sqrt{I}\right)$$

where I is the *radical* of I, given by

$$\sqrt{I} = \{f \mid f^m \in I \text{ for some } m \in \mathbb{N}\}$$

Given varieties $V \subseteq \mathbb{A}^n_k$, $W \subseteq \mathbb{A}^m_k$, a *morphism* $\varphi : V \to W$ is a function $V \to W$, such that if we write $\varphi = (f_1, \dots, f_m)$, each f_i is a restriction of a polynomial in $k[x_1, \dots, x_n]$. An *isomorphism* $\varphi : V \to W$ is a morphism, such that $\psi : W \to V$ also a morphism, $\psi \circ \varphi = \text{id}_V$, $\varphi \circ \psi = \text{id}_W$.

The basic correspondence is

$$\frac{\text{affine varieties over } k}{\text{isomorphism}} \leftrightarrow \text{finitely generated } k\text{-algebras } A \text{ without nilpotent elements}$$

<u>How?</u> Given a variety V (representing an isomorphism class of such), then we can write $V = \mathbb{V}(I)$. Moreover, we can assume that I is a radical ideal. We map

$$V \mapsto \frac{k[x_1, \ldots, x_n]}{I}$$

Conversely, if A is a finitely generated nilpotent-free k-algebra, then by definition,

$$A \cong \frac{k[y_1, \ldots, y_n]}{J}$$

where J is a radical ideal.

Note that we need to check that both maps are well-defined.

> **Notation 1.1.1.** The algebra associated to V is (classically) denoted $k[V]$, and called the *coordinate ring of V*.

Compatibility of morphisms

Note that we have a correspondence

$$\{\text{morphisms } V \to W\} \leftrightarrow \{k\text{-algebra homomorphisms } k[W] \to k[V]\}$$

Zariski topology

Suppose $V = \mathbb{V}(I) \subseteq \mathbb{A}^n_k$ is an affine variety, with coordinate ring $k[V]$. The *Zariski topology* has closed sets

$$V \cap \mathbb{V}(S)$$

where $S \subseteq k[x_1, \ldots, x_n]$.

If $V \cong W$ as varieties, then the topological spaces with the Zariski topology are also homeomorphic.

Nullstellensatz

Fix a variety V, and let $k[V]$ be its coordinate ring. Given a point $p \in V$, we can produce a homomorphism

$$\begin{aligned} \mathrm{ev}_p : k[V] &\to k \\ \mathrm{ev}_p(f) &= f(p) \end{aligned}$$

Moreover, ev_p is surjective, for example, by taking the constant functions. With this,

$$\mathfrak{m}_p = \ker(\mathrm{ev}_p) \trianglelefteq k[V]$$

is a maximal ideal. So we get a map

$$\{\text{points } p \in V\} \to \{\text{maximal ideals in } k[V]\}$$

<u>Hilbert's Nullstellensatz</u> $\implies$ the above map is a bijection.

Therefore, points of V and maximal ideals in $k[V]$ are "the same thing".

1.2 Limitations

What is an abstract variety?

That is, what is a topological space X, such that we have an open cover $\{U_i\}$, where each U_i is an affine variety, which is compatible on overlaps? For example, we have projective space $\mathbb{P}^n$.

Example 1.2.1 (non-algebraically closed fields)
Take the ideal
$$I = \left\langle x^2 + y^2 + 1 \right\rangle \trianglelefteq \mathbb{R}[x, y]$$
Observe $\mathbb{V}(I) = \varnothing \subseteq \mathbb{R}^2$, but I is prime, so radical. Hence the Nullstellensatz fails in this case.

On what topological space is $R[x, y]/\left\langle x^2 + y^2 + 1 \right\rangle$ "naturally" the set of functions?

Or more generally, $\mathbb{Z}, \mathbb{Z}[x]$ and so on.

Example 1.2.2 (why restrict to radical ideals, or nilpotent free algebras)
Let $C = \mathbb{V}(y - x^2) \subseteq \mathbb{A}_k^2$, and $D = \mathbb{V}(y) \subseteq \mathbb{A}_k^2$. Then
$$C \cap D = \mathbb{V}(x, y) = \{(0, 0)\}$$
If we set $D_\delta = \mathbb{V}(y - \delta)$, then $C \cap D_\delta$ is two points, for all $\delta \neq 0$.

In this case, what happens is that $C \cap D_0$ is one point, but it has multiplicity 2. Variety theory cannot tell the two apart, which is why we need scheme theory.

Lecture 3

1.3 Spectrum of a ring

Let A be a (commutative, unital) ring.

Definition 1.3.1 (Zariski spectrum)
The *Zariski spectrum of A* is
$$\operatorname{Spec} A = \{\mathfrak{p} \trianglelefteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

Given a ring homomorphism $\varphi : A \to B$, we have a induced map (of sets)
$$\begin{aligned} \varphi^{-1} : \operatorname{Spec}(B) &\to \operatorname{Spec} A \\ \mathfrak{q} &\mapsto \varphi^{-1}(\mathfrak{q}) \end{aligned}$$

Warning: This would fail if we considered maximal ideals instead. That is, the preimage of a maximal ideal need not be maximal. For example, consider the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$. The maximal ideal $\langle 0 \rangle$ in $\mathbb{Q}$ has preimage $\langle 0 \rangle$ in $\mathbb{Z}$, which is not maximal.

Given $f \in A$ and $\mathfrak{p} \in \operatorname{Spec} A$, we have an induced
$$\overline{f} \in A/\mathfrak{p}$$
by taking the quotient. Informally, we can evaluate any $f \in A$ at points $\mathfrak{p} \in \operatorname{Spec}(A)$, with the caviat that the codomain of the evaluation map depends on $\mathfrak{p}$.

Example 1.3.2
Let $A = \mathbb{Z}$, then
$$\operatorname{Spec}(\mathbb{Z}) = \{p \mid p \text{ a prime number}\} \cup \{0\}$$
Choose an element, say $132 \in \mathbb{Z}$. Given a prime p, we can look at $132 \mod p \in \mathbb{Z}/p$.

Takeaway:

$$\mathrm{Spec}(\mathbb{Z}) \rightsquigarrow \text{space}$$

$$132 \in \mathbb{Z} \rightsquigarrow \text{a function}$$

$$132 \mod p \rightsquigarrow \text{value of the function at } p$$

Example 1.3.3

$A = \mathbb{R}[x]$, then

$$\mathrm{Spec}\,\mathbb{R}[x] = \frac{\mathbb{C}}{\text{complex conjugation}} \cup \{0\}$$

Exercise: Draw $\mathrm{Spec}\,\mathbb{Z}[x]$ and $\mathrm{Spec}\,k[x]$ for any field k.

Example 1.3.4 (for sanity)

If $A = \mathbb{C}[x]$, then

$$\mathrm{Spec}\,A = \mathbb{C} \cup \{0\}$$

Given $a \in \mathbb{C}$, we have the maximal ideal $\langle z - a \rangle$.

1.4 Topology

Fix $f \in A$, then define

$$\mathbb{V}(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid f \equiv 0 \pmod{\mathfrak{p}}\} \subseteq \mathrm{Spec}(A)$$

or equivalently, $f + \mathfrak{p} = 0 \in A/\mathfrak{p}$, or $f \in \mathfrak{p}$. Similarly, for $J \trianglelefteq A$ an ideal,

$$\mathbb{V}(J) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid J \subseteq \mathfrak{p}\}$$

Proposition 1.4.1. The sets $\mathbb{V}(J) \subseteq \mathrm{Spec}\,A$ ranging over all ideals $J \trianglelefteq A$, form the closed sets of a topology on $\mathrm{Spec}\,A$. This topology is called the *Zariski topology*.

Proof. Easy facts that $\varnothing = \mathbb{V}(1)$ and $\mathrm{Spec}\,A = \mathbb{V}(0)$ are closed. Since

$$\mathbb{V}\left(\sum_\alpha I_\alpha\right) = \bigcap_\alpha \mathbb{V}(I_\alpha)$$

the intersections of closed sets is closed. Finally,

$$\mathbb{V}(I_1 \cap I_2) = \mathbb{V}(I_1) \cup \mathbb{V}(I_2)$$

$\supseteq$ is clear, conversely, $I_1 I_2 \subseteq I_1 \cap I_2 \subseteq \mathfrak{p}$, then by primality of $\mathfrak{p}$, $I_1 \subseteq \mathfrak{p}$ or $I_2 \subseteq \mathfrak{p}$. □

Example 1.4.2

Consider $\mathrm{Spec}\,\mathbb{C}[x, y]$. A few observations:

- $0 \in \mathrm{Spec}\,\mathbb{C}[x, y]$ is dense in the Zariski topology. That is,

  $$\overline{\{\langle 0 \rangle\}} = \mathrm{Spec}\,\mathbb{C}[x, y]$$

  since every prime ideal contains 0. This is true in $\mathrm{Spec}(A)$ for any integral domain A.

- Consider the prime ideal $\langle y^2 - x^3 \rangle$. Consider a maximal ideal

$$\mathfrak{m}_{a,b} = \langle x - a, y - b \rangle$$

When is $\mathfrak{m}_{a,b} \in \overline{\{\langle y^2 - x^3 \rangle\}}$? This holds if and only if $b^2 = a^3$. See examples sheet 1.

Remark 1.4.3. Points are not closed in general. In fact, we have that the closure of the point $\mathfrak{p}$ is $\mathbb{V}(\mathfrak{p})$, which is $\{\mathfrak{p}\}$ if and only if $\mathfrak{p}$ is maximal. That is, closed points correspond to maximal ideals.

1.5 Functions on opens

Let $f \in A$. Define the *distinguished open corresponding to f* to be

$$U_f = \operatorname{Spec}(A) \setminus \mathbb{V}(f)$$

Hartshorne uses the notation $D(f)$, which seems to be a bit more common?

Example 1.5.1

If $A = \mathbb{C}[x]$, then by the fundamental theorem of algebra ,

$$\operatorname{Spec} A = \mathbb{C} \cup \{\langle 0 \rangle\}$$

Take $f = x$. We have a bijection

$$\begin{aligned} \operatorname{Spec}(A) &\leftrightarrow \mathbb{C} \cup \{\langle 0 \rangle\} \\ (x - a) &\leftarrow\!\shortmid a \in \mathbb{C} \\ 0 &\leftarrow\!\shortmid 0 \end{aligned}$$

Then $\mathbb{V}(x) = \{\mathfrak{p} \in \operatorname{Spec} A \mid x \in \mathfrak{p}\} = \{\langle x \rangle\}$. So

$$U_x = \operatorname{Spec}(A) \setminus \{\langle 0 \rangle\}$$

As a picture, $\operatorname{Spec}(\mathbb{C}[x])$ is a line (corresponding to $\mathbb{C}$) with a "generic point" $\{0\}$.

More generally, suppose we fix $a_1, \ldots, a_r \in \mathbb{C}$, with

$$U = \operatorname{Spec}(A) \setminus \{\langle x - a_i \rangle\}_{i=1}^r$$

then $U = U_f$ where $f = \prod_{i=1}^r (x - a_i)$.

Lemma 1.5.2. The distinguished opens U_f for all $f \in A$ form a basis for the Zariski topology on $\operatorname{Spec} A$.

Proof. Examples sheet 1. Given any open set $U \subseteq \operatorname{Spec}(A)$, write $Z = \operatorname{Spec}(A) \setminus U$. Since Z is closed, we must have that $Z = \mathbb{V}(J)$ for some ideal J. But then we have that

$$U = \bigcup_{f \in J} U_f$$

□

A bit of commutative algebra: Given $f \in A$, the *localisation of A at f* is

$$A_f = \frac{A[x]}{\langle xf - 1 \rangle} = A\left[\frac{1}{f}\right]$$

Lemma 1.5.3. The distinguished open $U_f \subseteq \mathrm{Spec}(A)$ is naturally homeomorphic to $\mathrm{Spec}(A_f)$, via the ring homomorphism

$$j : A \to A_f$$

which induces

$$j^{-1} : \mathrm{Spec}(A_f) \to \mathrm{Spec}\, A$$

Lecture 4

Proof. Primes in A_f are in bijection with primes of A, which miss f, via j^{-1}. To see this, the preimage of a prime ideal is prime, and so if $\mathfrak{q} \trianglelefteq A_f$ is prime, then $j^{-1}(\mathfrak{a}) \trianglelefteq A$ is prime. For the converse, given $\mathfrak{p} \trianglelefteq A$, define

$$\mathfrak{p}_f = j(\mathfrak{p})A_f = \langle j(\mathfrak{p}) \rangle$$

Claim 1.5.4. $\mathfrak{p}_f$ is a prime exactly when $f \notin \mathfrak{p}$.

Proof of claim. To see this, f is a unit in A_f, and so if $f \in \mathfrak{p}$, then $\mathfrak{p}_f$ contains a unit, and so $\mathfrak{p}_f = 1$.

But if $f \notin \mathfrak{p}$, then

$$\frac{A_f}{\mathfrak{p}_f} \cong \left(\frac{A}{\mathfrak{p}}\right)_{\overline{f}}$$

where $\overline{f} = f + \mathfrak{p}$. So

$$(A/\mathfrak{p})_{\overline{f}} \subseteq \mathrm{Frac}(A/\mathfrak{p})$$

and so it is an integral domain. □

Finally, we can check that this defines a bijection, but this is just basic properties of ideals. □

This is also done in Commutative Algebra, although some work is needed to specialise to this case.

Facts about distinguised opens

- $U_f \cap U_g = U_{fg}$. To see this, note that for any prime $\mathfrak{p}$, $fg \in \mathfrak{p}$ if and only if $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.
- $U_{f^n} = U_f$ for all $n \geq 1$. This follows by repeated application of the above.
- The rings A_f and A_{f^n} (for $n \geq 1$) are isomorphic. To see this,

  $$A_f = \frac{A[x]}{\langle xf - 1 \rangle} \quad \text{and} \quad A_{f^n} = \frac{A[y]}{\langle yf^n - 1 \rangle}$$

  The isomorphism is given by

  $$y \mapsto x^n \quad \text{and} \quad x \mapsto f^{n-1}y$$

  which define inverse maps. Informally, inverting f and inverting f^n are the same, since we can write $f^{-n} = (f^{-1})^n$, and $f^{-1} = f^{n-1}(f^n)^{-1}$.
- Containment: $U_f \subseteq U_g$ if and only if f^n is a multiple of g for some $n \geq 1$. (Recall if $f = gf'$, then certainly $U_f \subseteq U_g$)

  Proof. The "if" direction is clear. Conversely, if $U_f \subseteq U_g$. That is,

  $$\mathbb{V}(f) \supseteq \mathbb{V}(g)$$

  The set $\mathbb{V}(f)$, by definition, is the set of all primes, containing f.

  Claim 1.5.5. $\sqrt{\langle f \rangle} \subseteq \sqrt{\langle g \rangle}$.

  To see this, the radical of an ideal I is the intersection of all primes containing I, which we will see in Commutative Algebra. □

Foreshadowing

Fix A, we've made an assignment

$$\begin{aligned} \text{distinguised opens in } \operatorname{Spec} A &\to \text{rings} \\ U_f &\mapsto A_f \end{aligned}$$

The association is functorial, that is, if $U_{f_1} \subseteq U_{f_2}$, then we can assume $f_1^n = f_2 f_3$, so $U_{f_1} = U_{f_1^n} = U_{f_2 f_3} \subseteq U_{f_2}$, and so there is a homomorphism

$$A_{f_2} \to A_{f_1}$$

which is the restriction map.

Can we extend this association to all open sets? See notes.

2 Sheaves

2.1 Presheaves

Let X be a topological space.

Definition 2.1.1 (presheaf)

A *presheaf of abelian groups* is an association

$$\begin{aligned} \{\text{Open set in } X\} &\leftrightarrow \text{Abelian Groups} \\ U &\mapsto \mathcal{F}(U) \end{aligned}$$

and for $U \subseteq V$ opens, we have a homomorphism

$$\operatorname{res}_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$$

called the *restriction map*, such that

1. $\operatorname{res}_U^U = \mathrm{id}$,
2. $\operatorname{res}_U^V \operatorname{res}_V^W = \operatorname{res}_U^W$ for $U \subseteq V \subseteq W$.

Example 2.1.2

For any space X, take

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$$

with $\operatorname{res}_U^V(f) = f|_U$.

We can make an analogous definition for presheaves of rings, sets, R-modules, and so on.

Definition 2.1.3 (morphism of presheaves)

A *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ *of presheaves* on X is, for each $U \subseteq X$ open, a homomorphism

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

compatible with restrictions, that is, the diagram

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \\ \downarrow{\scriptstyle \operatorname{res}_U^V} & & \downarrow{\scriptstyle \operatorname{res}_U^V} \\ \mathcal{F}(U) & \xrightarrow[\varphi(U)]{} & \mathcal{G}(U) \end{array}$$

commutes, for all $U \subseteq V$.

In terms of category theory, the opens in X form a category, with morphisms given by inclusions. In this case, a presheaf is a contravariant functor $\text{Opens}(X) \to \text{Ab}$, and a morphism is a natural transformation.

A morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is *injective* (resp. *surjective*) if $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (resp. surjective) for all $U \subseteq X$[1].

Lecture 5

2.2 Sheaves

Definition 2.2.1 (sheaf)

A *sheaf* is a presheaf $\mathcal{F}$ on X, such that

S1. Supppose $U \subseteq X$ open, $\{U_i\}$ an open cover of U, then for $s \in \mathcal{F}(U)$, with

$$s|_{U_i} = \text{res}^U_{U_i} s = 0$$

for all i, then $s = 0$. Intuitively, we can tell whether two sections are the same by looking locally.

S2. Suppose $U \subseteq X$ is open, $\{U_i\}$ an open cover of U, given $s_i \in \mathcal{F}(U_i)$, with

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

then there exists $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$. That is, we can glue "coherent" sections together.

Remark 2.2.2. These axioms imply that $\mathcal{F}(\varnothing) = 0$.

Definition 2.2.3 (morphism of sheaves)

A *morphisms of sheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is just a morphism of the underlying presheaves.

Example 2.2.4

If X is a topological space, and

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \text{ continuous}\}$$

Then $\mathcal{F}$ is a sheaf. This is also true for smooth functions on a manifold, and in fact if M is a manifold, the sheaf $C^\infty(M)$ determines the smooth structure on M.

Example 2.2.5 (non-example)

Let $X = \mathbb{C}$ with the usual Euclidean topology. Let

$$\mathcal{F}(U) = \{f : U \to \mathbb{C} \text{ holomorphic and bounded}\}$$

The issue here is that function being bounded is not a local property. That is, we can glue bounded functions together and get an unbounded function.

Example 2.2.6 (non-example)

Fix an abelian group G, and set

$$\mathcal{F}(U) = G$$

[1]Note that the definition of injectivity is the same for sheaves and presheaves, but the definition of surjectivity will be different. Once we mention the definition of injectivity/surjectivity of morphisms between sheaves, we will forget this definition.

This is called the *constant presheaf*. If U_1, U_2 are disjoint, then

$$\mathcal{F}(U_1 \cup U_2) = \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) = G \oplus G \neq G$$

(unless $G = 0$ that is).

