Algebraic Topology

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Lecture 1

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1 Introduction

Let X, Y be topological spaces, $f, g : X \to Y$ be continuous functions. Note we will use the terminology 'map := continuous function'. We say that f *is homotopic to* g, written $f \simeq g$, if there exists a map $F : X \times I \to Y$, such that

$$F|_{X\times\{0\}} = f \quad and \quad F|_{X\times\{1\}} = g$$

Here $I = [0, 1]$ carries its Euclidean topology. An exercise (sheet 1) is that this defines an equivalence relation on the set of maps $X \to Y$.

Definition 1.0.1 (homotopy equivalence)
We say that $f : X \to Y$ is a *homotopy equivalence* if there exists a function $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We say that g is a *homotopy inverse* to f.

The same exercise on sheet 1 shows that homotopy equivalence is an equivalence relation on topological spaces.

Example 1.0.2

1. If $f : X \to Y$ is a homeomorphism, then f is also a homotopy equivalence.
2. $i : \{0\} \hookrightarrow \mathbb{R}^n$ is a homotopy equivance.
3. The inclusion $i : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus 0$ is a homotopy equivalence.

Definition 1.0.3
A space X is *contractible* if it is homotopy equivalent to a point.

Idea of Algebraic Topology: Study spaces up to homotopy equivalence. Broadly, we are interested in "connectivity properties" of topological spaces.

Example 1.0.4

1. We say a space X is *path-connected* if any two maps $\{*\} \to X$ are homotopic. For example, $\mathbb{R}$ is path connected, $\mathbb{R} \setminus \{0\}$ is not. A corollary of this is the intermediate value theorem.
2. We say a path connected space X is *simply connected* if every map $f : S^1 \to X$ is homotopic to a constant map. Equivalently, every two maps $S^1 \to X$ are homotopic. Or equivalently (again) every continuous map $S^1 \to X$ extends to a continuous map $D^2 \to X$. For example, $\mathbb{R}^2$ is simply connected, by $\mathbb{R}^2 \setminus \{0\}$ is not. If $\gamma : S^1 \to \mathbb{R}^2 \setminus \{0\}$ is continuous, it has a *winding number*, or *degree* $\deg(\gamma) \in \mathbb{Z}$, such that
   - It is invariant under homotopy,
   - If $\gamma_n(t) = \exp(2n\pi it)$, then $\deg(\gamma_n) = n$

   In particular, taking $n = 0$ gives the constant map. Therefore, a constant map has degree 0. A corollary of this is the fundamental theorem of algebra.
3. We say a path connected space X is k*-connected*, if for all $i \leq k$, every map $S^i \to X$ is homotopic to a constant map. In this case, $\mathbb{R}^n$ is $(n-1)$-connected, but $\mathbb{R}^n \setminus 0$ is not. More precisely, maps from $S^{n-1} \to \mathbb{R}^n \setminus 0$ have a degree $\deg(f) \in Z$, such that degree is a homotopy invariant, and the degree of the inclusion map is 1, and the degree of the constant map is 0. A corollary of this is the Brouwer fixed point theorem.

Lecture 2

2 (Co)homology

2.1 Co(chain) complexes

We will define invariants of topological spaces in two steps:

(a) We associate to X a (co)chain complex,

(b) We take the (co)homology of the complex

The topology enters in step (a), whereas (b) is just homological algebra.

Definition 2.1.1 (chain complex)

A *chain complex* (C_*, ∂) consists of a sequence $\{C_i\}_{i\in\mathbb{Z}}$ of abelian groups, along with *boundary homomorphisms* $\partial_n : C_n \to C_{n-1}$ for all n, satisfying $\partial_n \circ \partial_{n+1} = 0$. We also write this conditions as $\partial^2 = 0$.

Often we will be lazy and just write ∂. Also, if we have several chain complexes, we might use subscripts or superscripts to distinguish the boundary homomorphisms.

Definition 2.1.2 (homology group)

The *i-th homology group* of a chain complex (C_*, ∂) is

$$\mathrm{H}_i(C_*, \partial) = \frac{\ker(\partial_i : C_i \to C_{i-1})}{\mathrm{im}(\partial_{i+1} : C_{i+1} \to C_i)}$$

We write

$$\mathrm{H}_*(C_*, \partial) = \bigoplus_{i\in\mathbb{Z}} \mathrm{H}_i(C_*, \partial)$$

Definition 2.1.3 (cochain complex)

A *cochain complex* (C^*, ∂) consists of a sequence $\{C^i\}_{i\in\mathbb{Z}}$ of abelian groups, along with *boundary homomorphisms* $\partial^n : C^n \to C^{n+1}$ for all n, satisfying $\partial^{n+1} \circ \partial^n = 0$. We also write this conditions as $\partial^2 = 0$.