Example 2.2.7 (constant sheaf)
Let G be an abelian group, with the discrete topology. Then define

$$\mathcal{F}(U) = \{f : U \to G \text{ continuous}\} = \{f : U \to G \text{ locally constant}\}$$

This is called the *constant sheaf*.

Example 2.2.8
If V is an irreducible variety, define

$$\mathcal{O}_V(U) = \{f \in k(V) \mid f \text{ is regular at all } p \in U\}$$

Here, regular at p means $f = g/h$, $g, h \in k[V]$, with $h(p) \neq 0$. $\mathcal{O}_X$ is called the *structure sheaf* of V. This is a sheaf as being regular is a local condition.

2.3 Basic constructions

Definition 2.3.1 (section)
Let $\mathcal{F}$ be a sheaf, a *section* s of $\mathcal{F}$ on U is an element $s \in \mathcal{F}(U)$.

Definition 2.3.2 (stalks)
Fix $p \in X$, and $\mathcal{F}$ a presheaf on X. The *stalk of* $\mathcal{F}$ *at* p is

$$\mathcal{F}_p = \frac{\{(U, s) \mid U \text{ open neighbourhood of } p, s \in \mathcal{F}(U)\}}{\sim}$$

where $(U, s) \sim (V, t)$ if there exists $W \subseteq U \cap V$ an open neighbourhood of p, such that

$$s|_W = t|_W$$

Elements of $\mathcal{F}_p$ are called *germs*.

Example 2.3.3
Consider the affine line $\mathbb{A}^1$, then

$$\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \;\middle|\; g(0) \neq 0 \right\} \subseteq k(t)$$

Equivalently, we can consider this as the localisation $k[t]_{(t)}$.

Proposition 2.3.4. Suppose $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, such that for all $p \in X$, the induced map

$$f_p : \mathcal{F}_p \to \mathcal{G}_p$$

is an isomorphism. Then f is an isomorphism.

Note that

$$f_p((U,s)) = (U, f_U(s))$$

which is well defined by definition of a morphism of sheaves. That is, if we have that $(U,s) \sim (V,t)$, then there exists $W \subseteq U \cap V$ an open neighbourhood of p, such that $s|_W = t|_W$. But then $f_W(s|_W) = f_U(s)|_W$, and so $f_U(s)|_W = f_V(t)|_W$, and $(U, f_U(s)) \sim (V, f_V(t))$.

Proof. We will show that

$$f_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

is an isomorphism for all U. Once we know this, we will define f^{-1} by

$$(f^{-1})_U = f_U^{-1}$$

f_U is injective. Suppose $s \in \mathcal{F}(U)$ is such that $f_U(s) = 0$. By injectivity of f_p, $(U,s) = 0$ in $\mathcal{F}_p$ for all $p \in U$. This means that for all $p \in U$, there exists an open neighbourhood U_p of p such that $s|_{U_p} = 0$. This defines a cover of U on which s vanishes. Hence by S1, $s = 0$.

f_U is surjective. Given $t \in \mathcal{G}(U)$, for each $p \in U$, we have $(U_p, s_p) \in \mathcal{F}_p$, with $f_p(U_p, s_p) = (U,t) \in \mathcal{G}_p$[2]. By shrinking U_p if necessary, we can assume $f_{U_p}(s_p) = t|_{U_p}$. For points $p, q \in U$,

$$f_{U_p \cap U_q}(s_p|_{U_p \cap U_q} - s_q|_{U_p \cap U_q}) = t|_{U_p \cap U_q} - t|_{U_p \cap U_q} = 0$$

Thus by injectivity of $f_{U_p \cap U_q}$,

$$s_p|_{U_p \cap U_q} = s_q|_{U_p \cap U_q}$$

Thus by S2, there exists $s \in \mathcal{F}(U)$, with $s|_{U_p} = s_p$. Now

$$f_U(s)|_{U_p} = f_U(s|_{U_p}) = f|_{U_p}(s_p) = t|_{U_p}$$

Thus, $f_U(s) = t$ by S1. □

Remark 2.3.5. There is an asymmetry in the proof here, we need to show injectivity before surjectivity. That is, we needed to show uniqueness before existence. Looking at the proof, we used gluing to construct the section, and so we needed to check compatibility, which is the uniqueness condition.

Lecture 6

Exercises:

1. There is an injective map

$$\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p$$
$$s \mapsto ((U,s))_{p \in U}$$

This is essentially the proof of injectivity above, where we can check that the section is zero at stalk level.

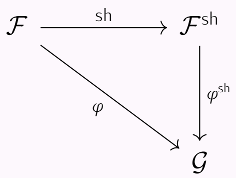
2. Given morphisms $\varphi, \psi : \mathcal{F} \to \mathcal{G}$, with $\varphi_p = \psi_p$ for all $p \in X$, we have that $\varphi = \psi$. Let $U \subseteq X$ be open, $s \in \mathcal{F}(U)$ a section. Then for all $p \in U$, there exists a neighbourhood U_p of p, on which $\varphi_{U_p}(s|_{U_p}) = \psi_{U_p}(s|_{U_p})$. Then by S1 we are done.

Definition 2.3.6 (sheafification)

Suppose $\mathcal{F}$ is a presheaf on X, then a morphism $\mathrm{sh} : \mathcal{F} \to \mathcal{F}^{\mathrm{sh}}$, where $\mathcal{F}^{\mathrm{sh}}$ is a sheaf, is a *sheafiifcation* if for any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there exists a unique morphism $f^{\mathrm{sh}} : \mathcal{F}^{\mathrm{sh}} \to \mathcal{G}$, making

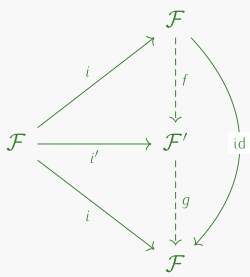
[2] There is a slight abuse of notation here. Strictly speaking this should be $\sim$, but we don't distinguish between a germ and a representative.

the diagram



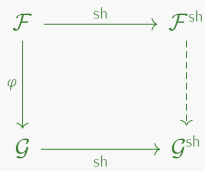
commute.

Remark 2.3.7. (i) We have defined sheafification by a universal property, and so as usual, it is unique up to unique isomorphism. Suppose $(i, \mathcal{G}), (i', \mathcal{G}')$ are sheafifications of $\mathcal{F}$. Consider the diagram



By uniqueness, the vertical composition has to be id, and so $g \circ f = \text{id}$. Similarly, we must have that $f \circ g = \text{id}$, and so f is an isomorphism.

(ii) A morphism on presheaves $\mathcal{F} \to \mathcal{G}$ induces a morphism of sheaves $\mathcal{F}^{\text{sh}} \to \mathcal{G}^{\text{sh}}$. Basically, apply the universal property to $\text{sh} \circ \varphi : \mathcal{F} \to \mathcal{G}^{\text{sh}}$, to get



Proposition 2.3.8. Sheafification exists.

Construction. Given a presheaf $\mathcal{F}$ on X, define

$$\mathcal{F}^{\text{sh}}(U) = \left\{ f : U \to \bigsqcup_{p \in U} \mathcal{F}_p \;\middle|\; (*) \right\}$$

where the condition $(*)$ is that: $f(p) \in \mathcal{F}_p$ and for every $p \in U$ there exists an open neighbourhood $V_p \subseteq U$ of p, and $s \in \mathcal{F}(V_p)$ such that $(V_p, s) = f(q) \in \mathcal{F}_q$ for all $q \in V_p$.

It is clear that this is a sheaf, and we leave as an exercise to check that this satisfies the universal property.

Perhaps a better way to think about this is to consider $f \in \prod_{p \in U} \mathcal{F}_p$, and write f_p for the p-th coordinate. The condition becomes for every $p \in U$, there exists an open neighbourhood $V_p \subseteq U$ of p, and $s \in \mathcal{F}(V_p)$ such that $(V_p, s) = f_q \in \mathcal{F}_q$ for all $q \in V_p$.

We will check that $\mathcal{F}^{\text{sh}}$ is a sheaf. First, we check that it is a presheaf. For $U \subseteq V$, the projection map

$$\prod_{p \in V} \mathcal{F}_p \to \prod_{p \in U} \mathcal{F}_p$$

defines the restriction map $\mathcal{F}^{\text{sh}}(V) \to \mathcal{F}^{\text{sh}}(U)$. The only non-trivial part follows by taking $\widetilde{V_p} = V_p \cap U$.

Next, we need to check the sheaf axioms. If we have an open cover of U by U_i, and $s \in \mathcal{F}^{\text{sh}}(U)$ such that $s|_{U_i} = 0$ for all i. But the restriction map is the projection, and so the result is clear. Similarly, gluing is clear.

Finally, we check the universal property. Let $\mathcal{G}$ be a sheaf, $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism. Let $U \subseteq X$ be open, $s \in \mathcal{F}^{\text{sh}}(U)$ a section. For $p \in U$, define

$$t_p = \varphi_p(s_p) \in \mathcal{G}_p$$

We need to show that the t_p glue together to a section $t \in \mathcal{G}(U)$. For each $p \in U$, we have an open set V_p, and a section $s'_p \in \mathcal{F}(V_p)$, such that $(V_p, s'_p) = s_q$ for all $q \in V_p$. We need to show that

$$s'_p|_{V_p \cap V_q} = s'_q|_{V_p \cap V_q}$$

This will show the result as $t_p = \varphi_{V_p}(s'_p)_p$. But we can check this statement locally.

The fact that φ factors is clear, and so all that remains is uniqueness. But working at stalk level, the above is the only way to define t_p. ☐

Corollary 2.3.9. The stalks of $\mathcal{F}$ and $\mathcal{F}^{\mathrm{sh}}$ coincide.

Proof. From the definition of sheafification. More formally, we have

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\ \mathrm{sh}\ } & \mathcal{F}^{\mathrm{sh}} \\ \downarrow & & \downarrow \mathrm{pr} \\ \mathcal{F}_p & \overset{\sim}{\text{———}} & \left(\mathcal{F}^{\mathrm{sh}}\right)_p \end{array}$$

where pr denotes the projection map to the p-th coordinate, and the bottom map is an isomorphism. ☐

Exercise: Find a non-zero presheaf $\mathcal{F}$ with $\mathcal{F}^{\mathrm{sh}} = 0$. One rather trivial example is to let $X = \varnothing$, with $\mathcal{F}(\varnothing) = G$ for some non-zero abelian group G. Any sheaf on X must have $G = 0$.

2.4 Kernels, Cokernels, etc.

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Then we can define presheaves $\ker(\varphi), \operatorname{im}(\varphi), \operatorname{coker}(\varphi)$, where on an open set U, we define

$$\begin{aligned} \ker(\varphi)(U) &= \ker(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)) \\ \operatorname{coker}(\varphi)(U) &= \operatorname{coker}(\varphi_U) \\ \operatorname{im}(\varphi)(U) &= \operatorname{im}(\varphi_U) \end{aligned}$$

These are all presheaves.

Exercise: The presheaf kernel for a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ between sheaves is a sheaf. We will verify the sheaf axioms. Using the fact that $\mathcal{F}$ is a sheaf, S1 is clear. Now suppose $U \subseteq X$ open, U_i an open cover, with $s_i \in \ker(\varphi)(U_i)$, such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all i, j. Using S2 for $\mathcal{F}$, we obtain $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. It suffices to show that $s \in \ker(\varphi_U)$. Let $t = \varphi_U(s)$. Then $t|_{U_i} = \varphi_{U_i}(s_i) = 0$, and so $t = 0$ by S1 for $\mathcal{G}$.

However, in general, $\operatorname{coker}(\varphi)$ is not a sheaf.

Example 2.4.1

Let $X = \mathbb{C}$ with the usual Euclidean topology, and let

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{C} \text{ holomorphic}\}$$

with addition, and set

$$\mathcal{O}_X^*(U) = \{f : U \to \mathbb{C} \text{ holomorphic nowhere vanishing}\}$$

with multiplication. We have a morphism of sheaves $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$, given by

$$\exp(f)(z) = \exp(f(z))$$

The kernel of exp is $2\pi i\mathbb{Z}$, where $\mathbb{Z}$ is the constant sheaf. But the cokernel is not a sheaf. Let $U_1 = \mathbb{C} \setminus [0, \infty)$ and $U_2 = \mathbb{C} \setminus (-\infty, 0]$. Let $U = U_1 \cup U_2 = \mathbb{C} \setminus 0$.

Let $f(z) = z$. Then $f \in \mathcal{O}_X^*(U)$, and is not in the image of $\exp : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$, since $\log(z)$ is not defined on U. Thus f defines a non-zero section of $\operatorname{coker}(\exp)(U)$. But if we restrict to U_i, then f is in the image of exp. With this, $f|_{U_i} = 1 \in \operatorname{coker}(\exp)(U_i)$. Hence $\operatorname{coker}(\exp)$ does not satisfy S1.

Definition 2.4.2 (sheaf image, sheaf cokernel)

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We define the *sheaf cokernel* and the *sheaf image* of φ to be $\operatorname{coker}(\varphi)^{\mathrm{sh}}$ and $\operatorname{im}(\varphi)^{\mathrm{sh}}$ respectively.

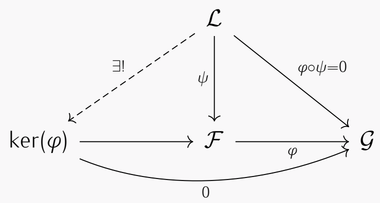
We say that a morphism $f : \mathcal{F} \to \mathcal{G}$ is *injective* if $\ker(f) = 0$, and *surjective* if $\operatorname{im}(f) = \mathcal{G}$.
Warning: $f : \mathcal{F} \to \mathcal{G}$ being surjective does *not* imply that the map $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all U[3].

Remark 2.4.3. A crucial fact is that there exists an exact sequence

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1$$

That is, $2\pi i\mathbb{Z}$ is the kernel of exp, and $\operatorname{coker}(\exp) = 1$.

Remark 2.4.4. $\ker(\varphi)$ and $\operatorname{coker}(\varphi)$ satisfies the category theoretic definitions (i.e. in abelian categories).
Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism, then $\ker(\varphi)$ is the object such that given $\psi : \mathcal{L} \to \mathcal{F}$, with $\varphi \circ \psi = 0$, then ψ factors through $\ker(\varphi)$.



It is easy to check this by working locally, that is, in open sets. There is an analogous definition for the cokernel.
With this, we see that the sheaf of abelian groups is an abelian category.

Remark 2.4.5.
1. We can also define subsheaves, $\mathcal{F} \subseteq \mathcal{G}$ if there exists inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ compatible with restrictions. For example, $\ker(\varphi : \mathcal{F} \to \mathcal{G})$ is a subsheaf of $\mathcal{F}$.
2. If $\mathcal{F} \subseteq \mathcal{G}$ is a sub-presheaf, then we can define the quotient presheaf $\mathcal{G}/\mathcal{F}$ by

$$(\mathcal{G}/\mathcal{F})(U) = \mathcal{G}(U)/\mathcal{F}(U)$$

We need to verify that this is a presheaf. If $U \subseteq V$ open, define

$$\operatorname{res}_U^V(s + \mathcal{F}(V)) = s|_U + \mathcal{F}(U)$$

This is well defined, since any element of $\mathcal{F}(V)$, restricted to U, will be in $\mathcal{F}(U)$. The quotient sheaf is the sheafification of this.

Lecture 7

2.5 Moving between spaces

Given $f : X \to Y$ continuous, with sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y.

Definition 2.5.1 (pushforward, direct image)

Define the *presheaf pushforward* $f_*\mathcal{F}$ by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

[3] i.e. a surjective morphism of sheaves need not be surjective as a morphism of presheaves...

for an open $U \subseteq Y$.

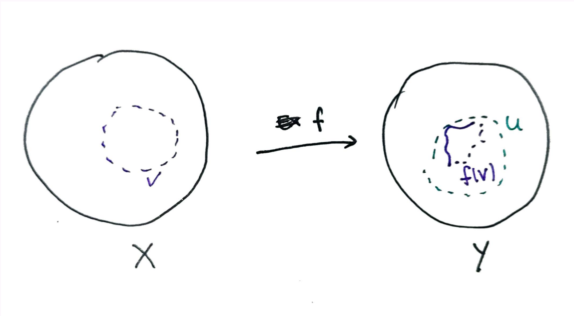
Proposition 2.5.2. The presheaf pushforward of a sheaf is a sheaf.

Proof. Trivial. The fact that it is a presheaf is clear by definition. The sheaf axioms follow from the fact that if U_i is an open cover of U, then $f^{-1}(U_i)$ is an open cover of $f^{-1}(U)$. □

Definition 2.5.3 (inverse image)
The *inverse image presheaf* $(f^{-1}\mathcal{G})^{\text{pre}}$ is defined by

$$(f^{-1}\mathcal{G})^{\text{pre}}(V) = \frac{\{(s_U, U) \mid f(V) \subseteq U, s_U \in \mathcal{G}(U)\}}{\sim}$$

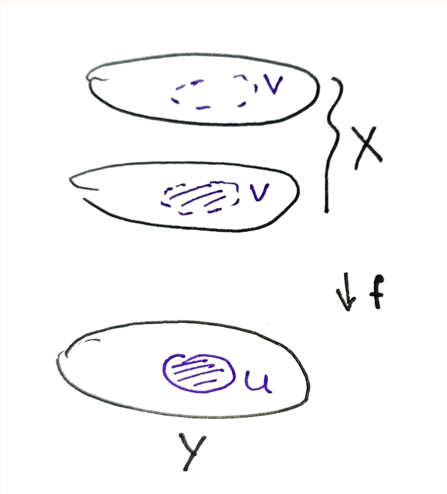
where $\sim$ denotes sections which agree on a smaller open set containing $f(V)$.



The *inverse image sheaf* is defined by

$$f^{-1}\mathcal{G} = \left((f^{-1}\mathcal{G})^{\text{pre}}\right)^{\text{sh}}$$

Example 2.5.4 (was sheafification necessary?)
Let Y be any topological space, $X = Y \sqcup Y$. Let $\mathcal{G} = \underline{\mathbb{Z}}$ be the consatnt sheaf, and $\mathcal{F} = (f^{-1}\mathcal{G})^{\text{pre}}$.



Fix $U \subseteq Y$ open, and $V = f^{-1}(U)$. By definition, $\mathcal{F}(U) = \mathcal{G}(V) = \mathbb{Z}$ assuming U is connected. But $V = U \sqcup U$, and so

$$\mathcal{F}^{\text{sh}}(V) = \mathcal{G}(U) \times \mathcal{G}(U) = \mathbb{Z}^2$$

Note in this case f is an open map, so the reason for requiring sheafification was not because f does not have to be open.

Example 2.5.5

Let $\mathcal{F}$ be a sheaf on X, $\pi : X \to \{*\}$. Then $\pi_*\mathcal{F}$ is a sheaf on a point. But a sheaf on a point is just an abelian group, in particular, specifically

$$\mathcal{F}(\pi^{-1}(*)) = \mathcal{F}(X)$$

Notation 2.5.6. For a sheaf $\mathcal{F}$ on X, write

$$\mathcal{F}(X) = \Gamma(X, \mathcal{F}) = \mathrm{H}^0(X, \mathcal{F})$$

for the global sections, or the zeroth cohomology with coefficients in $\mathcal{F}$.