Definition 2.1.4 (cohomology group)

The *i-th cohomology group* of a cochain complex (C^*, ∂) is

$$\mathrm{H}^i(C^*, \partial) = \frac{\ker(\partial^i : C^i \to C^{i+1})}{\mathrm{im}(\partial^{i-1} : C^{i-1} \to C^i)}$$

We write

$$\mathrm{H}^*(C^*, \partial) = \bigoplus_{i\in\mathbb{Z}} \mathrm{H}_i(C^*, \partial)$$

Notation 2.1.5. Elements of $\ker(\partial)$ are called *cycles* (in a chain complex), *cocycles* (in a cochain complex). Elements of $\mathrm{im}(\partial)$ are called *boundaries* (and *coboundary* resp.).

Elements of H_* are called *homology classes*, and elements of H^* are called *cohomology classes*.

We will call ∂ the *differential*.

Definition 2.1.6 (chain map)

Given chain complex (C_*, ∂), (D_*, ∂), a *chain map* $f : C_* \to D_*$ comprises group homomorphism $f_i : C_i \to D_i$, such that

$$\begin{tikzcd} C_i \arrow[r, "f_i"] \arrow[d, "\partial"'] & D_i \arrow[d, "\partial"] \\ C_{i-1} \arrow[r, "f_{i-1}"'] & D_{i-1} \end{tikzcd}$$

commutes.

Lemma 2.1.7. A chain map $f : C_* \to D_*$ induces homomorphisms $f_* : \mathrm{H}_i(C_*) \to \mathrm{H}_i(D_*)$.

Proof. Let $a \in \mathrm{H}_i(C_*)$, and we can choose a cycle $\alpha \in C_i(C_*)$, with $\partial\alpha = 0$. In this case,

$$\partial f_i(\alpha) = f_{i-1}(\partial\alpha) = f_{i-1}(0) = 0$$

So $f_i(\alpha) \in D_i$ is a cycle. We set $f_*(a) = [f_i(\alpha)] \in \mathrm{H}_i(D_*)$. Next, we need to show that this is independent of choices. Suppose $a = [\alpha] = [\alpha']$. Then $\alpha - \alpha' \in \mathrm{im}(\partial)$. Therefore, we can write $\alpha - \alpha' = \partial\tau$. In this case,

$$f_i(\alpha) - f_i(\alpha') = f_i(\partial\tau) = \partial f_{i+1}(\tau) \in \mathrm{im}(\partial)$$

So we conclude $[f_i(\alpha)] = [f_i(\alpha')] \in \mathrm{H}_i(D_*)$. It is clear that f_* defines a homomorphism. □

Correspondingly, we can define *cochain maps* $C^* \to D^*$, that is, $f^i : C^i \to D^i$ such that

$$\begin{tikzcd} C^{i+1} \arrow[r, "f^{i+1}"] & D^{i+1} \\ C^i \arrow[u, "\partial"] \arrow[r, "f^i"'] & D^i \arrow[u, "\partial"'] \end{tikzcd}$$

commutes.

Exercise: This construction is functorial. That is, $(\mathrm{id}_{C_*})_* = \mathrm{id}_{\mathrm{H}_*}$, and if we have $f : C_* \to D_*, g : D_* \to E_*$, then $(g \circ f)_* = g_* \circ f_*$. This is obvious.

Goal: To associate to a topological space X (co)chain complex $C_*(X)$ and $C^*(X)$. The definition works for any X, but the theories are better behaved for "nicer" X.

2.2 Singular (co)homology

We will develop singular (co)homology.

Definition 2.2.1 (standard simplex, face)

The *standard simplex* is

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1 \right\}$$

The i-th face Δ^n is

$$\Delta^n_i = \Delta^n \cap \{t_i = 0\}$$

and we have a canonical homeomorphism

$$\begin{aligned} \delta_i : \Delta^{n-1} &\to \Delta^{n-1}_i \\ (t_0, \ldots, t_{n-1}) &\mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \end{aligned}$$

Definition 2.2.2 (singular simplex, singular chain complex)
If X is a topological space, a *singular n-simplex* in X is a continuous map $\sigma : \Delta^n \to X$.

The *singular chain complex* is given by the the free abelian groups

$$C_i(X) = \bigoplus_{\sigma \text{ singular } n\text{-simplex}} \mathbb{Z}\sigma$$

We define the boundary map $C_i(X) \to C_{i-1}(X)$ via

$$\partial\sigma = \sum_{j=0}^{i} (-1)^j \sigma|_{\Delta^i_j}$$

where

$$\sigma|_{\Delta^i_j} := \sigma \circ \delta_j : \Delta^{n-1} \to X$$

Note if $\{v_i\}_{i=0}^n$ are $n+1$ ordered points in $\mathbb{R}^{n+1}$, and if $\{v_i - v_0\}_{i=1}^n$ are linearly independent, then the convex hull of $v_0, \ldots, v_n$ is an n-simplex, which we will call

$$[v_0, \ldots, v_n]$$

is given by the map

$$\begin{aligned} \Delta^n &\to \mathbb{R}^{n+1} \\ t &\mapsto \sum_i t_i v_i \end{aligned}$$

We orient the edges of the standard simplex (and thus any simplex $[v_0, \ldots, v_n]$) by saying that $v_i < v_j$ if $i < j$.

More concretely, for $n = 2$, we have the oriented edges $v_0 \to v_1, v_0 \to v_2, v_1 \to v_2$.

Lemma 2.2.3. $\partial^2 = 0$.

Proof. If σ is an n-simplex, defined on $[v_0, \ldots, v_n]$, then

$$\partial\sigma = \sum_j (-1)^j \sigma|_{[v_0, \ldots, \widehat{v_j}, \ldots, v_n]}$$

and so

$$\partial(\partial\sigma) = \sum_{i<j} (-1)^i (-1)^j \sigma|_{[v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_n]} + \sum_{i>j} (-1)^{i+1} (-1)^j \sigma|_{[v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_n]} = 0$$

since the signs cancel. □

Definition 2.2.4
The *singular homology groups of X*, $\mathrm{H}_i(X)$, are the homology groups of the singular chain complex.

Remark 2.2.5. Since all we used is the topology on X, $\mathrm{H}_i(X)$ is clearly a topological (read homeomorphism) invariant of X.

Moreover, if $f : X \to Y$ is continuous, $\sigma : \Delta^n \to X$ is an n-simplex on X, then $f \circ \sigma$ is an n-simplex on Y. The fact that

$$(f \circ \sigma) \circ \delta_j = f \circ (\sigma \circ \delta_j)$$

shows that f induces a chain map.

Lecture 3

Definition 2.2.6 (singular cochain complex)
The *singular cochain complex* of X has

$$C^i(X) = \mathrm{Hom}_{\mathbb{Z}}(C_i(X), \mathbb{Z})$$

and the boundary operator

$$\partial^* \psi(\sigma) = \psi(\partial\sigma)$$

for $\psi \in C^i(X), \sigma \in C_{i+1}(X)$. That is, it is adjoint to $\partial : C_i(X) \to C_{i-1}(X)$.

Since $\partial^2 = 0, (\partial^*)^2 = 0$. The *singular cohomology* is

$$\mathrm{H}^*(X) = \bigoplus_{i \geq 0} \mathrm{H}^i(C^*, \partial^*)$$

If $f : X \to Y$ continuous, then f induces a chain map $f_* : C_*(X) \to C_*(Y)$ via $f_*(\sigma) = f \circ \sigma$, satisfying $\partial f_* = f_* \partial$. Similarly, $f : X \to Y$ defines a *pullback map* $f^* : C^*(Y) \to C^*(X)$, given by

$$f^*(\psi)(\tau) = \psi(f_*(\tau))$$

Again, we have that

$$\partial^* f^*(\psi)(\tau) = f^*\psi(\partial\tau) = \psi(f_*\partial\tau) = \psi(\partial f_*\tau) = (\partial^*\psi)(f_*\tau) = f^*(\partial^*\psi)(\tau)$$

So f^* is a cochain map, i.e. $\partial^* f^* = f^* \partial^*$. Therefore, we have an induced homomorphism on cohomology,

$$f^* : \mathrm{H}^*(Y) \to \mathrm{H}^*(X)$$

Warning: by definition,

$$C^i(X) = \mathrm{Hom}_{\mathbb{Z}}(C_i(X), \mathbb{Z})$$

but it is not true (in general) that

$$\mathrm{H}^i(X) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_i(X), \mathbb{Z})$$

It's easy to show that we always have a surjection

$$\mathrm{H}^i(X) = \mathrm{Hom}(\mathrm{H}_i(X), \mathbb{Z})$$

But in general, this is not an isomorphism if $\mathrm{H}_i(X)$ has torsion.