For $p \in X$, $i : \{p\} \to X$, $\mathcal{G}$ a sheaf on p, i.e. an abelian group A. Consider $i_*(\mathcal{G})$. The sheaf on X is

$$(i_*\mathcal{G})(U) = \begin{cases} 0 & p \notin U \\ A & p \in U \end{cases}$$

We call this the *skyscraper at p with value A*.

3 Schemes

tl;dr: Spec(A) with a sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$, with $\mathcal{O}_{\mathrm{Spec}(A)}(U_f) = A_f$. Globalise to get a scheme.

Definition 3.0.1 (localisation)

Let A be a ring, $S \subseteq A$ closed under multiplication. The *localisation of A at S* is

$$S^{-1}A = \frac{\{(a, s) \mid a \in A, s \in S\}}{\sim}$$

where $(a, s) \sim (a', s')$ if and only if there exists $s'' \in S$ such that

$$s''(as' - a's) = 0$$

For example, consider $S = \{1, f, f^2, \dots\}$, or $S = A \setminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal.

Remark 3.0.2. Note that the natural map $A \to S^{-1}A$ need not be injective.

What is going to happen? We will define a sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ on Spec(A), such that

- The stalk at a prime $\mathfrak{p}$ is the localisation $(A \setminus \mathfrak{p})^{-1}A$,
- if U_f is a distinguised open, then $\mathcal{O}_{\mathrm{Spec}(A)}(U_f) = A_f$.

3.1 Sheaf on a base

Fix a topological space X, and $\mathcal{B}$ a basis for the topology.

Definition 3.1.1 (sheaf on a base)

A *sheaf F on the base $\mathcal{B}$* consists of assignments $B_i \mapsto F(B_i)$, with restriction maps $F(B_I) \to F(B_j)$ whenever $B_J \subseteq B_i$, satisfying the usual relations when $B_i \subseteq B_j \subseteq B_k$ and $B_i = B_j$. Moreover, we have the additional axioms:

SB1. If $B = \bigcup_i B_i$, with $B, B_i \in \mathcal{B}$, $f, g \in F(B)$, with $f|_{B_i} = g|_{B_i}$ for all i, then $f = g$.

SB2. If $B = \bigcup_i B_i$, with $B, B_i \in \mathcal{B}$, with $f_i \in F(B_i)$, such that for all $B' \subseteq B_i \cap B_j$, $f_i|_{B'} = f_j|_{B'}$, then there exists $f \in F(B)$ with $f|_{B_i} = f_i$.

Proposition 3.1.2. Let F be a sheaf on a base $\mathcal{B}$ of X. Then this uniquely determines a sheaf $\mathcal{F}$, by $\mathcal{F}(B_i) = F(B_i)$, agreeing with restriction maps.

Proof. We will define the stalks of $\mathcal{F}$ first. Set

$$\mathcal{F}_p = \frac{\{(s_B, B) \mid p \in B \in \mathcal{B}, s_B \in F(B)\}}{\sim}$$

where $(s_B, B) \sim (s_{B'}, B')$ if there exists $B'' \subseteq B \cap B'$ such that $s_B|_{B''} = s_{B'}|_{B''}$.

We can then use the trick from sheafification, that is,

$$\mathcal{F}(U) = \left\{(f_p \in \mathcal{F}_p)_{p \in U} \mid (*)\right\}$$

where the condition $(*)$ is such that: for every $p \in U$, there exists a basic open B containing p, and $s \in F(B)$, with $s_q = f_q$ for all $q \in B$. This is clearly a sheaf. As before, the projection maps define the restriction maps, making it into a presheaf. Checking the sheaf axioms is the same as sheafification.

By the sheaf axioms, the natural maps $F(B) \to \mathcal{F}(B)$ are isomorphisms. □

Lecture 8

Recall $\mathrm{Spec}(A)$ is a topological space, with distinguised open sets $\{U_f\}_{f \in A}$. Moreover, $U_f = U_g$ if and only if $f^n = ga$ and $g^m = fb$ for some $m, n \in \mathbb{N}, a, b \in A$. Thus, if $U_f = U_g$, then $A_f \cong A_g$. Therefore, the assignment

$$U_f \mapsto A_f$$

is well-defined.

Proposition 3.1.3. The assignment $U_f \to A_f$ defines a sheaf of rings on the base $\{U_f\}$ of $\mathrm{Spec}(A)$.

An immediate consequence is that $\mathrm{Spec}(A)$ inherits a sheaf of rings, denoted $\mathcal{O}_{\mathrm{Spec}(A)}$, called the *structure sheaf*.

Prelude. Suppose $\{U_{f_i}\}_{i \in I}$ covers $\mathrm{Spec}(A)$, then there exists a finite subcover. That is, $\mathrm{Spec}(A)$ is quasicompact. This is on examples sheet 1, but also:

Since the U_{f_i} cover, there is no prime ideal $\mathfrak{p} \trianglelefteq A$ containing all f_i. Equivalently,

$$\sum_{i \in I} \langle f_i \rangle = \langle 1 \rangle$$

But we can write 1 as a finite sum $1 = \sum a_i f_i$. But then if $J \subseteq I$ are the indices with $a_i \neq 0$, then $\{U_{f_i}\}_{i \in J}$ cover. □

Proof of proposition 3.1.3. We need to check SB1 and SB2. We will check these for the basis open $B = \mathrm{Spec}(A)$. The general case is similar (replace A with A_f).

SB1: Suppose we have a cover

$$\mathrm{Spec}(A) = \bigcup_{i=1}^n U_{f_i}$$

By the prelude, it suffices to consider the finite case. Given $s \in A$ such that $s|_{U_{f_i}} = 0$ for all i, then by the definition of localisation, we have that $f_i^m s = 0$ for some m large enough. But $\langle 1 \rangle = \langle f_i^m \rangle_{i=1}^n$ for any $m > 0$, as the U_{f_i} cover, which then implies the $U_{f_i^m}$ cover.

With this,

$$s = s \cdot 1 = s \cdot \sum_{i=1}^n r_i f_i^m = \sum_{i=1}^n r_i s_i f_i^m = 0$$

SB2: Say

$$\mathrm{Spec}(A) = \bigcup_{i \in I} U_{f_i}$$

and choose elements in each A_{f_i}, which agree on $A_{f_i f_j}$. That is, we have $s_i \in A_{f_i}$, with the images of s_i, s_j in $A_{f_i f_j}$ agreeing. We need to build $s \in A$ with these localisations.

First suppose I is finite. On A_{f_i}, we have the element

$$\frac{a_i}{f_i^{\ell_i}} \in A_{f_i}$$

We will write $g_i = f_i^{\ell_i}$, and note that $U_{f_i} = U_{g_i}$.

On overlaps, restrict to $A_{g_i g_j}$. Then we have that

$$(g_i g_j)^{m_{ij}}(a_i g_j - a_j g_i) = 0$$

Rewriting this using algebra and the fact that $U_f = U_{f^k}$ for all k. By taking the largest, we can assume $m = m_{ij}$. We write

$$b_i = a_i g_i^m \qquad h_i = g_i^{m+1}$$

Using this, the element we chose in A_{f_i}, becomes b_i/h_i. But now $U_{f_i} = U_{h_i}$ cover $\mathrm{Spec}(A)$, and so we can write

$$1 = \sum_i r_i h_i$$

Now we construct

$$r = \sum r_i b_i$$

with r_i as above. This restricts correctly to b_i/h_i on U_{h_i} (i.e. in the localisation A_{h_i}).

When I is infinite, choose a finite subcover, or equivalently, $\langle f_1, \dots, f_n \rangle = A$, and use the above to build $r \in A$. But given $\langle f_1, \dots, f_n, f_\alpha \rangle = A$, the same construction gives a new element r'. But $r' = r$ by SB1. □

Definition 3.1.4 (structure sheaf)

The *structure sheaf on* $\mathrm{Spec}(A)$ is the sheaf associated to the sheaf on the base sending

$$U_f \mapsto A_f$$

The sheaf is denoted $\mathcal{O}_{\mathrm{Spec}(A)}$.

Remark 3.1.5.

$$\mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} = A_{\mathfrak{p}}$$

Proof. To see this, we wll define an isomorphism $\varphi : \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} \to A_{\mathfrak{p}}$. A general element of the stalk is of the form (U_f, s), where $f \in A$, with $\mathfrak{p} \in U_f$, and $s \in A_f$. Note that $\mathfrak{p} \in U_f$ is equivalent to $f \notin \mathfrak{p}$. In this case, we have a natural map $A_f \to A_{\mathfrak{p}}$, which we will write as $s \mapsto s_{\mathfrak{p}}$.

Define

$$\begin{aligned} \varphi_{\mathfrak{p}} : \mathcal{O}_{\mathrm{Spec}(A),\mathfrak{p}} &\to A_{\mathfrak{p}} \\ (U_f, s) &\mapsto s_{\mathfrak{p}} \end{aligned}$$

The fact that this is independent of the choice of representative follows from properties of localisation. For an element $x = s/f$, where $f \in A \setminus \mathfrak{p}$, we have that $x = \varphi_{\mathfrak{p}}(U_f, s)$, essentially by definition. Thus it remains to check injectivity.

Say $\varphi_{\mathfrak{p}}(U_f, s) = 0$, with $s = a/f^n$. Then $s_{\mathfrak{p}}$ is given by $a/f^n \in A_{\mathfrak{p}}$. If this is zero, then there exists $t \in A \setminus \mathfrak{p}$ such that $ta = 0$. Now $tf \in A \setminus \mathfrak{p}$, and so

$$(U_f, s) = \left(U_{tf}, \frac{t^n a}{(tf)^n}\right) = (U_{tf}, 0)$$

as required. □

Definition 3.1.6 (ringed space)
A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X, with a sheaf of rings $\mathcal{O}_X$ on X.

We note that any open subset of a ringed space is naturally a ringed space, with

$$\mathcal{O}_U(V) = \mathcal{O}_X(V)$$

for all $V \subseteq U \subseteq X$ open.

Definition 3.1.7 (isomorphism of ringed spaces)
An *isomorphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is

- a homeomorphism $\pi : X \to Y$,
- an isomorphism of sheaves on Y, $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$.

Remark 3.1.8. We could have also chosen $\pi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Definition 3.1.9 (affine scheme)
An *affine scheme* is a ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$.

Definition 3.1.10 (scheme)
A *scheme* is a ringed space $(X, \mathcal{O}_X)$, that is locally isomorphic to an affine scheme. That is, for every $x \in X$, there exists $x \in U \subseteq X$ open, with $(U, \mathcal{O}_U)$ being an affine scheme.

Another way to say this is that we have an open cover of X by affine schemes, and this is the way which we will often think about it.

3.2 Examples of schemes

Example 3.2.1
$\mathrm{Spec}(A)$ is a scheme.

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Example 3.2.2 (Open subschemes)
Let X be a scheme, $U \subseteq X$ an open subset. We write $i : U \hookrightarrow X$ for the inclusion map. Using this, we can define

$$\mathcal{O}_U = \mathcal{O}_X|_U := i^{-1}\mathcal{O}_X$$

Proposition 3.2.3. The ringed space $(U, \mathcal{O}_U)$ is a scheme.

A simple case of this is to take $X = \mathrm{Spec}(A)$, and the distinguished open $U = U_f$. Then

$$(U, \mathcal{O}_U) \cong (\mathrm{Spec}(A_f), \mathcal{O}_{\mathrm{Spec}(A_f)})$$

Proof. Let $p \in U \subseteq X$. Since X is a scheme, we can find V_p such that V_p is isomorphic to an affine scheme. Take $V_p \cap U$ with structure sheaf via restriction. Note however $V_p \cap U$ may not be affine.

Since V_p is affine, say $V_p \cong \mathrm{Spec}(B)$. The distinguished opens form a basis for the Zariski topology on $\mathrm{Spec}(B)$. So we've reduced to the simple case as above. □

We define *affine space*

$$\mathbb{A}_k^n = \mathrm{Spec}(k[x_1, \dots, x_n])$$

Example 3.2.4

Take $U = \mathbb{A}_k^{n^2} \setminus \{\det(x_{ij}) = 0\}$. That is, "$U$ is $\mathrm{GL}(n, k)$" roughly speaking. Eventually we will show multiplication $U \times U \to U$ is a morphism of schemes.

Example 3.2.5 (a non-affine scheme)

Let $X = \mathbb{A}_k^2$, $U = X \setminus \{\langle x, y \rangle\}$. Roughly speaking, we have $k^2 \setminus 0$.

Claim 3.2.6. U is not affine.

Proof. Suppose it was. We can compute $\mathcal{O}_U(U)$. Write $U_x = \mathbb{A}_k^2 \setminus \mathbb{V}(x)$, and define $U_y = \mathbb{A}_k^2 \setminus \mathbb{V}(y)$. Observe $U = U_x \cup U_y$, and

$$U_x \cap U_y = \mathbb{A}_k^2 \setminus \mathbb{V}(xy)$$

With this,

$$\begin{aligned} \mathcal{O}_U(U_x) &= k[x, x^{-1}, y] \\ \mathcal{O}_U(U_y) &= k[x, y, y^{-1}] \\ \mathcal{O}_U(U_x \cap U_y) &= k[x, x^{-1}, y, y^{-1}] \end{aligned}$$

and the restriction maps $\mathcal{O}_U(U_x) \to \mathcal{O}_U(U_{xy})$ are the obvious ones. By the sheaf axioms,

$$\mathcal{O}_U(U) = k[x, x^{-1}, y] \cap k[x, y, y^{-1}]$$

where we compute the intersection in $k[x, x^{-1}, y, y^{-1}]$. This means that

$$\mathcal{O}_U(U) = k[x, y]$$

Contradiction. One way to see this is that in U, there exists a maximal ideal in the global section ring with empty vanishing locus, namely the maximal ideal $\langle x, y \rangle$. □

We will show the "by sheaf axioms" part of the above. There is a natural map from $\mathcal{O}_U(U)$ to the intersection, given by restriction. Thus, all we need to show is that this is an isomorphism.

First of all, note that the restriction maps $\mathcal{O}_U(U_x) \to \mathcal{O}_Y(U_{xy})$ and $\mathcal{O}_U(U_y) \to \mathcal{O}_U(U_{xy})$ are injective. hence if s vanishes when restricted to U_{xy}, then it vanishes on U_x and U_y. Thus, the map $\mathcal{O}_U(U) \to \mathcal{O}_U(U_{xy})$ is injective by S1. Surjectivity follows essentially immediately from S2, as compatibility is true by definition.

For general topological spaces and sheaves, the surjectivity part is always true, but injectivity does not have to be true. One example would be the sheaf of continuous functions on $\mathbb{R}$.

Example 3.2.7

More generally, let X be a scheme, $f \in \Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$. Fix $p \in X$, then we can consider the stalk $\mathcal{O}_{X,p}$. This is of the form $A_\mathfrak{p}$, where A is a ring and $\mathfrak{p} \trianglelefteq A$ is a prime ideal. In particular, $A_\mathfrak{p}$ has a unique maximal ideal, namely $\mathfrak{p}A_\mathfrak{p}$. We say that f *vanishes at* p if its image in $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is zero. Equivalently, if f is in $\mathfrak{p}A_\mathfrak{p}$.

Here, we're using an isomorphism V_p open to $\mathrm{Spec}(A)$. For $f \in \Gamma(X, \mathcal{O}_X)$, the set $\mathbb{V}(f) \subseteq X$ which is the *vanishing locus* of f is well defined.

3.3 Interlude – gluing sheaves

Let X be a topological space, with a cover $\{U_\alpha\}$. Suppose we have sheaves $\mathcal{F}_\alpha$ on U_α, and isomorphisms of sheaves

$$\phi_{\alpha\beta} : \mathcal{F}_\alpha|_{U_\alpha \cap U_\beta} \to \mathcal{F}_\beta|_{U_\alpha \cap U_\beta}$$

Such that $\varphi_{\alpha\alpha} = \text{id}$, $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$, and the cocycle condition

$$\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma} \text{ on } U_\alpha \cap U_\beta \cap U_\gamma$$

Proposition 3.3.1. We can build a sheaf $\mathcal{F}$ on X. Given $V \subseteq X$ open, define

$$\mathcal{F}(V) = \left\{(s_\alpha)_\alpha \mid s_\alpha \in \mathcal{F}_\alpha(U_\alpha \cap V) \text{ such that } \phi_{\alpha\beta}(s_\alpha|_{V\cap U_\alpha \cap U_\beta}) = s_\beta|_{V \cap U_\alpha \cap U_\beta}\right\}$$

Moreover, $\mathcal{F}|_{U_\alpha} = \mathcal{F}_\alpha$ on U.

$\mathcal{F}$ is a presheaf. Given $(s_\alpha) \in \mathcal{F}_V$, $W \subseteq V$ open, we can take

$$(s_\alpha)|_W = \left(\text{res}^{V\cap U_\alpha}_{W \cap U_\alpha}(s_\alpha)\right)_\alpha$$

We need to check that this lies in $\mathcal{F}(W)$. But this follows from the sheaf axioms.

$\mathcal{F}$ is a sheaf. The sheaf axioms are clear. This is basically just using the sheaf axioms on the s_α.

Restriction. We need to build an isomorphism $\mathcal{F}|_{U_\gamma} \cong \mathcal{F}_\gamma$. Define a morphism $\mathcal{F}_\gamma \to \mathcal{F}|_{U_\gamma}$, where for $V \subseteq U_\gamma$, $s \in \mathcal{F}_\gamma(V)$, define the image to be

$$\left(\phi_{\gamma\alpha}(s|_{V\cap U_\alpha})\right)_\alpha$$

We need to check that this is in $\mathcal{F}|_{U_\gamma}(V) = \mathcal{F}(V)$. But this follows from the cocycle condition, as

$$\phi_{\alpha\beta} \circ \phi_{\gamma\alpha}(s|_{V\cap U_\alpha \cap U_\beta}) = \phi_{\gamma\beta}(s|_{V\cap U_\alpha\cap U_\beta})$$

It is easy to see that this is an isomorphism. □

3.4 More schemes

Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be schemes, with opens $U \subseteq X, V \subseteq Y$ and an isomorphism $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$ as ringed spaces. We can glue both topological spaces and schemes, that is,

$$S = \frac{X \sqcup Y}{U \sim V}$$

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By definition of the quotient topology, the images of X and Y in S form an open cover, with intersection being the image of U (or V). We can then glue the structure sheafs of these open sets as before.

Note that in this case, there is no cocycle condition to check.

Example 3.4.1 (bug-eyed line, line with two origin)
Let k be a fieodl, and $X = \text{Spec}(k[t])$, $Y = \text{Spec}(k[u])$. Set

$$U = \text{Spec}(k[t, t^{-1}]) = \text{Spec}(k[t]_t) \quad V = \text{Spec}(k[u]_u)$$

These are distinguised opens. We have natural isomorphism

$$U \to V$$
$$t \leftarrow\!\shortmid u$$

of rings, which is formally, induced by the above map $k[u]_u \to k[v]_v$, then apply the contravariant functor.