2.2.1 Basic computations

Lemma 2.2.7 (homology of a point). Let $X = \{\bullet\}$. Then

$$\mathrm{H}_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. By definition, $C_i(X)$ is the free abelian groups on continuous map $\Delta^i \to \{\bullet\}$. But there is only one such map, namely the constant map (for $i \geq 0$). Hence the chain complex looks like

$$\cdots \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow 0$$

where each $C_i(X)$ is $\mathbb{Z}$. Computing the boundary maps,

$$\partial(\sigma_n) = \begin{cases} \sigma_{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Hence the chain complex is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} 0$$

We can then compute the homology groups explicity. $\square$

Lemma 2.2.8. If we write

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

as a disjoint union of path components, then

$$\mathrm{H}_i(X) = \bigoplus_{\alpha \in A} \mathrm{H}_i(X_\alpha)$$

Moreover, if X is path connected (and nonempty), then $\mathrm{H}_0(X) = \mathbb{Z}$.

Proof. Since Δ^i is path connected, the image of a continuous map $\sigma : \Delta^i \to X$ must lie within some X_α. Moreover, so does the image of all of the boundary faces $\sigma \circ \delta_j$. So in fact

$$(C_i(X), \partial) = \left(\bigoplus_{\alpha \in A} C_i(X_\alpha), \partial \right)$$

Note that a general element of $C_i(X)$ is a finite linear combination of i-simplicies in X. So its image can only meet finitely many of the X_α.

Now suppose X is path connected. Define a function

$$\begin{aligned} \varepsilon : C_0(X) &\to \mathbb{Z} \\ \sum n_i \sigma_i &\mapsto \sum n_i \end{aligned}$$

Since X is nonempty, ε is surjective. On the other hand, if $\tau \in C_1(X)$, then $\varepsilon(\tau(1) - \tau(0)) = 0$. So by linearity,

$$\mathrm{im}(\partial : C_1(X) \to C_0(X)) \subseteq \ker(\varepsilon)$$

Recall

$$\mathrm{H}_0(X) = \frac{C_0(X)}{\mathrm{im}(\partial : C_1(X) \to C_0(X))}$$

and so ε descends to a map $\mathrm{H}_0(X) \to \mathbb{Z}$. Now we will use the fact that X is path connected. Suppose

$$\sum n_i \sigma_i \in \ker(\varepsilon)$$

and fix a base point $p \in X$. Since the σ_i are 0-simplicies, they correspond to points in X. Moreover, since X is path connected, we can choose paths

$$\tau_i : \Delta^1 \to X$$

such that $\tau_i(1) = \sigma_i$ and $\tau_i(0) = p$. In this case,

$$\partial \left(\sum n_i \tau_i \right) = \sum n_i \partial \tau_i = \sum n_i \sigma_i - (\sum n_i) p$$

But we assumed $\sum n_i = 0$, and so $\ker(\varepsilon) \subseteq \mathrm{im}(\partial : C_1(X) \to C_0(X))$. This then gives the required result, as the induced map