On the level of topological spaces, $X = Y = \mathbb{A}^1_k$, $U = \mathbb{A}^1_k \setminus \{\langle t\rangle\}$, which is $\mathbb{A}^1_k$ with a point removed. In this case,

$$\frac{X \sqcup Y}{\sim}$$

looks like a line with two origins.

The open sets in the scheme are:

1. Suppose $W \subseteq X \subseteq S$ or $W \subseteq Y \subseteq S$, these are 'nice' open sets.

2. If $W = S \setminus \{p_1, \dots, p_r\}$, where $p_i \in U \cup V$. The simplest case is when $W = S$.

   What is $\mathcal{O}_S(S)$? We can use the sheaf axioms to show that $\mathcal{O}_S(S) \cong k[t]$ as above. With this, we see that S is not an affine scheme.

Example 3.4.2 (projective line)
Let $X = \operatorname{Spec}(k[t])$, $Y = \operatorname{Spec}(k[s])$, $U = \operatorname{Spec}(k[t, t^{-1}])$ and $V = \operatorname{Spec}(k[s, s^{-1}])$ as above. We now glue along $s \mapsto t^{-1}$, and we call the result $\mathbb{P}^1_k$ for the resulting scheme (for now).

Proposition 3.4.3. $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$

Proof. The only elements of $k[t, t^{-1}]$ which are polynomials in t and t^{-1} are the constants. Note here we used the same trick as before, which is that the global sections can be computed as the intersection in nice cases. □

In particular, $\mathbb{P}^1$ is not affine.

Example 3.4.4
We can similarly build $S = \mathbb{A}^2_k$ with a doubled origin. This has the property where there exists affine open subschemes $U_1, U_2 \subseteq S$, such that $U_1 \cap U_2$ is not affine.

Proposition 3.4.5 (gluing schemes). Given schemes $(X_i)_{i \in I}$, open subschemes $X_{ij} \subseteq X_i$, with $X_{ii} = X_i$, isomorphisms $f_{ij} : X_{ij} \to X_{ji}$, such that $f_{ii} = \mathrm{id}$, satisfying the cocycle conditions

$$f_{ij} = f_{ji}^{-1}$$
$$f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$$

Then there exists a unique scheme X, with an open cover by X, glued along $X_{ij} \sim X_{ji}$.

This is on examples sheet 1. On the other hand, it's basically one big tautology, where everything is true by definitions.

Example 3.4.6 (projective space)
Let A be any ring,

$$X_i = \operatorname{Spec}\left(A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]\right)$$

and

$$X_{ij} = X_j \setminus \mathbb{V}\left(\frac{x_j}{x_i}\right)$$

with isomorphisms

$$X_{ij} \to X_{ji}$$
$$\frac{x_k}{x_i} \mapsto \frac{x_k}{x_j} \cdot \left(\frac{x_i}{x_j}\right)^{-1}$$

The resulting scheme is called projective n-space $\mathbb{P}^n_A$.

<u>Exercise:</u> $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$. This is essentially the same idea as for $\mathbb{P}^1$. That is, we can compute it as an intersection

$$\bigcap_{i=0}^{n} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

in $A[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$. But by considering say the degree in x_i, we can see that this is just A.

3.5 The Proj construction

Definition 3.5.1 ($\mathbb{Z}$-grading)
A $\mathbb{Z}$-*grading* on a ring A is a decomposition of

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

as abelian groups, such that multiplication respects the grading, that is:

$$A_i A_j \subseteq A_{i+j}$$

Example 3.5.2
Let $A = k[x_0, \ldots, x_n]$, we write A_d for the set of degree d-homogeneous polynomials (and 0).

If $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous ideal (i.e. generated by homogeneous elements, of possibly different degrees). Then $k[x_0, \ldots, x_n]/I$ is also naturally graded.

Note by definition A_0 is always a subring. Throughout, we will make the **assumption** that the degree 1 elements generate A as an A_0-algebra. That is,

$$A = A_0[A_1]$$

as A_0 algebras.

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Moreover, we will assume $A_i = 0$ for $i < 0$. Define

$$A_+ = \bigcup_{i>0} A_i \subseteq A$$

for the *irrelevant ideal*. Strictly speaking we want the ideal generated by the positive degree elements, since in the case when $A = A_0$, we would want this to be the zero ideal, and not the empty set.

Definition 3.5.3 (homogeneous element, homogeneous ideal)
A *homogeneous element* is $f \in A_d$ for some d. An ideal $I \subseteq A$ is called *homogeneous* if it is generated by homogeneous elements.

Definition 3.5.4 (Proj(A))
The set Proj(A) is the set of all homogeneous primes in A, which do not contain A_+.

If $I \trianglelefteq A$ is homogeneous, then we can define

$$\mathbb{V}(I) = \{\mathfrak{p} \in \operatorname{Proj}(A) \mid I \subseteq \mathfrak{p}\}$$

We can define the Zariski topology on Proj(A) to have closed sets $\mathbb{V}(I)$ where I is homogeneous.

Let $f \in A_1$ and $U_f = \operatorname{Proj}(A) \setminus \mathbb{V}(f)$. Then observe that $\{U_f\}_{f \in A_1}$ covers Proj(A), since by assumption, elements of $A_1 \subseteq A$ generate $\langle 1 \rangle \trianglelefteq A$.

The ring $A[1/f] = A_f$ is naturally $\mathbb{Z}$-graded, by saying $\deg(f^{-1}) = -\deg(f)$.

Example 3.5.5
Let $A = k[x_0, x_1]$, $f = x_0$, then in

$$A[1/f] = k[x_0, x_1, x_0^{-1}]$$

we have degree zero elements $\lambda \in k$, but also

$$\frac{x_1^d}{x_0^d} \text{ for all } d \geq 0$$

Proposition 3.5.6. There is a natural bijection

$$(\{\text{homog. primes in } A_f\} \leftrightarrow) \{\text{homog. primes in } A \text{ which miss f}\} \leftrightarrow \{\text{primes in } (A_f)_{\deg=0}\}$$

Proof/Construction. First, observe that homogeneous primes in A missing f are naturally in bijection with homogeneous primes in A_f, where we use the fact that f is homogeneous. Suppose $\mathfrak{q} \subseteq (A_f)_{\deg=0}$ is a prime. Then let $\Psi(\mathfrak{q})$ be the ideal generated by

$$\bigcup_{d \geq 0} \left\{ a \in A_d \;\middle|\; \frac{a}{f^d} \in \mathfrak{q} \right\} \subseteq A$$

$\Psi(\mathfrak{q})$ is a prime. Conversely, let $\mathfrak{p} \subseteq A$ be a homogeneous prime missing f, take

$$\varphi(\mathfrak{p}) = \mathfrak{p} \cdot A[1/f] \cap (A[1/f])_{\deg=0}$$

Note that the first part is the contraction, and the second part is the extension of ideals with respect to localisation (almost).

In particular, $\varphi \circ \Psi = \text{id}$. For $\Psi \circ \varphi = \text{id}$, we will prove $\mathfrak{p} = \Psi(\varphi(\mathfrak{p}))$ by showing both containments.

Suppose $\mathfrak{p} \in U_f \subseteq \text{Proj}(A)$, if $a \in \mathfrak{p} \cap A_d$, then (we can assume without loss of generality[4])

$$\frac{a}{f^d} \in \varphi(\mathfrak{p})$$

and so $a \in \Psi(\varphi(\mathfrak{p}))$. Conversely, if $a \in \Psi(\varphi(\mathfrak{p}))$, then

$$\frac{a}{f^d} \in \varphi(\mathfrak{p})$$

for some d. So there exists $b \in \mathfrak{p}$, such that

$$\frac{b}{f^e} = \frac{a}{f^d}$$

But then

$$f^k(f^d b - f^e a) = 0$$

By primality, $f^{e+k} \notin \mathfrak{p}$, and so $a \in \mathfrak{p}$. □

Remark 3.5.7. The bijection we constructed above is order preserving. That is, it defines a homeomorphism

$$U_f \to \text{Spec}\left((A_f)_{\deg=0}\right)$$

That is, $\text{Proj}(A)$ is covered by open sets, each homeomorphic to $\text{Spec}((A_f)_{\deg=0})$ for some f. If $f, g \in A_1$, then $U_f \cap U_g$ is naturally homeomorphic to

$$\text{Spec}\left((A[1/f])_{\deg=0}[f/g]\right) = \text{Spec}((A[f^{-1}, g^{-1}])_{\deg=0})$$

Take the open cover $\{U_f\}$, withj structure sheaf $\mathcal{O}_{(\text{Spec}(A_f))_{\deg=0}}$ on each U_f, and we have isomorphisms on $U_f \cap U_g$ given by the above. The cocycle condition follows from the properties of localisation, and so $\text{Proj}(A)$ is a scheme.

Definition 3.5.8 (projective space)

Let $A = k[x_0, \ldots, x_n]$ with the standard grading, then we denote

$$\mathbb{P}_k^n = \text{Proj}(A)$$

[4]We can write a as a *sum* of such elements.

for the *projective space*.

We will see that this is the same as the gluing construction of projective space, but we don't have a notion of "same" yet.

4 Morphisms

We have seen some maps, which *should* be morphisms. For example, for $U \subseteq X$ an open subscheme, $U \hookrightarrow X$. Or if $f : A \to B$ is a ring homomorphism, then we should have a morphism of schemes $\text{Spec}(B) \to \text{Spec}(A)$.

4.1 Morphisms of schemes and locally ringed spaces

Given a scheme $(X, \mathcal{O}_X)$, the stalks $\mathcal{O}_{X,p}$ are local rings, that is there is a unique maximal ideal $\mathfrak{m}_{X,p}$. Given a function $f \in \mathcal{O}_X(U)$, $p \in U$, we can ask: Does f vanish at p. That is, is $f_p \in \mathfrak{m}_{X,p}$?

Definition 4.1.1 (morphism of ringed spaces)
A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is

1. a continuous map $f : X \to Y$,
2. a morphism of sheaves of rings on Y, $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$.

Intuitively, if $s \in \mathcal{O}_Y(U)$ is a function on $U \subseteq Y$, then $f^\sharp(s) \in f_*\mathcal{O}_X$ is the function $s \circ f$.

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Warning: It is possible to find $(f, f^\sharp)$ a morphism between schemes $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$, with $U \subseteq Y$ open, $q \in U$, $h \in \mathcal{O}_Y(U)$, such that h vanishes at q, and

$$f^\sharp(h) \in \mathcal{O}_X(f^{-1}(U))$$

which does not vanish at $p \in X$ with $f(p) = q$.

Observe that: given a morphism $f : X \to Y$ of ringed spaces, for $p \in X$, we have an induced map

$$f^\sharp : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

That is, given $s \in \mathcal{O}_{Y,f(p)}$, we can represent it as (U, s) where $U \subseteq Y$ open, $f(p) \in U$ and $s \in \mathcal{O}_Y(U)$. With this, $f^\sharp(s) \in \mathcal{O}_X(f^{-1}(U))$, and so the pair $(f^{-1}(U), f^\sharp(s))$ defines an element of $\mathcal{O}_{X,p}$.

Definition 4.1.2 (locally ringed space)
Let $(X, \mathcal{O}_X)$ be a ringed space. It is *locally ringed* if for all $p \in X$, $\mathcal{O}_{X,p}$ is a local ring. A *morphism of locally ringed spaces*

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a morphism of ringed spaces, such that if $\mathfrak{m}_p$ denotes the maximal ideal in $\mathcal{O}_{X,p}$ and $\mathfrak{m}_{f(p)}$ is the maximal ideal of $\mathcal{O}_{Y,f(p)}$, then

$$f^\sharp(\mathfrak{m}_{f(p)}) \subseteq \mathfrak{m}_p$$

This is true for functions, since if s vanishes at q, then for any p with $f(p) = q$, we would like $s \circ f$ to vanish at p. That is, $f^\sharp(s)$ vanishing.

Definition 4.1.3 (morphism of schemes)
A *morphism of schemes* $X \to Y$ is a morphism between the locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Theorem 4.1.4. There is a natural bijection

$$\{\text{morphisms of schemes } \operatorname{Spec}(B) \to \operatorname{Spec}(A)\} \leftrightarrow \{\text{ring homomorphisms } A \to B\}$$

Recall that a section $s \in \mathcal{F}(U)$ is a coherent collection of elements $s(p) \in \mathcal{F}_p$ for all $p \in U$.

Proof. We'll show that every ring homomorphism induce a morphism of schemes, and every morphism between schemes arises via this construction.

Given a ring homomorphism $\varphi : A \to B$, we have an associated continuous map

$$\widetilde{\varphi} = \varphi^{-1} : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

We will now build $f^\sharp : \mathcal{O}_{\operatorname{Spec}(A)} \to \widetilde{\varphi}_* \mathcal{O}_{\operatorname{Spec}(B)}$.

At stalk level, take the map

$$\begin{aligned} A_{\varphi^{-1}(\mathfrak{p})} &\to B_{\mathfrak{p}} \\ \frac{a}{s} &\mapsto \frac{\varphi(a)}{\varphi(s)} \end{aligned}$$

induced by φ. Note that if $s \notin \varphi^{-1}(\mathfrak{p})$, then $\varphi(s) \notin \mathfrak{p}$. This is automatically local, i.e. it sends the maximal ideal $\mathfrak{m}_{f^{-1}(p)} \trianglelefteq A_{\mathfrak{p}}$ to the maximal ideal $\mathfrak{m}_p \trianglelefteq B_{\mathfrak{p}}$.

Given $U \subseteq \operatorname{Spec}(A)$, we need to define

$$\varphi^\sharp : \mathcal{O}_{\operatorname{Spec}(A)}(U) \to \mathcal{O}_{\operatorname{Spec}(B)}(\widetilde{\varphi}^{-1}(U))$$

where $\widetilde{\varphi}^{-1}(U)$ means take the preimage of U under φ^{-1}. An element $s \in \mathcal{O}_{\operatorname{Spec}(A)}(U)$ is a collection of assignments $(\mathfrak{p} \mapsto s(\mathfrak{p}))_{\mathfrak{p} \in U}$, where $\mathfrak{p} \in U, s_{\mathfrak{p}} \in A_{\mathfrak{p}}$. We define

$$\varphi^\sharp : (s \mapsto s(\mathfrak{p}))_{\mathfrak{p} \in U} \mapsto \left(\mathfrak{q} \mapsto \varphi_{\mathfrak{q}}(s(\varphi^{-1}(\mathfrak{q})))\right)_{\mathfrak{q} \in \widetilde{\varphi}^{-1}(U)}$$

We can check that this glues.

Conversely, suppose $(f, f^\sharp) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a morphism of schemes. Using the fact that we have a ring homomorphism

$$A = \mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) \to \mathcal{O}_{\operatorname{Spec}(B)}(\operatorname{Spec}(B)) = B$$

we get a ring homomorphism $g : A \to B$. We need to check that $\widetilde{g} = f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, and the construction from the first part gives the correct map on strcture sheaves.

The maps on stalks are compatible with restriction. That is, the diagram

$$\begin{array}{ccc} \Gamma(\mathcal{O}_{\operatorname{Spec}(A)}, \operatorname{Spec}(A)) & \longrightarrow & \Gamma(\mathcal{O}_{\operatorname{Spec}(B)}, \operatorname{Spec}(B)) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\operatorname{Spec}(A), f(\mathfrak{p})} & \longrightarrow & \mathcal{O}_{\operatorname{Spec}(B), \mathfrak{p}} \end{array}$$

commutes. Equivalently, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ g\ } & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow[f^\sharp]{} & B_{\mathfrak{p}} \end{array}$$

commutes. Since the morphism is local, $(f^\sharp)^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = f(\mathfrak{p})A_{\mathfrak{p}}$. By commutativity of the diagram, $g^{-1} = f$, and the structure sheaf maps agree at stalk level by construction. □

4.2 Housekeeping

Definition 4.2.1 (open immersion, closed immersion)
Let X, Y be schemes. A morphism of schemes $f : X \to Y$ is an *open immersion* if f induces an isomorphism of X onto an open subscheme of Y, $(U, \mathcal{O}_Y|_U)$ where $U \subseteq Y$ is open.

A morphism $g : X \to Y$ is a *closed immersion* if the map on topological space is a homeomorphism onto a closed subset of Y, and $\mathfrak{g}^\sharp : \mathcal{O}_Y \to g_*\mathcal{O}_X$ is surjective.

Intuitively, if we think about $X \subseteq Y$ as a closed subset, then the surjectivity condition says that every function on X is given by the restriction of a function on Y. Equivalently, every function on X extends to a function on a neighbourhood of X.

Example 4.2.2
Consider the ring homomorphism

$$k[x] \mapsto \frac{k[t]}{t^2}$$

Taking spectra, we have a closed immersion.

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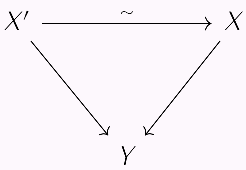
Definition 4.2.3 (closed subscheme)
Let Y be a scheme. Then a *closed subscheme* of Y is an equivalence class of closed immersions $X \to Y$, where $(X \to Y) \sim (X' \to Y)$ are equivalent if there is an isomorphism making the triangle



commute.

Example 4.2.4 (typical example of closed immersion)
If A is a ring, $I \trianglelefteq A$ is an ideal, then the natural map

$$\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$$

is a closed immersion.

To see this, note that the image of the natural map is $\mathbb{V}(I) \subseteq \mathrm{Spec}(A)$, which is a closed subset, and by the correspondence theorem, it is a homeomorphism onto its image.

For surjectivity, we note that it suffices to check at stalk level, since a sequence of sheaves is exact if and only if it is exact at stalk level. Let $f : \mathrm{Spec}(A) \to \mathrm{Spec}(A/I)$ be the map, $\mathfrak{p} \in \mathrm{Spec}(A)$. Let $(U, s) \in (f_*\mathcal{O}_{\mathrm{Spec}(A/I)})_\mathfrak{p}$. That is, $\mathfrak{p} \in U \subseteq \mathrm{Spec}(A)$, $s \in f_*\mathcal{O}_{\mathrm{Spec}(A/I)}(U) = \mathcal{O}_{\mathrm{Spec}(A/I)}(f^{-1}(U))$. By shrinking U, we may assume $U = U_a$, for some $a \in A$, is a distinguished open.

In this case, $f^{-1}(U_a) = U_{a+I}$ essentially by definition, and so $s \in (A/I)_{a+I}$. Thus, this reduces to the fact that the natural map

$$A_a \to (A/I)_{a+I}$$

is surjective, which is a fact in commutative algebra.

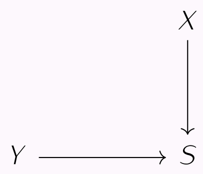
4.3 Fibre products

A fibre product will simultaneuously capture/generalise

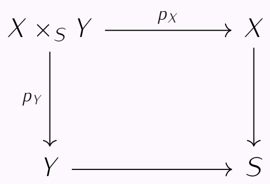
- product of schemes,

- if $X_1, X_2 \subseteq Y$ are closed subschemes, then $X_1 \cap X_2$ is also a closed subscheme,
- given a morphism $f : X \to Y$, and a subscheme $Z \subseteq Y$, the preimage $f6{-}1(Z)$ is a subscheme of X. One special case would be the preimage of a point.

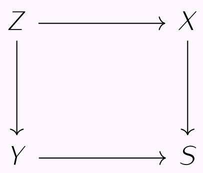
Definition 4.3.1 (fibre product)
Consider a diagram



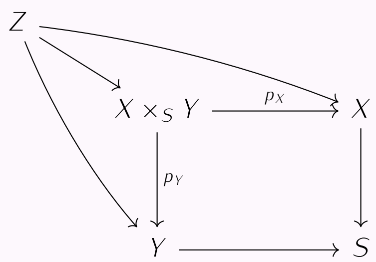
The fibre product is a scheme $X \times_S Y$ filling in the diagram



such that for any other scheme Z, with commuting diagram



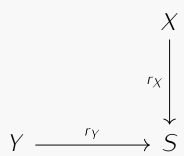
there exists a unique morphisms of schemes $Z \to X \times_S Y$, making



commute.

As usual, if $X \times_S Y$ exists, then it is unique up to unique isomorphism.

Remark 4.3.2. We can similarly define the fibre product for sets. If we have



Then

$$X \times_S Y = \{(x, y) \in X \times Y \mid r_X(x) = r_Y(y)\}$$

In particular, if S is a single point, then $X \times_S Y$ is $X \times Y$.

The fibre product also makes sense for topological spaces, with the same definition as above, and with the subspace topology.

Say $r_X : X \to S$ is a map of sets, $Y = \{*\}$, with $r_Y(*) = s$. Then $X \times_Y S$ is just $r_X^{-1}(s)$. Finally, if r_X, r_Y are inclusion of subsets, then the fibre product is the intersection.

Theorem 4.3.3. Fibre product of schemes exist.

For full details, see Hartshorne Chapter 2 Theorem 3.3.

Proof. **Step 1: Let X, Y, S be affine schemes.** Say $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $S = \mathrm{Spec}(R)$. Then the fibre product $X \times_S Y$ exists, and it is isomorphic to

$$\mathrm{Spec}(A \otimes_R B)$$

That is, we will check that the universal property is satisfied. That is, given any scheme Z with morphisms

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

there exists a unique morphism $Z \to \mathrm{Spec}(A \otimes_R B)$. If Z is affine, then it is clear, using the corresponding morphisms of rings, and the universal property of tensor products.

Fact (Examples sheet 2): A scheme theoretic map $Z \to \mathrm{Spec}(A \otimes_R B)$ is the same data as $A \otimes_R B \to \Gamma(Z, \mathcal{O}_Z)$. In fact, if X is a scheme, then a morphism $X \to \mathrm{Spec}(A)$ is the same as a ring homomorphism $A \to B = \mathcal{O}_X(X)$. To see this, one direction is clear by taking global sections. Now given a ring homomorphism $f : A \to B$, for each open affine $U = \mathrm{Spec}(C) \subseteq X$, we get a ring homomorphism $A \to C$, which is the same as a morphism $U \to \mathrm{Spec}(A)$. It suffices to show that these glue together to give a morphism $X \to \mathrm{Spec}(A)$. But this is clear, since $U \cap V$ is covered by distinguised opens in both U and V, and then it follows by properties of localisation.

Step 2: Now let X, Y, S be any schemes. If $X \times_S Y$ exists, $U \subseteq X$ is an open subscheme, then $U \times_S Y$ also exists. To see this, take the inverse image of U under the projection $X \times_Y S \to X$, with the open subscheme structure.

If X is covered by opens X_i, if $X_i \times_S Y$ exists for all i, then $X \times_S Y$ exists. This is because the schemes will glue together. Note that in this case there are no cocycle conditions.

Step 3: Let X be any scheme, S, Y are affine, then by steps 1 and 2, $X \times_S Y$ exists.

Step 4: Let X, Y be any scheme, S is affine. This is because we can exchange X and Y in the above.

Step 5: Let X, Y, S be any schemes. Say S is covered by affines S_i. Let X_i, Y_i be the preimages of S_i in X and Y respectively. Since the S_i are affine, $X_i \times_{S_i} Y_i$ exists. By the universal property, $X_i \times_{S_i} Y_i = X_i \times_S Y_i$. Finally, we glue these together to get $X \times_S Y$. □

Example 4.3.4

$\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{C})$, where $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{Z})$ is induced by $\mathbb{Z} \hookrightarrow \mathbb{C}$, and $\mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ is induced locally by the inclusion $\mathbb{Z} \to \mathbb{Z}[x_1/x_0, \dots, x_n/x_0]$.

For this, recall from commutative algebra that

$$\mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}[x]$$

Example 4.3.5

Let $C = \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{\langle y - x^2 \rangle}\right)$, $L = \mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{\langle y \rangle}\right)$. We have natural maps $C \to \mathbb{A}^2_{\mathbb{C}}$ and $L \to \mathbb{A}^2_{\mathbb{C}}$. In fact, the morphisms are closed immersions. By some algebra:

$$C \times_{\mathbb{A}^2_{\mathbb{C}}} L \cong \mathrm{Spec}\left(\frac{C[x]}{\langle x^2 \rangle}\right)$$

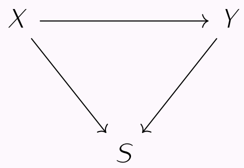
In this case, we have one 'point' at the intersection, but it keeps track of the multiplicity.

Lecture 14

For the definitions which are omitted from lectures, see Examples Sheet 2 or Hartshorne.

Definition 4.3.6 (base scheme, scheme over)

In scheme theory, we often fix a scheme S, and we refer to it as the *base scheme*. We then work over a fixed base, and consider schemes X with a fixed morphism $X \to S$, called *schemes over* S. These form a category $\mathbf{Sch}/S$, with morphisms being commuting triples



A typical example would be $S = \mathrm{Spec}(k)$, or $S = \mathrm{Spec}(\mathbb{Z})$. The product of X, Y in $\mathbf{Sch}/S$ is the fibre product $X \times_S Y$.

4.4 Separated morphisms

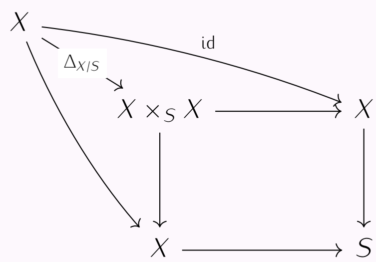
For motivation, recall that a topological space X is Hausdorff if and only if the diagonal $\Delta_X \subseteq X \times X$ is closed.

Definition 4.4.1 (diagonal)

Let $X \to S$ be a morphism of schemes. Then the *diagonal* is the morphism

$$\Delta_X/S : X \to X \times_S X$$

induced by the following diagram



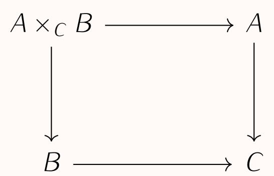
If X, S are clear, we will just write Δ.

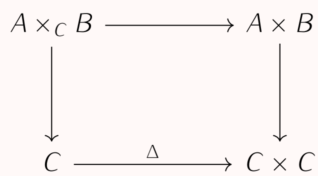
Example 4.4.2

If $U, V \subseteq X$ are open subschemes, where $S = \mathrm{Spec}(k)$, k a field, then

$$\Delta^{-1}(U \times_S V) = U \cap V$$

For some abstract nonsense, the diagrams





are "the same".

Definition 4.4.3 (separated)
A morphism $X \to S$ is *separated* if $\Delta_{X/S} : X \to X \times_S X$ is a closed immersion.

That is, the algebraic geometer's version of Hausdorff.

Example 4.4.4
Say $X = \operatorname{Spec}(\mathbb{C}[t])$ and $S = \operatorname{Spec}(\mathbb{C})$, $X \to S$ induced by $\mathbb{C} \to \mathbb{C}[t]$, then

$$X \times_S X = \operatorname{Spec}(\mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[u]) = \operatorname{Spec}(\mathbb{C}[t, u])$$

The diagonal map Δ is induced by taking Spec of

$$\begin{aligned} \mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[t] &\to \mathbb{C}[t] \\ f \otimes g &\mapsto fg \end{aligned}$$

To see that Δ is closed, the map above is clearly surjective[a]. More generally,

$$\mathbb{A}^n_k \to \operatorname{Spec}(k)$$

is separated.

[a] and a surjective ring homomorphism is "the same" as a quotient.

Proposition 4.4.5. Let $X \to S$ be a morphism of schemes. Then there exists a factorisation of $\Delta_{X/S}$, with

$$X \xrightarrow{\text{closed imm.}} U \xrightarrow{\text{open imm.}} X \times_S X, \qquad \Delta_{X/S} : X \to X \times_S X$$

That is, it is a *locally closed immersion*.

Proof. Let $g : X \to S$ be the morphism of schemes. Say S is covered by open affines $\{V_i\}$, and suppose X is covered by affine opens $\{U_{ij}\}$, where for fixed i,

$$g^{-1}(V_i) = \bigcup_j U_{ij}$$

We have morphisms $U_{ij} \to V_i$ induced by

$$\begin{array}{ccccc} U_{ij} & \longrightarrow & g^{-1}(V_i) & \xrightarrow{g} & V_i \\ & & \downarrow & & \downarrow \\ & & X & \longrightarrow & S \end{array}$$

Now observe $U_{ij} \times_{V_i} U_{ij}$ is an affine open in $X \times_S X$, and their union over i, j contains the image of $\Delta_{X/S}$, and

$$\Delta^{=1}(U_{ij} \times_{V_i} U_{ij}) = U_{ij} \subseteq X$$

Take

$$U = \bigcup_{i,j} U_{ij} \times_{V_i} U_{ij}$$

The second map is clearly an open immersion. Now observe that to check a morphism $T \to T'$ is a closed immersion, it suffices to check this locally on the codomain. That is, for U_{ij} affine, the diagonal gives a map $U_{ij} \to U_{ij} \times_{V_i} U_{ij}$, which is clearly[5] a closed immersion. □

Proposition 4.4.6. If $X \to S$ is a morphism of affine schemes, then $\Delta_{X/S}$ is a closed immersion.

Proof. For $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(B)$, $X \to S = \mathrm{Spec}(B \to A)$, then the map

$$A \otimes_B A \to A$$

is surjective. □

Example 4.4.7

Recall the bug-eyed line from example 3.4.1. That is,

$$X = \frac{\mathbb{A}^1_k \sqcup \mathbb{A}^1_k}{\sim}$$

where we glued along $U = \mathbb{A}^1_k \setminus \{0\}$, using

$$\begin{aligned} k[u, u^{-1}] &\to k[t, t^{-1}] \\ u &\mapsto t \end{aligned}$$

We claim that this is not separated over $S = \mathrm{Spec}(k)$. We can compute $X \times_S X$ using a gluing construction of the fibre product, giving a plane with doubled axes and 4 origins. But the diagonal only contains two out of the four origins, and this is not a closed subset.

Example 4.4.8 (to check/wait for)

Open and closed immersions are always separated. In the closed immersion case, the key observation is that

$$\frac{A}{I} \otimes_A \frac{A}{I} = \frac{A}{I+I} = \frac{A}{I}$$

and so the diagonal map is just the identity.

An easy consequence of proposition 4.4.6 is that if $X \to S$ is a morphism of schemes, if $\mathrm{im}(\Delta_{X/S})$ is closed as a topological subspace, then $X \to S$ is separated.

To see this, a locally closed immersion where the image is closed is a closed immersion.

Proposition 4.4.9. Let k be a field, $X \to \mathrm{Spec}(k)$ is morphism of schemes, and $U, V \subseteq X$ be affine opens. If $X \to \mathrm{Spec}(k)$ is separated, then $U \cap V$ is also affine.

Lecture 15

Proposition 4.4.10. Composition of separated morphisms is separated.

Example 4.4.11 (base change)

Composition of separated morphisms is separated. Suppose $X \to S$ is separated, $S' \to S$ arbitrary, then

[5] See example 4.4.4, the idea is the same once we unfold the definitions.

the map $X \times_S S' \to S'$ coming from the fibre product is also separated.

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

This will be on examples sheet 3.

Intuitively, "a morphism is separated if each fibre is Hausdorff".

Proposition 4.4.12. Let A be a ring, then the morphism $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ is separated.

Proof. We would like to show that in the following diagram

$$\begin{array}{ccccc} \mathbb{P}^n_R & \xrightarrow{\Delta} & \mathbb{P}^n_R \times_R \mathbb{P}^n_R & \longrightarrow & \mathbb{P}^n_R \\ & & \downarrow & & \downarrow \\ & & \mathbb{P}^n_R & \longrightarrow & \operatorname{Spec}(R) \end{array}$$

the map Δ is closed. By abuse of notation, we write $A \times_R B$ for $A \times_{\operatorname{Spec}(R)} B$. It suffices to check this on an open cover of $\mathbb{P}^n_R \times_R \mathbb{P}^n_R$. Let $A = R[x_0, \dots, x_n]$ with the usual grading, and let $U_i = \operatorname{Spec}((A[1/x_i])_{\deg=0})$. From our discussion of Proj, the U_i's cover $\mathbb{P}^n_R$. Now

$$U_i \times_R U_j = \operatorname{Spec}\left(R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_0}{y_j}, \cdots, \frac{y_n}{y_j}\right]\right)$$

Observe the restriction of Δ to $\Delta^{-1}(U_i \times U_j)$ is precisely

$$U_i \cap U_j \to U_i \times_R U_j$$

given on rings by

$$R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]\left[\frac{x_i}{x_j}\right] \leftarrow R\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_0}{y_j}, \cdots, \frac{y_n}{y_j}\right]$$

by sending y_k to x_k. This is clearly a surjection, and the $U_i \times_R U_j$ cover, and so the map is closed. □

Let $k = \overline{k}$ be an algebraically closed field, $X \to \operatorname{Spec}(k)$ be a scheme over $\operatorname{Spec}(k)$. We say X is of *finite type* if there exists an open cover $\{U_\alpha\}$ of affines covering X, with each $\mathcal{O}_X(U_\alpha)$ being a finitely generated k-algebra.

We say X is *reduced* if for all opens $U \subseteq X$, $\mathcal{O}_X(U)$ has no non-zero nilpotents (i.e. it is a reduced ring).

Definition 4.4.13 (variety)
$X \to \operatorname{Spec}(k)$ is a *variety* if it is reduced, of finite type and separated.

Example 4.4.14
Part II AG.

4.5 Properness

Definition 4.5.1 (finite type)

Let $f : X \to S$ be a morphism of schemes, then f is of *finite type* if there exists an affine open cover of S by $\{V_\alpha\}$, where $V_\alpha = \operatorname{Spec}(A_\alpha)$, and corresponding covers $\{U_{\alpha\beta}\}$ of $f^{-1}(V_\alpha)$ by open affines, with each $U_{\alpha\beta} = \operatorname{Spec}(B_{\alpha\beta})$, such that $B_{\alpha\beta}$ is a finitely generated A_α-algebra, and for each α, we can cover $f^{-1}(V_\alpha)$ by finitely many $U_{\alpha\beta}$.

Definition 4.5.2 (universally closed)

Suppose $f : X \to S$ is a closed map (topologically). It is *universally closed* if for any $S' \to S$, the induced map $X \times_S S' \to S'$ is also closed.

Definition 4.5.3 (proper)

We say that f is *proper* if it is separated, finite type and universally closed.

Example 4.5.4 (check/wait)

Closed immersions are proper.

Example 4.5.5 (non-example)

The obvious map $\mathbb{A}^1_k \to \operatorname{Spec}(k)$ is not proper. It is clearly separated and finite type, and so it suffices to show that it is not univerally closed. The map $\mathbb{A}^1_k \to \operatorname{Spec}(k)$ is closed. Now consider the base extension

$$\begin{array}{ccc} \mathbb{A}^2_k & \longrightarrow & \mathbb{A}^1_k \\ \downarrow & & \downarrow \\ \mathbb{A}^1_k & \longrightarrow & \operatorname{Spec}(k) \end{array}$$

Intuitively, the map $\mathbb{A}^2_k \to \mathbb{A}^1_k$ is the projection map. But this is not necessarily a closed map. For "example", consider $xy = 1$. The projection onto the x-axis is $\mathbb{A}^1_k \setminus \{0\}$.

More precisely, let $Z = \mathbb{V}(xy - 1)$, then the projection of Z is not Zariski closed.

Observe if $X \to S$ is proper, then any base extension $X \times_S S' \to S'$ is also proper.

Lecture 16

Notation 4.5.6. If the morphism is $X \to \operatorname{Spec}(k)$, often we say "$X$ is proper", or "X is separated".

Example 4.5.7

Line with two origins is neither separated nor universally closed.

Proposition 4.5.8. Let R be any ring, then the map $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ is proper.

Observe that a morphism $X \to S$ being universally closed is stable under base extension. Since we already saw $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ is separated, and finite type is clear. Hence all we need to check is the case $R = \mathbb{Z}$, since

$$\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(R)$$

Proof. We must show that for any $Y \to \text{Spec}(Z)$, the base extension $\mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec}(\mathbb{Z})} Y \to Y$ is closed. But Y is covered by affine schemes of the form $\text{Spec}(R)$, and closedness is local on the target, it suffices to show $\mathbb{P}^n_R \to \text{Spec}(R)$ is closed. Let $Z \subseteq \mathbb{P}^n_R$ be Zariski closed. That is, $Z = \mathbb{V}(g_1, \dots)$ of homogeneous polynomials g_i. If $\pi : \mathbb{P}^n_R \to \text{Spec}(R)$ is the map, then we would like to show that $\pi(Z)$ is a closed set in $\text{Spec}(R)$.

That is, we need equations for $\pi(Z)$. Equvialently, we need to characterise those primes $\mathfrak{p} \trianglelefteq R$ such that $\pi^{-1}(\mathfrak{p}) \cap Z$ is non-empty.

Let $K(\mathfrak{p}) = \text{Frac}(R/\mathfrak{p})$, and then we have a morphism $\text{Spec}(K(\mathfrak{p})) \to \text{Spec}(R)$. We would like to know for which $\mathfrak{p}$ is $Z_{\mathfrak{p}} = Z \times_{\text{Spec}(R)} \text{Spec}(K(\mathfrak{p}))$ non-empty.

What is $Z_{\mathfrak{p}}$? We take the equations $g_1, g_2, \dots$, which are homogeneous polynomials with coefficients in R. Reducing mod $\mathfrak{p}$, we get $\overline{g_1}, \overline{g_2}, \dots$, which has coefficients in $K(\mathfrak{p})$.

So $Z_{\mathfrak{p}}$ is non-empty if and only if $\overline{g_1}, \overline{g_2}, \dots$ cut out more than origin in $\mathbb{A}^{n+1}_{K(\mathfrak{p})}$. Thus,

$$Z_{\mathfrak{p}} \text{ is non-empty} \iff \sqrt{\langle \overline{g_1}, \overline{g_2}, \dots \rangle} \not\supseteq \langle x_0, \dots, x_n \rangle$$

where $\mathbb{P}^n_R = \text{Spec}(R[x_0, \dots, x_n])$.

Equivalently, for all positive integers d,

$$\langle x_0, \dots, x_n \rangle^d \not\subseteq (\overline{g_1}, \overline{g_2}, \dots)$$

Write $A = R[x_0, \dots, x_n]$ with the usual grading. Non-containment is equivalent to the map

$$\bigoplus_i A_{d-\deg(g_i)} \to A_d$$

given by

$$(f_i)_i \mapsto \sum_i f_i g_i$$

being non-surjective mod$\mathfrak{p}$ (equivalently in $K(\mathfrak{p})$), for all d. The condition is given by vanishing of maximal minors of the matrix associated to the above map, which is infinitely many polynomials in the g_i. ☐

From now on, all schemes will be assumed to be Noetherian. That is, it has a finite cover by open subschemes of the form $\text{Spec}(R)$, where R is a Noetherian ring.

4.6 Valuative criteria (for separatedness and properness)

Recall a *discete valuation ring* is a local PID[6].

Example 4.6.1

$\mathbb{C}[\![t]\!]$ is a DVR, and so is

$$\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \;\middle|\; g(0) \neq 0 \right\}$$

Moreover, so are $\mathbb{Z}_{\langle \mathfrak{p} \rangle}$ (localisation) and $\mathbb{Z}_p$ (p-adic integers).

Let A be a discrete valuation ring, then $\text{Spec}(A)$ consists of two points, $0 \trianglelefteq A$ and the maximal ideal $\mathfrak{m} \trianglelefteq A$. The topology on $\text{Spec}(A)$ has:

- $\{0\}$ is dense,
- $\mathfrak{m}$ is closed.

Any generator π of $\mathfrak{m}$ is called a *uniformiser* or a *uniformising parameter*.

In $\mathbb{C}[\![t]\!]$, $\langle t \rangle$ is a maximal ideal, and the units are power series with non-zero constant coefficient. Intuitively, $\mathbb{C}[t]$ is the line, $\mathbb{C}[\![t]\!]$ is the "germ of the curve at 0".

Any element $a \in A$ can be written as $u\pi^k$, where $u \in A$ is a unit, k is unique. The integer k is called the *valuation of a*. This gives a map

$$\text{val} : A \setminus 0 \to \mathbb{N}$$

Lecture 17

[6] As we are assuming the ring is Noetherian.

$K = \mathrm{Frac}(A)$ is a *valued field* and val extends to a map

$$\mathrm{val} : K^\times \to \mathbb{Z}$$

with $\mathrm{val}(a/b) = \mathrm{val}(a) - \mathrm{val}(b)$.

If we take $A = k[\![t]\!]$, then $K = k(\!(t)\!)$, and the valuation of

$$a_\ell t^\ell + a_{\ell+1} t^{\ell+1} + \cdots$$

with $a_\ell \neq 0$ is ℓ.

In this case, we have an open immersion $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$. As an analogy, we can think of $\mathrm{Spec}(K)$ as the punctured unit disc, and $\mathrm{Spec}(A)$ as the unit disc. Intuitively, if we map $\mathrm{Spec}(K)$ to a "compact" space, we can extend it to $\mathrm{Spec}(A)$ by "filling in the origin".

Fianlly, recall (sequential) compactness and Hausdorff can be stated in terms of sequences. $\mathrm{Spec}(K)$ will be out version of sequences.

Theorem 4.6.2 (valuative criterion)**.** If $f : X \to Y$ is a morphism of schemes, then f is separated if and only if for any discrete valuating ring A, with fraction field K, given the following diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ \downarrow & \overset{g}{\nearrow} & \downarrow f \\ \mathrm{Spec}(A) & \longrightarrow & Y \end{array}$$

with the solid arrows, there exists *at most* one choice of g. Similarly, f is universally closed if and only if there exists *at least* one choice of g.

Proof. Omitted, therefore non-examinable. □

Corollary 4.6.3.
(i) $\mathbb{P}^n_R \to \mathrm{Spec}(R)$ is proper,

(ii) $\mathbb{A}^n_R \to \mathrm{Spec}(R)$ is not proper, but it is separated,

(iii) closed immersions are proper, and so if we have $f : Z \to \mathbb{P}^n_R$ is closed, then the induced map $Z \to \mathrm{Spec}(R)$ is proper,

(iv) composition of proper (resp. separated) morphisms is proper (resp. separated),

(v) if $f : X \to Y$ is proper, $Y' \to Y$ arbitrary, then the map $X \times_Y Y' \to Y'$ is also proper.

Proof. For (i) and (iv), see Dhruv's notes. For (v) see Hartshorne. Otherwise omitted. However we will verify some of the statements in some examples.

$\mathbb{A}^1_k \to \mathrm{Spec}(k)$ **is not proper (i.e. not universally closed)**. Say $\mathbb{A}^1_k = \mathrm{Spec}(k[x]), A = k[\![t]\!], K = k(\!(t)\!)$.

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & \mathbb{A}^1_k \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

Let $\varphi : \mathrm{Spec}(K) \to \mathbb{A}^1_k$ be induced by the map on rings,

$$\begin{aligned} k[x] &\to k(\!(t)\!) \\ x &\mapsto \frac{1}{t} \end{aligned}$$

This doesn't factor through $k[\![t]\!]$, and so it does not extend.

Exercise: Use valuative criteria to show that if $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ is proper, then $\mathrm{Spec}(A)$ is finite.

Observe if $\mathbb{A}^1_k$ is replaced by $\mathbb{P}^1_k$, then there is always an an affine chart in $\mathbb{P}^1_k$, such that the map above looks like $x \mapsto t$. □

5 Modules over $\mathcal{O}_X$

5.1 Motivation

Example 5.1.1 (please forget all scheme theory)

Consider the variety

$$\mathbb{C}\mathrm{P}^n = \frac{\mathbb{C}^{n+1} \setminus 0}{\mathbb{C}^*}$$

On this, we have a structure sheaf $\mathcal{O}_{\mathbb{C}\mathrm{P}^n}$, and if $U \subseteq \mathbb{C}\mathrm{P}^n$ is Zariski open, then

$$\mathcal{O}_{\mathbb{C}\mathrm{P}^n}(U) = \left\{ \frac{P(x)}{Q(x)} \;\middle|\; P, Q \text{ homogenous of the same degree, } P/Q \text{ regular on U} \right\}$$

For any integer d, we can consider the sheaf of abelian groups $\mathcal{O}_{\mathbb{C}\mathrm{P}^n}(d)$, given by

$$\mathcal{O}_{\mathbb{C}\mathrm{P}^n}(d)(U) = \left\{ \frac{P(x)}{Q(x)} \;\middle|\; P, Q \text{ homogenous with } \deg(P) - \deg(Q) = d, P/Q \text{ regular on U} \right\}$$

In fact, $\mathcal{O}_{\mathbb{C}\mathrm{P}^n}(d)(U)$ is an $\mathcal{O}_{\mathbb{C}\mathrm{P}^n}(U)$ module in the natural way.

Example 5.1.2 (please remember all of scheme theory)

Let A be a ring, M an A-module. Define a sheaf $\mathcal{F}_M$ on $\mathrm{Spec}(A)$ of abelian groups, if $U_f \subseteq \mathrm{Spec}(A)$ is a distinguished open, then we can set

$$\mathcal{F}_M(U) = M_f$$

which is the localisation. On general opens, use sheaf on a base construction.

Another way to think about this as the Algebraic Geometry analogue of vector bundles.

5.2 Definitions

Fix a ringed space $(X, \mathcal{O}_X)$.

Definition 5.2.1 (sheaf of $\mathcal{O}_X$-modules)

A *sheaf of $\mathcal{O}_X$-modules* is a sheaf $\mathcal{F}$ of groups, along with a map $\mathcal{O}_X(U) \to \mathcal{F}(U)$ making $\mathcal{F}(U)$ into an $\mathcal{O}_X(U)$-module. Moreover, we require this to be compatible with restrictions, i.e.

$$(r \cdot m)|_V = r|_V \cdot m|_V$$

Similarly, we can define a sheaf of $\mathcal{O}_X$-algebras. A *morphism between sheaves of $\mathcal{O}_X$-modules* is defined in the usual way, that is, a morphism between sheaves of abelian groups compatible with the $\mathcal{O}_X$-module structure. Lecture 18

Example 5.2.2 (sheaf associated to a module)

If $X = \mathrm{Spec}(A)$, M an A-module, then we have a sheaf M^{sh} on X, such that

$$M^{\mathrm{sh}}(U_f) = M_f$$

and we extend it to all opens.

This is essentially the same as the construction of the structure sheaf from a sheaf on a base. Also note Hartshorne uses $\widetilde{M}$ for this sheaf.

We have basic operations:

- give a morphism $f : \mathcal{F} \to \mathcal{G}$ of (sheaves of) $\mathcal{O}_X$-modules, we have $\ker(f)$, $\operatorname{coker}(f)$, $\operatorname{im}(f)$,
- we can take direct sums, direct products, tensor product, Homs,

which extend in the "natural way". Note coker, im and tensor product and Homs requires sheafification.

The sheaf tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ has

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$$

and we can then sheafify this.

If $f : X \to Y$ is a morphism of ringed spaces (or schemes), and given an $\mathcal{O}_X$-module $\mathcal{F}$, the *pushforward* $f_*\mathcal{F}$ is a sheaf of abelian groups, but we have a map $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. This gives $f_*\mathcal{F}$ and $\mathcal{O}_Y$-module structure.

Given $U \subseteq Y$ open, $a \in \mathcal{O}_Y(U)$, $m \in f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, we define

$$am := f^\sharp(a)m$$

where we note that $f^\sharp(a) \in \mathcal{O}_X(f^{-1}(U))$. Conversely, if $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, then we define the *pullback* sheaf

$$f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_y} \mathcal{O}_X$$

where the $f^{-1}\mathcal{O}_Y$-module structure on $\mathcal{O}_X$ is defined via the adjoint to $f^\#$. See examples sheet 1 Q14. That is, if X, Y are spaces, $f : X \to Y$ a continuous map, $\mathcal{F}$ a sheaf on X, $\mathcal{G}$ a sheaf on Y, then we have a natural bijection

$$\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \leftrightarrow \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

Using this, a homomorphism $f^{-1}\mathcal{O}_Y \to f^*\mathcal{G}$ is the same as a homomorphism $\mathcal{O}_Y \to f_*f^*\mathcal{G}$, which we defined above.

5.3 $\mathcal{O}_X$-modules on schemes and quasi-coherence

Definition 5.3.1 ((quasi-)coherent sheaf)

A *quasi-coherent sheaf $\mathcal{F}$ (on a scheme X)* of $\mathcal{O}_X$-modules is a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, such that there exists a cover of X by affines U_i, such that $\mathcal{F}|_{U_i}$ is the sheaf associated to a module over the ring $\mathcal{O}_X(U_i)$.

If the module over the $\mathcal{O}_X(U_i)$ can be taken to be finitely generated, we say that $\mathcal{F}$ is *coherent*[a].

[a] Recall we assumed our schemes are Noetherian.

Example 5.3.2

On any scheme, $\mathcal{O}_X$ is quasi-coherent (in fact coherent). More generally, $\mathcal{O}_X^{\oplus n}$ is coherent. On the other hand, $\mathcal{O}_X^{\oplus I}$ is quasi-coherent but not coherent if I is infinite.

Example 5.3.3

If $i : X \hookrightarrow Y$ is a closed module, then $i_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_Y$-module. Say $U = \operatorname{Spec}(A) \subseteq Y$ is affine, then $X \cap U \hookrightarrow U$ gives an ideal $I \trianglelefteq A$, which is the kernel of the map on structure sheafs $\mathcal{O}_Y(U) \twoheadrightarrow \mathcal{O}_X(X \cap U)$. On U, $i_*\mathcal{O}_X|_U$ is the sheaf associated to the A-module A/I.

Proposition 5.3.4. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for any open affine $U = \operatorname{Spec}(A) \subseteq X$, $\mathcal{F}|_U$ is the sheaf associated to a module over A.

Similarly, $\mathcal{F}$ is coherent if and only if each $\mathcal{F}|_U$ is finitely generated as an A-module.

Lemma 5.3.5. Let $X = \mathrm{Spec}(A)$ be a scheme, $f \in A$, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Let $s \in \Gamma(X, \mathcal{F})$. Then

(i) if s restricts to zero on U_f, then $f^n s = 0$ for some n.

(ii) if $t \in \mathcal{F}(U_f)$, then for some n, $f^n t$ is the restriction of a global section of $\mathcal{F}$.

Proof. There exists some cover of X by affine opens $V = \mathrm{Spec}(B)$, such that

$$\mathcal{F}_V = M^{\mathrm{sh}}$$

for a B-module M. But we can cover V by distinguished affines of the form U_g for some $g \in A$. In this case,

$$\mathcal{F}|_{U_g} = \left(M \otimes_B A_g\right)^{\mathrm{sh}}$$

since $\mathcal{F}|_V$ is already quasi-coherent. But recall that $\mathrm{Spec}(A)$ is quasi-compact, i.e. every open cover has a finite subcover, and so finitely many g_i, U_{g_i} and M_i will suffice to cover X by open such that

$$\mathcal{F}|_{U_{g_i}} = M_i^{\mathrm{sh}}$$

Result then follows from formal properties of localisation. See Hartshorne for details. □

Proof of proposition 5.3.4. Let $\mathcal{F}$ be a quasicoherent sheaf on X. Given $U \subseteq X$ open, $\mathcal{F}|_U$ is also quasi-coherent. Hence we can reduce to the case when $X = \mathrm{Spec}(A)$. Take $M = \mathcal{F}(X)$, and M^{sh} the associated sheaf. We claim that $M^{\mathrm{sh}} \cong F$.

Let $\alpha : M^{\mathrm{sh}} \to \mathcal{F}$ be the map given by restricting global sections (e.g. via stalks). Moreover, α is an isomorphism at stalk level. But this is just the lemma. □

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Facts (proofs omitted and so non-examinable):

- images, kernels and cokernels of maps of quasi-coherent (resp. coherent) sheaves remain quasi-coherent (resp. coherent).
- if $f : X \to S$ is a morphism of schemes, $\mathcal{F}$ on S is quasi-coherent (resp. coherent). Then $f^*\mathcal{F}$ is quasi-coherent (resp. coherent).
- if $f : X \to S$ is a morphism of schemes, $\mathcal{G}$ a quasi-coherent sheaf on X, then $f_*\mathcal{G}$ is quasi-coherent on S. In general, if $\mathcal{G}$ is coherent, then $f_*\mathcal{F}$ need not be coherent. For example, take the natural map $\mathbb{A}^1_k \to \mathrm{Spec}(k)$, then $f_*\mathcal{O}_{\mathbb{A}^1_k}$ is a quasi-coherent sheaf on $\mathrm{Spec}(k)$, i.e. a k-vector space. In particular, it is $k[x]$, which is *not* a finite dimensional k-vector space.

  Observe if we took $\mathbb{P}^1 \to \mathrm{Spec}(k)$ instead, then $f_*\mathcal{O}_{\mathbb{P}^1}$ is just the sheaf associated to k.

  More generally, if $\mathcal{G}$ is a coherent sheaf on X, $f : X \to S$ is proper, then $f_*\mathcal{F}$ is cohereent. We will prove this for closed immersions on examples sheet 3.

Source of examples: Let A be an $\mathbb{N}$-graded ring (with the usual assumptions), we built $\mathrm{Proj}(A)$ which is a scheme. This was covered by $\mathrm{Spec}(A[1/f]_0)$ for $f \in A_1$.

Definition 5.3.6

Let M be a graded A-module, that is,

$$M = \bigoplus_{d \in \mathbb{Z}} M_d$$

where M_d is an abelian group. M is an A-module, and $A_i M_j \subseteq M_{i+j}$. Consider the sheaf determined by the association

$$\mathrm{Proj}(A) \supseteq U_f \mapsto M[1/f]_0$$

That is, the degree zero part of localisation of M at f. This gives a quasi-coherent sheaf on $\mathrm{Proj}(A)$, by the same arguments as in the construction of $\mathrm{Proj}(A)$.

Notation 5.3.7. Let X be a scheme, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. We say

- $\mathcal{F}$ is *free* if $\mathcal{F} \cong \mathcal{O}_X^{\bigoplus I}$ for some indexing set I. Note these are coherent if and only if the indexing set I is finite.
- $\mathcal{F}$ is called a *(algebraic) vector bundle* if there exists an open cover $\{U_i\}$, such that $\mathcal{F}|_{U_i}$ is free. This is also known as *locally free*.
- A *line bundle* or an *invertible sheaf* is a vector bundle which is locally isomorphic to $\mathcal{O}_X$.

5.4 Coherent sheaves on projective space

Definition 5.4.1

Let A be a graded ring, M a graded A-module. Let d be an integer, and define the *twisting* $M(d)$ for the module such that

$$M(d)_k = M_{k+d}$$

Let $X = \text{Proj}(A)$, then the sheaf $\mathcal{O}_X(d)$ is the sheaf associated to the graded module $A(d)$ for $d \in \mathbb{Z}$. We will call $\mathcal{O}_X(1)$ the *twisting sheaf*.

Remark 5.4.2. $\mathcal{O}_X(d) = \mathcal{O}_X(1)^{\otimes d}$. This follows from the fact that tensor product on graded modules acts additively on the grading. In particular, if A is a graded ring, M, N graded A-modules, then $M \otimes N$ is a graded module, with

$$(M \otimes N)_k = \bigoplus_{i+j=k} M_i \otimes N_j$$

Moreover, by definition,

$$M^{\text{sh}} \otimes N^{\text{sh}} = (M \otimes N)^{\text{sh}}$$

and so it suffices to show that $A(d) = A(1)^{\otimes d}$. We show this by induction on d. The case $d = 1$ is clear. Now

$$(A(1) \otimes A(d))_k = \bigoplus_{i+j=k} A(1)_i \otimes A(d)_j = \bigoplus_{i+j=k} A_{i+1} \otimes A_{j+d}$$

But the right hand side is precisely A_{d+k+1}.

Let $X = \text{Proj}(k[x_1, \dots, x_n]) = \mathbb{P}^n_k$. Then global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$ are naturally identified with homogeneous degree d polynomials in the x_i. In particular, if $d < 0$, then there are no non-zero global sections of $\mathcal{O}_{\mathbb{P}^n}(d)$.

Definition 5.4.3 (globally generated)

An $\mathcal{O}_X$-module $\mathcal{F}$ is called *globally generated* or *generated by global sections* if it is a quotient of $\mathcal{O}_X^r$ for some r. That is, if there exists a surjective map $\mathcal{O}_X^r \to \mathcal{F}$.

Equivalently, there exists $s_1, \dots, s_r \in \mathcal{F}(X)$, such that the s_i generate the stalks $\mathcal{F}_p$ over $\mathcal{O}_{X,p}$ for all p.

Let $i : X \to \mathbb{P}^n_R$ be a closed immersion. $\mathcal{O}_X(1)$ be the restriction of $\mathcal{O}_{\mathbb{P}^n}(1)$ to X. That is, $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^n}(1)$.

Theorem 5.4.4 (Serre)**.** Let $\mathcal{F}$ be a coherent sheaf on X. Then there exists $d_0 \in \mathbb{Z}$, such that for all $d \geq d_0$, the sheaf

$$\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$$

is globally generated.

In particular, every $\mathcal{F}$ as above is a quotient of a vector bundle.

Proof. By formal properties, it is equivalent to show the statement for $i_*\mathcal{F}$. That is, $i_*\mathcal{F}(d)$ is globally generated on $\mathbb{P}^n_R$. More precisely, $i_*\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n_k$, and $i_*\mathcal{F}(d) = i_*(\mathcal{F}(d))$. Moreover, the global sections of both are the same.

Strategy: First we will cover by affines $U_i = \text{Spec}(R[x_0/x_i, \dots, x_n/x_i])$. Then $i_*\mathcal{F}|_{U_i}$ is a sheaf associated to a module M_i. Choose generators $\{s_{ij}\}$ for M_i. Finally, we will clear denominators by multiplying by x_i^d for some large d, and extend them to generators of global sections of $\mathcal{F}(d)$.

Write $\mathbb{P}^n_R = \mathrm{Proj}(R[x_0, \ldots, x_n])$, and cover $\mathbb{P}^n_R$ by U_i, where

$$U_i = \mathrm{Spec}\left(R\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]\right)$$

Now $\mathcal{F}|_{U_i} = M_i^{\mathrm{sh}}$, where M_i is a finitely generated R_i-module.

Choose generators $\{s_{ij}\}$ for M_i.

Claim 5.4.5. The sections $\{x_i^d s_{ij}\}_j$ of $\mathcal{F}(d)(U_i) = (\mathcal{F} \otimes \mathcal{O}(d))(U_i)$ are restrictions of global sections $t_{ij} \in \Gamma(\mathbb{P}^n, \mathcal{F}(d))$ for all sufficiently large d.

Proof. Left as an exercise. Say $s_{ij} \in M_i = \mathcal{F}_i(U_i)$, and let $x_i \in \mathcal{O}(1)$. We claim that $x_i^d s_{ij} \in (\mathcal{F} \otimes \mathcal{O}(d))|_{U_i}$ is the restriction of a global section.

In the case when $X = \mathbb{P}_1$, we can cover $U_1 = \mathbb{P}^1 \setminus \{0\}$ and $U_2 = \mathbb{P}^2 \setminus \{\infty\}$. Restrict s_{1j} to $U_1 \cap U_2$. By lemma 5.3.5, this is a restriction after multiplying by a high power of x_i. ☐

On U_i, s_{ij} gobally generate M_i^{sh}, but we have a morphism of sheaves

$$\begin{aligned} \cdot x_i^d : \mathcal{F} &\to \mathcal{F}(d) \\ s &\mapsto s \otimes x_i^d = x_i^d s \end{aligned}$$

On each U_i, this restricts to an isomorphism for $\mathcal{F}|_{U_i} \to \mathcal{F}(d)|_{U_i}$, since x_i is invertible on U_i. Since the s_{ij} generate $\mathcal{F}|_{U_i}$, the $x_i^d s_{ij}$ generate $\mathcal{F}(d)|_{U_i}$. With this, the t_{ij} globally generate. ☐

Corollary 5.4.6. With the notation as above, $\mathcal{F}$ is a quotient of $\mathcal{O}(-d)^{\oplus N}$ for some sufficiently large N, $d \in \mathbb{Z}$.

Proof. In the theorem, we have $\mathcal{O}_X^{\oplus N} \to \mathcal{F}(d)$. Now tensor with $\mathcal{O}(-d)$. ☐

6 Divisors

In rings, we have two classes of special ideals. Principal (prime) ideals and height 1 prime ideals. Recall if $\mathfrak{p} \in \mathrm{Spec}(R)$, the height of $\mathfrak{p}$, $\mathrm{ht}(\mathfrak{p})$ is the largest n such that there exists a chain of inclusions

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$$

In particular, if R is an integral domain, $\mathfrak{p} \in \mathrm{Spec}(R)$ has height 1 if and only if there is no non-zero prime strictly contained in $\mathfrak{p}$.

Example 6.0.1

In $\mathbb{C}[x, y]$, then $\langle 0 \rangle$ has height 0, $\langle x \rangle$ has height 1, $\langle xy \rangle$ has height 2.

Remark 6.0.2. In a UFD, height 1 prime and principal prime ideals are the same.

We will now globalise both notions.

- height 1 primes correspond to Weil divisors.
- principal ideals correspond to Cartier divisors.

Definition 6.0.3 (generic point)

If X is an integral scheme, $U = \mathrm{Spec}(A)$ is an open affine in X, then the ideal $0 \in \mathrm{Spec}(A)$ is called the *generic point* of X. This is true for any U open affine. We denote this as η or η_X.

This is well defined since any two affine opens intersect, by irreducibility (integral schemes are irreducible).

In this case, $\mathcal{O}_{X,\eta_X} = \mathrm{Frac}(A)$ is a field, and this is independent of the choice of A. We denote this as $k(X)$, the *function field* of X.

6.1 Topological facts

Definition 6.1.1 (dimension, codimension)

For a topological space X, the *dimension* of X is the length of the longest chain of nonempty closed irreducible subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

in X.

Let $Z \subseteq X$ be closed and irreducible. The *codimension* of Z in X is the longest chain

$$Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$$

Lemma 6.1.2. If X is a Noetherian topological space, then every closed $Z \subseteq X$ has a decomposition into finitely many irreducible closed subsets.

Proof. Essentially the same as the proof in Part II AG which says that a variety is a union of finitely many irreducible components. Moreover, the decomposition is unique. □

Definition 6.1.3 (regular in codimension 1)

Let X be a Noetherian integral separated scheme, then X is *regular in codimension* 1 if for all $Y \subseteq X$ closed irreducible codimension 1, let η_Y denote the generic point of Y, then $\mathcal{O}_{X,\eta_Y}$ is a discrete valuation ring (i.e. a local PID).

6.2 Weil divisors

Assume X is a (Noetherian) integral separated and regular in codimension 1 scheme.

Definition 6.2.1 (prime divisor, Weil divisor)

A *prime divisor* on X is an integral closed subscheme of codimension 1. A *Weil divisor* is an element of the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors.

We will write $D \in \mathrm{Div}(X)$ as

$$D = \sum_i n_{Y_i}[Y_i]$$

where Y_i are prime. A Weil divisor D is *effective* if all $n_{Y_i} \geq 0$.

Proposition 6.2.2. Let $f \in \mathcal{O}_{X,\eta_X}^\times = k(X)^\times$. For every $Y \subseteq X$ a prime divisor, the ring $\mathcal{O}_{X,\eta_Y}$ is a DVR[a], and we can calculate the valuation $v_Y(f)$ of f in the DVR. We define the *divisor*

$$\mathrm{div}(f) = \sum_{Y \subseteq X \text{ prime Weil divisor}} v_Y(f)[Y]$$

Then $\mathrm{div}(f)$ is a Weil divisor.

[a] as X is regular in codimension 1

First, if X is integral, choose $U \subseteq X$, $U = \text{Spec}(A)$, then $\mathcal{O}_{X,\eta} = \text{Frac}(A)$. Since η is contained in every open affine, $\mathcal{O}_{X,\eta}$ allows arbitraty denominators.

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Proof. We just need to check that the sum is finite. Let $f \in k(X)^\times$, and choose A such that $U = \text{Spec}(A)$ is an affine open (so $k(X) = \text{Frac}(A)$), and $f \in A$. We can assume this by localising at the denominators. Geometrically, f is regular on U. For this, note that the poles of f are the zeroes of $1/f$, which is a closed subset.

In this case, $X \setminus U$ is closed of codimension at least 1, and so we have only finitely many prime divisors of X, which are contained in $X \setminus U$. On U, f is regular, i.e. $\nu_Y(f) \geq 0$. But $\nu_Y(f) > 0$ if and only if $Y \subseteq \mathbb{V}(f) \subseteq U$. By the same argument, only finitely many Y are contained in $\mathbb{V}(f)$. □

Definition 6.2.3 (principal divisor)

A Weil divisor of the form $\text{div}(f)$ is called *principal*. In $\text{Div}(X)$, the set of principal divisors form a subgroup $\text{Prin}(f)$.

Definition 6.2.4 ((Weil divisor) class group)

The *(Weil divisor) class group* of X is

$$\text{Cl}(X) = \frac{\text{Div}(X)}{\text{Prin}(X)}$$

Proposition 6.2.5. Some basic facts

(i) If A is a Noetherian domain, then A is a UFD if and only if A is integrally closed, and $\text{Cl}(\text{Spec}(A)) = 0$. Moreover, there exists A such that $\text{Spec}(A)$ has non-trivial class group.

In particular, $\text{Cl}(\mathbb{A}^n_k) = 0$.

(ii) $\text{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$,

(iii) if $Z \subseteq X$ is closed, with $U = X \setminus Z$, then there exists a surjective map

$$\begin{aligned} \text{Cl}(X) &\to \text{Cl}(U) \\ [Y] &\mapsto [Y \cap U] \end{aligned}$$

where on the right hand side, we set $[\varnothing] = 0$.

(iv) if Z has codimension at least two, then the map $\text{Cl}(X) \to \text{Cl}(U)$ is an isomorphism.

(v) if $Z \subseteq X$ is integral, closed, codimension 1, then there exists an exact sequence

$$\mathbb{Z} \longrightarrow \text{Cl}(X) \longrightarrow \text{Cl}(U) \longrightarrow 0$$

where the map from $\mathbb{Z}$ to $\text{Cl}(X)$ sends 1 to $[Z]$.

We call (iii), (iv), (v) *excision*.

Proof of (ii). Let $D \subseteq \mathbb{P}^n$ be integral closed codimension 1. Then $D = \mathbb{V}(f)$ where f is homogeneous, of degree d. Define $\deg(D) = d$.

Now extend linearly to get a homomorphism $\deg : \text{Div}(\mathbb{P}^n_k) \to \mathbb{Z}$. We claim that this is an isomorphism $\text{Cl}(D) \to \mathbb{Z}$. First, if $f = g/h$ is a rational function on $\mathbb{P}^n_k$, i.e. a ratio of homogeneous polynomials of the same degree, then

$$\deg(\text{div}(f)) = 0$$

For surjectivity, take $H = \mathbb{V}(X_0)$, where X_0. For injectivity, say

$$D = \sum_i n_{Y_i}[Y_i]$$

If $\sum n_{y_i} \deg(Y_i) = 0$, write $Y_i = \mathbb{V}(g_i)$, where g_i is homoegrnous. Set

$$f = \prod_i g_i^{n_{Y_i}}$$

Then f is a homoegenous rational function of degree zero. □

Proof of excision. For (iii), $k(X)$ and $k(U)$ are naturally isomorphic, and so principal divisors are sent to principal divisors, and the map is well defined. For surjectivity, for $D \subseteq U$ a prime Weil divisor, its closure $\overline{D}$ in X is a prime Weil divisor on X, with $\overline{D} \cap U = D$.

For (iv), Z does not even enter into the definitions. Equivalently, there isn't even a prime divisor contained in Z.

For (v), the kernel of the restriction $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$, is just divisors in X contained in Z. □

6.3 Cartier divisors

We would like to study things locally looking like a principal ideals. Recall a height 1 prime in a UFD is principal.

> **Definition 6.3.1** (Cartier divisor)
> A *Cartier divisor* is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$.

Let X be a scheme, take the presheaf

$$U = \mathrm{Spec}(A) \mapsto S^{-1}A$$

where S is the set of all non-zero divisors[7]. Sheafify this, and call the result $\mathcal{K}_X$. This is a sheaf of rings, and take $\mathcal{K}_X^* \subseteq \mathcal{K}_X$ for the subsheaf of invertible elements. This is a sheaf of abelian groups. Similarly, $\mathcal{O}_X^*$ is the subsheaf of $\mathcal{O}_X$, consisting of invertible elements.

Practically, every section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ can be described by data

$$\{(U_i, f_i)\}$$

where the U_i is a cover of X, f_i is a section of $\mathcal{K}_X^*(U_i)$, such that on $U_i \cap U_j$, we have that

$$\frac{f_i}{f_j} \in \mathcal{O}_X^*(U_i \cap U_j)$$

This is how we should think about Cartier divisors, which is something which locally looks like the divisor of a rational function. With this in mind, the condition above becomes that on overlaps, the choice does not matter as their ratio is a unit, which has divisor zero.

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If X is integral, then $\mathcal{K}_X$ is a constant sheaf with $\mathcal{O}_{X,\eta_X} = \mathrm{Frac}(A)$, where $\mathrm{Spec}(A) \subseteq X$ is open.

We have a surjective sheaf homomorphism $\mathcal{K}^* \twoheadrightarrow \mathcal{K}^*/\mathcal{O}^*$, but a global section of $\mathcal{K}^*/\mathcal{O}^*$ need not be the image of a global section of $\mathcal{K}^*$.

> **Definition 6.3.2** (principal Cartier divisors, Cartier class group)
> The image of $\Gamma(X, \mathcal{K}_X^*)$ in $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is called the set of *principal Cartier divisors*. The quotient
>
> $$\frac{\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)}{\mathrm{im}(\Gamma(X, K^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}^*))}$$
>
> is called the *Cartier class group of* X.

[7] non-(zero divisors).

Proposition 6.3.3. Let X be an integral, Noetherian, separated, regular in codimension 1 scheme. Given a Cartier divisor $\mathcal{D} \in \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$, we get a Weil divisor by the rule: If $Y \subseteq X$ is prime Weil, it has a generic point η_Y. Now represent $\mathcal{D}$ by $\{U_i, f_i\}$, set

$$n_Y = v_Y(f_i)$$

for some U_i containing η_Y. We then have a divisor

$$\sum_{Y \subseteq X \text{ codimension 1 integral}} n_Y[Y]$$

Proof. If η_Y is contained in U_i and U_j, then the valuations of f_i and f_j differ by $v_Y(f_i/F_j)$, but f_i/f_j is a unit, so it has valuation 0. Thus, this also tells us that it is independent of the choice of representative. □

Proposition 6.3.4. If X is Noetherian, integral, separated, and all local rings $\mathcal{O}_{X,x}$ are UFDs[a], then the association

$$\{\text{Cartier divisors}\} \to \{\text{Weil divisors}\}$$

constructed above is a bijection, and respects principal divisors. That is, it defines an isomorphism of class groups.

[a]This is called *locally factorial*. Note this implies that X is regular at codimension 1

Sketch proof. All height 1 primes in a UFD are principal. For $x \in X$, $\mathcal{O}_{X,x}$ is a UFD, and so given a Weil divisor D, we can restrict it to $\operatorname{Spec}(\mathcal{O}_{X,x}) \to X$. That is, we take a fibre product. On $\operatorname{Spec}(\mathcal{O}_{X,x})$, D is given by $\mathbb{V}(f_x)$ as $\mathcal{O}_{X,x}$ is a UFD, and prime Weil corresponds to height one primes, which are principal. Now f_x extends to a neighbourhood U_x of x. Now glue these f_x to form a Cartier divisor. □

Definition 6.3.5

Given a Cartier divisor $\mathcal{D}$ on X, with representatives $\{U_i, f_i\}$, let $L(\mathcal{D}) \subseteq \mathcal{K}_X$ be the sub $\mathcal{O}_X$-module generated on U_i by f_i^{-1},

Note that this is well defined, as on overlaps f_i/f_j is a unit.

Keep in mind if A is an integral domain, $X = \operatorname{Spec}(A)$, $\mathcal{D} = \{(X, f)\}$ where $f \in A$, then $A_f \subseteq \operatorname{Frac}(A)$ is an A-module, generated by $1/f$.

Proposition 6.3.6. The sheaf $L(\mathcal{D})$ is a line bundle, i.e. it is locally free of rank 1[a].

[a]locally (free of rank 1)

Proof. On U_i, we have an isomorphism

$$\mathcal{O}(U_i) \to L(\mathcal{D})(U_i)$$
$$1 \mapsto \frac{1}{f_i}$$

□

Important exercise: If $X = \mathbb{P}^n_k$, D to be the Weil divisor given by $\mathbb{V}(x_0)$. Say $\mathcal{D}$ is the correspoding Cartier divisor. Now show that

$$\mathcal{O}_{\mathbb{P}^n}(1) \cong L(\mathcal{D})$$

Proof. We have an open cover of $\mathbb{P}^n_k$ by $U_0, \ldots, U_n$ the standard opens. We claim that the representatives for $\mathcal{D}$ are $(U_i, f_i = x_0/x_i)$. On overlaps,

$$\frac{f_i}{f_j} = \frac{x_j}{x_i} \in \mathcal{O}^*_{\mathbb{P}^n}(U_i \cap U_j)$$

and so this is a well defined Cartier divisor. Hence it suffices to show that it corresponds to the hyperplane $H = \mathbb{V}(x_0)$.

Now on

$$U_i = \mathrm{Spec}(A^i) \quad \text{where} \quad A^i = k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

we can compute the stalks, which are just given by localisation. In particular, if η is the generic point of some $Y \subseteq \mathbb{P}^n_k$ prime Weil divisor, then

$$v_Y(f_i) = \begin{cases} 0 & x_0/x_i \text{ is a unit in } \mathcal{O}_{X,\eta} = A^i_\eta \\ 1 & x_0/x_i \text{ is not a unit in } \mathcal{O}_{X,\eta} = A^i_\eta \end{cases}$$

In particular, this is zero if $\eta \notin H$, since x_0/x_i is non-vanishing. Now $v_Y(f_i) = 1$ only if Y is contained in H, and so by irreducibility, $Y = H$.

With this in mind, the sheaf $L(\mathcal{D})$ is then generated on U_i by $1/f_i = x_i/x_0$, i.e. $f_i A^i$. On the other hand, $\mathcal{O}_{\mathbb{P}^n}(1)$ is generated on U_i by x_i, i.e. we get $x_i A^i$. The isomorphism is then given by multiplication by x_0. □

Remark 6.3.7. A line bundle L on X has a inverse under tensor product, namely

$$L^{-1} = \mathrm{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$$

Moreover, tensor product of line bundles are line bundles. If all Weil divisors are Cartier, then $L(D+E) = L(D) \otimes L(E)$.

Proof. For the inverse part, it suffices to note that

$$L^{-1} \otimes L = \mathrm{Hom}(L, \mathcal{O}_X) \otimes L = \mathrm{Hom}(L, L) = \mathcal{O}$$

For the last equality, it suffices to note that L is a locally free rank 1 $\mathcal{O}_X$-module. □

Definition 6.3.8 (Picard group)

The *Picard group* on X, denoted by $\mathrm{Pic}(X)$, is the group of line bundles on X up to isomorphism, with group operation being tensor product.

Proposition 6.3.9. Under mild assumptions, for example $X \to \mathrm{Spec}(k)$ being projective, or X is integral, then the map

$$\begin{aligned} \text{Cartier divisors on } X &\to \mathrm{Pic}(X) \\ \mathcal{D} &\mapsto L(\mathcal{D}) \end{aligned}$$

is surjective, and the kernel is exactly the principal Cartier divisors.

Proof. Omitted. See Abelian Varieties for more details. □

7 Sheaf cohomology - a survival guide

We have seen that if $X = \mathbb{A}^2_k \setminus \{(0,0)\}$, then $\mathcal{O}_X(X) \cong k[x, y]$, and so it cannot be affine.

As an overview, given X a topological space, $\mathcal{F}$ a sheaf of abelian groups on X, we will build groups $\mathrm{H}^i(X, \mathcal{F})$ for $i \in \mathbb{N}$, known as the *sheaf cohomology of* $\mathcal{F}$, such that

1. $\mathrm{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$,
2. (functoriality) if $f : Y \to X$ is continuous, then we have an induced map

$$f^* : \mathrm{H}^i(X, \mathcal{F}) \to \mathrm{H}^i(Y, f^{-1}\mathcal{F})$$

3. Given a short exact sequence of sheaves on X,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

We get a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{F}) \longrightarrow \mathrm{H}^0(X, \mathcal{F}') \longrightarrow \mathrm{H}^0(X, \mathcal{F}'') \longrightarrow \mathrm{H}^1(X, \mathcal{F}) \longrightarrow \cdots$$

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We will omit the definition, see Hartshorne Chapter 3. For our purposes, two key aspects of the definition:

1. if X is an affine scheme, $\mathcal{F}$ is a quasi-coherent sheaf, then

$$\mathrm{H}^i(X, \mathcal{F}) = 0 \text{ for } i > 0$$

2. if X is a Noetherian separated scheme, then $\mathrm{H}^i(X, \mathcal{F})$ can be computed from the sections of $\mathcal{F}$ on an open affine cover $\{U_i\}$ and from the data of the restrictions to the intersections.

The second part is called Čech cohomology.

See Dhruv's notes for §7.1, 7.2.

7.3 Čech cohomology

Let X be a topological space, $\mathcal{F}$ a sheaf on X. Fix an open cover $\mathcal{U} = \{U_i\}_{i \in I}$, indexed by a well-ordered set I. In this case, Čech cohomology is attached to the triple $(X, \mathcal{F}, \mathcal{U})$.

We will write $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. The group of Čech p-cochains is

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p})$$

We have the differential

$$\mathrm{d} : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

where

$$(\mathrm{d}\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{k+1} (-1)^k \alpha_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}}$$

Easy exercise to see that $\mathrm{d}^2 = 0$. This makes $(C^p(\mathcal{U}, \mathcal{F}), \mathrm{d})$ into a cochain complex.

Definition 7.3.1 (Čech cohomology)
The *Čech cohomology* of $(X, \mathcal{F}, \mathcal{U})$ is the cohomology of the cochain complex $C^p(\mathcal{U}, \mathcal{F})$. We will write

$$\check{\mathrm{H}}^*(\mathcal{U}, \mathcal{F}) = \mathrm{H}^*(C^\bullet(\mathcal{U}, \mathcal{F}))$$

Example 7.3.2
Let $X = S^1$. Let $\mathcal{F}$ be the constant sheaf $\underline{\mathbb{Z}}$. Let $\mathcal{U} = \{U, V\}$, where U, V are the upper and lower halves of S^1.

In this case, the cochain groups are

$$C^0(\mathcal{U}, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U) \oplus \underline{\mathbb{Z}}(V) = \mathbb{Z}^2$$
$$C^1(\mathcal{U}, \underline{\mathbb{Z}}) = \underline{\mathbb{Z}}(U \cap V) = \mathbb{Z}^2$$

and the boundary map is

$$\mathrm{d}(a, b) = (b - a, b - a)$$

Hence

$$\check{H}^*(\mathcal{U}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Observe that this is the same as the singular cohomology groups of S^1. On the other hand, for a poorly chosen $\mathcal{U}$, then $\check{H}^*$ will not behave well. That is, $\check{H}^*$ depends on $\mathcal{U}$ in a crucial way.

Exercise: Take $X = \mathbb{P}^1_k$, $U = \mathbb{P}^1 \setminus 0$, $V = \mathbb{P}^1 \setminus \infty$. Then U, V cover. Show that

$$\check{H}^*(\mathcal{U}, \mathcal{O}_X) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 7.3.3. Let X be a Noetherian separated scheme, $\mathcal{U} = \{U_i\}$ be an affine cover, and so all $U_{i_0 \cdots i_p}$ are all affine. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{F}$, then

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$$

Proposition 7.3.4. Let $X = \mathbb{P}^n_k$, and

$$\mathcal{F} = \bigoplus_d \mathcal{O}_{\mathbb{P}^n}(d)$$

Then we have an isomorphism of graded k-vector spaces

- $H^0(X, \mathcal{F}) \cong k[x_0, \ldots, x_n]$
- $H^n(X, \mathcal{F}) \cong \frac{1}{x_0 \cdots x_n} k[x_0^{-1}, \ldots, x_n^{-1}]$,
- $H^p(X, \mathcal{F}) = 0$ for all other p.

In particular, $H^0(\mathbb{P}^n, \mathcal{O}(d))$ has dimension $\binom{n+d}{d}$, and $H^n(\mathbb{P}^n, \mathcal{O}(d))$ has dimension $\binom{-d-1}{n}$ (when these things make sense, zero otherwise).

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Proof. The claim for $H^0(X, \mathcal{F})$ follows from prior description, as

$$H^0(X, \mathcal{F}) = \mathcal{F} = \bigoplus_d \Gamma(\mathbb{P}^n, \mathcal{O}(d))$$

For H^n, choose the standard cover $\mathcal{U}$ by affine opens $U_i = \mathbb{P}^n \setminus \mathbb{V}(x_i)$. Observe

$$\mathcal{F}(U_{i_0 \cdots i_p}) = k[x_0, \ldots, x_n]_{x_{i_0} \cdots x_{i_p}}$$

This k-module is spanned over k by monomials

$$x_0^{k_0} \cdots x_n^{k_n}$$

where $k_{i_0}, \ldots, k_{i_p} \in \mathbb{Z}$, the rest are in $\mathbb{Z}_{\geq 0}$. The vector spaces in the associated Čech complex is:

$$\begin{aligned} \check{C}^{n-1} &= \bigoplus_{i=1}^{n} k[x_0, \ldots, x_n]_{x_0 \cdots \widehat{x_i} \cdots x_n} \\ \check{C}^n &= k[x_0, \cdots, x_n]_{x_0 \cdots x_n} \end{aligned}$$

Since $\mathcal{U}$ contains only $n+1$ elements, $\check{C}^{n+1}$ vanishes. So we can conclude that

$$\begin{aligned} H^n(\mathbb{P}^n, \mathcal{F}) &= \check{H}^n(\mathcal{U}, \mathcal{F}) \\ &= \frac{\check{C}^n}{\operatorname{im}(\check{C}^{n-1} \to \check{C}^n)} \\ &= \frac{\operatorname{span}_k \left\{ x_0^{k_0} \cdots x_n^{k_n} \mid k_i \in \mathbb{Z} \right\}}{\operatorname{span}_k \left\{ x_0^{k_0} \cdots x_n^{k_n} \mid k_i \geq 0 \text{ for some } i \right\}} \end{aligned}$$

Finally, for the intermediate cohomology groups, we will use the LES associated to a SES of sheaves. Moreover, we will induct on the dimension $n \geq 2$.

First, we have that $\mathbb{P}^{n-1}$ is isomorphic to the closed subscheme $\mathbb{V}(x_0) \subseteq \mathbb{P}^n$. Say $i : \mathbb{P}^{n-1} \to \mathbb{P}^n$ is the inclusion map. There is an associated "ideal sheaf sequence"

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow i_*\mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0$$

Recall we have the identification $\mathcal{O}_{\mathbb{P}^n}(-1) = L(-H)$, where $H = \mathbb{V}(x_0)$. See examples sheet 4 for more details. By formal properties of cohomology which we have asserted, we get an associated long exact sequence. Assume the result holds for $n-1$. We get three associated exact sequences:

$$\begin{array}{c} 0 \longrightarrow \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}) \longrightarrow \mathrm{H}^0(\mathbb{P}^n, \mathcal{F}_{\mathbb{P}^{n-1}}) \\ \swarrow \\ \mathrm{H}^1(\mathbb{P}^n, \mathcal{F}) \xrightarrow[\cdot x_0]{} \mathrm{H}^1(\mathbb{P}^n, \mathcal{F}) \longrightarrow 0 \end{array} \tag{a}$$

$$0 \longrightarrow \mathrm{H}^p(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} \mathrm{H}^p(\mathbb{P}^n, \mathcal{F}) \longrightarrow 0 \tag{b}$$

and

$$\begin{array}{c} 0 \longrightarrow \mathrm{H}^{n-1}(\mathbb{P}^n, \mathcal{F}) \xrightarrow{\cdot x_0} \mathrm{H}^{n-1}(\mathbb{P}^n, \mathcal{F}) \longrightarrow \mathrm{H}^{n-1}(\mathbb{P}^n, \mathcal{F}_{\mathbb{P}^{n-1}}) \\ \swarrow \\ \mathrm{H}^n(\mathbb{P}^n, \mathcal{F}) \xrightarrow[\cdot x_0]{} \mathrm{H}^n(\mathbb{P}^n, \mathcal{F}) \longrightarrow 0 \end{array} \tag{c}$$

Using (a) and (c), we can observe that the sequence (b) is also exact for $p = 1$ and $p = n-1$, by writing out the Čech cohomology. Multiplication by x_0 makes $\mathrm{H}^p(\mathbb{P}^n, \mathcal{F})$ into a $k[x_0]$-module. Next, we calculate the localisation of this module at x_0. That is, we would like to find

$$\mathrm{H}^p(\mathbb{P}^n, \mathcal{F})_{x_0}$$

Since localisation ie exact,

$$\mathrm{H}^p(\mathbb{P}^n, \mathcal{F})_{x_0} = \mathrm{H}^p(U_0, \mathcal{F}|_{U_0}) = 0$$

since U_0 is affine. Thus, for any $\alpha \in \mathrm{H}^p(\mathbb{P}^n, \mathcal{F})$, $x_0^k \alpha = 0$ for some $k > 0$. But multiplication by x_0 is an isomorphism on cohomology. So in fact $\alpha = 0$, and so $\mathrm{H}^p(\mathbb{P}^n, \mathcal{F}) = 0$. □

Given the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow i_*\mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0$$

we can tensor with $\mathcal{O}_{\mathbb{P}^n}(d)$, and we get an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow i_*\mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow 0$$

Since $\mathcal{O}_{\mathbb{P}^n}(d)$ is locally free, we can just check the above of stalks, there is no need to check that it is flat.

On Sheet 4, there are two more computations of sheaf cohomology. The first is that if

$$X = \mathbb{A}^2_k \setminus 0$$

then

$$\dim_k(\mathrm{H}^1(X, \mathcal{O}_X)) = \infty$$

and so X is not affine.

Next, let f_d be a degree d homogeneous polynomial in $k[x, y, z]$. Let $X_d = \mathbb{V}(f_d) \subseteq \mathbb{P}^2$. The Čech complex gives that

$$\dim_k \left(\mathrm{H}^0(X_d, \mathcal{O}_{X_d})\right) = 1$$
$$\dim_k \left(\mathrm{H}^1(X_d, \mathcal{O}_{X_d})\right) = \binom{d-1}{2}$$

The second is just the degree-genus formula if X_d is a smooth curve, but the above computation works in general. We call this the *arithmetic genus*.

Let X be proper over $\mathrm{Spec}(k)$, and $\mathcal{F}$ a coherent sheaf on X. We've seen that $\mathrm{H}^0(X, \mathcal{F})$ is a finite dimensional k-vector space. In fact, the same is true for all $\mathrm{H}^p(X, \mathcal{F})$.

Under the same hypotheses, if X has dimension n, then $\mathrm{H}^p(X, \mathcal{F}) = 0$ for $p > n$. Thus, in this setup, given $(X, \mathcal{F})$ there are finitely many numbers

$$h^p(X, \mathcal{F}) = \dim_k (\mathrm{H}^p(X, \mathcal{F}))$$

Definition 7.3.5 (Euler characteristic)
The *Euler characteristic* of $\mathcal{F}$ is

$$\chi(\mathcal{F}) = \chi(X, \mathcal{F}) = \sum_{p=0}^{d} (-1)^p h^i(X, \mathcal{F})$$

Now suppose

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of coherent sheaves on X, then the associated long exact sequence gives

$$\chi(\mathcal{F}') = \chi(\mathcal{F}) + \chi(\mathcal{F}'')$$

Lecture 25

7.4 Choice of cover

So far, given a Noetherian separated scheme X, a quasi-coherent sheaf $\mathcal{F}$ on X, and $\mathcal{U} = \{U_i\}$ a finite open affine cover, we've been using $\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F})$ to be equal to the sheaf cohomology groups $\mathrm{H}^i(X, \mathcal{F})$. We will prove that $\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F})$ is independent of the choice of cover $\mathcal{U}$.

Theorem 7.4.1. Let X be affine and $\mathcal{F}$ quasi-coherent. Then for any cover $\mathcal{U} = \{U_i\}$ by affine opens, the groups $\mathrm{H}^i(\mathcal{U}, \mathcal{F})$ are zero for $i > 0$.

Proof. Define the *sheafified Čech complex* as follows:

$$\mathcal{C}^p(\mathcal{F}) = \prod_{i_0 < \cdots < i_p} i_* \mathcal{F}|_{i_0 \cdots i_p}$$

where $i : U_{i_0 \cdots i_p} \hookrightarrow X$ is the inclusion. By what we have done previously, the $\mathcal{C}^p(\mathcal{F})$ are quasi-coherent sheaves. BY taking global sections,

$$\Gamma(X, \mathcal{C}^p(\mathcal{F})) = \check{C}^p(\mathcal{F})$$

The same formula used to build the Čech complex gives differentials

$$\mathrm{d} : \mathcal{C}^p(\mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{F})$$

which is a morphism of sheaves. Our goal is to show that the usual Čech complex

$$\check{C}^0(\mathcal{F}) \longrightarrow \check{C}^1(\mathcal{F}) \longrightarrow \cdots$$

is exact. From examples sheet 4 question 10, on affines, taking global sections preserves exactness. Thus, it suffices to prove instead exactness of

$$\mathcal{C}^0(\mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{F}) \longrightarrow \cdots$$

But the exactness of this can be check at the level of stalks. Let $q \in X$, and suppose $q \in U_j$. We now define

$$\kappa : \mathcal{C}^p_q(\mathcal{F}) \to \mathcal{C}^{p-1}_q(\mathcal{F})$$

where $\mathcal{C}^p_q(\mathcal{F})$ is the stalk of $\mathcal{C}^p(\mathcal{F})$ at q. This is defined by

$$\kappa(\alpha)_{i_0 \cdots i_p} = \alpha_{j i_0 \cdots i_{p-1}}$$

where by convention if $j i_0 \cdots i_{p-1}$ is not in increasing order, but $\sigma \in S_{p+1}$ makes it into increasing order, then

$$\alpha_{j i_0 \cdots i_p} = \operatorname{sign}(\sigma) \alpha_{\sigma(j)\sigma(i_0)\cdots\sigma(i_p)}$$

By direct calculation,

$$\mathrm{d}\kappa + \kappa \mathrm{d} = \mathrm{id}$$

on $\mathcal{C}^p$ for all p. This is a chain homotopy between id and 0. Hence the cochain complex $\mathcal{C}^p(\mathcal{F})$ is contractible. More concretely, for $\alpha \in \ker(\mathcal{C}^p \to \mathcal{C}^{p+1})$, then

$$\alpha = \mathrm{d}\kappa(\alpha) \in \operatorname{im}(\mathcal{C}^{p-1} \to \mathcal{C}^p)$$

□

Lemma 7.4.2. Let X be a scheme and $\mathcal{F}$ a quasi-coherent sheaf on X, Fix $\mathcal{U} = \{U_1, \cdots, U_k\}$, and $\widetilde{\mathcal{U}} = \{U_0, \cdots, U_k\}$. Then $\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F})$ and $\check{\mathrm{H}}^i(\widetilde{\mathcal{U}}, \mathcal{F})$ are naturally isomorphic.

Sketch proof. Let $C^p(\mathcal{F})$ and $\widetilde{C}^p(\mathcal{F})$ be the respective cochain groups. There are maps

$$\widetilde{C}^p(\mathcal{F}) \to C^p(\mathcal{F})$$

given by dropping the terms with U_0. To make this precise, observe that $\widetilde{\alpha} \in \widetilde{C}^p(\mathcal{F})$ can be considered as a pair (α, α_0) where $\alpha \in C^p(\mathcal{F})$ and $\alpha_0 \in C^{p-1}(\{U_1 \cap U_0, \cdots U_k \cap U_0\}, \mathcal{F}|_{U_0})$. The map is given by projection. This defines a chain map, and so we get an induced map on cohomology

$$\check{\mathrm{H}}^i(\widetilde{U}, \mathcal{F}) \to \check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F})$$

We leave as an exercise: By reducing to a calculation on U_0 which is affine, deduce from theorem 7.4.1 that these are isomorphism. □

Corollary 7.4.3. $\check{\mathrm{H}}^i(\mathcal{U}, \mathcal{F})$ is independent of the choice of $\mathcal{U}$.

Proof. If $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ are two covers, then so is $\mathcal{U} \cup \widetilde{\mathcal{U}}$. Use the above lemma. □

7.5 *Further topics in cohomology*

Concrete consequences of sheaf cohomology: Let $X_d \subseteq \mathbb{P}^3_k$ be the vanishing locus of f_d, where f_d is a degree d homogeneous polynomial, with $d \neq 2$. Then X_d is not isomorphic to a product (over $\operatorname{Spec}(k)$) of schemes of dimension 1.

Note we can have $X_2 \cong \mathbb{P}^1 \otimes_{\operatorname{Spec}(k)} \mathbb{P}^1$ by the Segre embedding. This is a consequence of the sheaf Künneth formula. In particular,

$$h^1(X_d, \mathcal{O}_{X_d}) = 0$$

Moreover, the different X_d for distinct d are non-isomorphic as schemes. This follows from calculating the Euler characteristic of X_d.

Finally, the next topic in sheaf cohomology would be duality theory. Given $i : Z \hookrightarrow X$ a closed immersion, we have the ideal sheaf $\mathcal{I}_Z = \ker(i^\sharp : \mathcal{O}_X \to i_* \mathcal{O}_Z)$, which is a coherent sheaf on X.

Definition 7.5.1 (conormal sheaf)
The *conormal sheaf* to the closed immersion $i : Z \hookrightarrow X$ is given by

$$i_* \left(\frac{\mathcal{I}_Z}{\mathcal{I}_Z^2} \right)$$

where $\mathcal{I}_Z^2$ is the sheaf given by the sheafification of the presheaf $U \mapsto \mathcal{I}_Z(U)^2$. We denote this as $\mathcal{N}_{\mathbb{Z}/X}^\vee$.

Definition 7.5.2 (cotangent sheaf)
If $X \to S$ is separated, then define the *cotangent sheaf*

$$\Omega_{X/S} = \mathcal{N}_{\Delta_{X/S}}^\vee$$

For $X \to \mathrm{Spec}(k)$, we say that X is *nonsingular* if Ω_X is locally free. The *dualising sheaf*, denoted ω_X, is the sheafification of

$$U \mapsto \Lambda^{\dim(X)} \Omega_X(U)$$

Theorem 7.5.3 (Serre duality). If X is nonsingular of dimension n, and if $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, then there exists an isomorphism

$$\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^{n-1}(X, (\mathcal{F}^\vee \otimes \omega_X)^\vee)$$

where $\mathcal{F}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

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