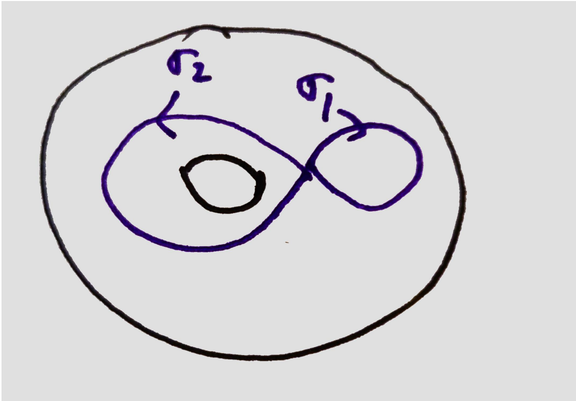
$$\mathrm{H}_0(X) \to \mathbb{Z}$$

is then an isomorphism. □

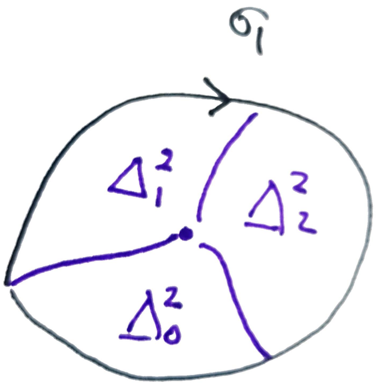
Informal conjecture: We can't compute anything else.

Therefore, we need to develop some more structure about (co)homology.

What are we doing? Consider the case of X being an annulus in $\mathbb{R}^2$. We have 1-simplices



(σ_1 null-homotopic, σ_2 winds around the origin)
In this case, $\partial\sigma_1 = \partial\sigma_2 = 0$, and so they define homology classes.



Moreover, $[\partial\sigma_1] = 0$, since we have triangles τ_0, τ_1, τ_2 such that

$$\partial(\tau_0 + \tau_1 + \tau_2) = \sigma_1$$

From this intuition, $[\sigma_2] \neq 0$. But to show even this, we need some structure theorems.

Lecture 4

2.3 Fundamental properties

(Co)homology is useful by virtue of various structural properties.

Theorem 2.3.1 (homotopy invariance). If $f, g : X \to Y$ are homotopic, then the induced maps f_*, g_* on homology, and the induced maps f^*, g^* on cohomology agree.

Corollary 2.3.2. If $X \simeq Y$, then $\mathrm{H}_*(X) \cong \mathrm{H}_*(Y)$, and if f is a homotopy equivalence, then f_* induces the isomorphism. For cohomology we have the similar statement, but with f^*.

Proof of the corollary. $X \simeq Y$ via f is saying that there exists $g : Y \to X$, such that $g \circ f \simeq \mathrm{id}_X$, and $f \circ g \simeq \mathrm{id}_Y$. That is,

$$g_* \circ f_* = \mathrm{id} \quad \text{and} \quad f_* \circ g_* = \mathrm{id}$$

□

We think of this as saying that (co)homology is "insensitive to inessential deformations".

Example 2.3.3
Recall $\{0\} \hookrightarrow \mathbb{R}^n$ is a homotopy equivalence, and so

$$\mathrm{H}_*(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

The other key structural property relies on some homological algebra.

Definition 2.3.4 (exact sequence)
An *exact sequence* is a chain complex with $H_*(C_*, \partial) = 0$. That is,

$$\mathrm{im}(\partial : C_{n+1} \to C_n) = \ker(\partial : C_n \to C_{n-1})$$

Similarly, a cochain complex is *exact* if $H^*(C^*, \partial) = 0$.
If A, B, C are abelian groups, and we have

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

Then this sequence is *exact at B* if $\ker(\beta) = \mathrm{im}(\alpha)$.

Definition 2.3.5 (short exact sequence)
A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

which is exact at all places. That is, α is injective, β is surjective, and $\mathrm{im}(\alpha) = \ker(\beta)$.

Example 2.3.6 (i) If

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

is exact, then α is an isomorphism.

(ii) If

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} G \xrightarrow{\beta} \mathbb{Z}/n \longrightarrow 0$$

is exact, then we could have $G = \mathbb{Z} \oplus \mathbb{Z}/n$, or we could have $G = \mathbb{Z}$, with $\alpha(x) = nx$.

Theorem 2.3.7 (Mayer-Vietoris)**.** If $X = A \cup B$, with A, B open subsets of X. Then there exists *Mayer-Vietoris boundary homomorphism*

$$\partial_{MV} : \mathrm{H}_{i+!}(X) \to \mathrm{H}_i(A \cap B)$$

for all i, so that the sequence

$$\cdots \longrightarrow \mathrm{H}_{i+1}(X) \xrightarrow{\partial_{MV}} \mathrm{H}_i(A \cap B) \xrightarrow{i_{A*} \oplus i_{B*}} \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) \xrightarrow{j_{A*} - j_{B*}} \mathrm{H}_i(X) \qquad \cdots$$

is exact, where

$$\begin{tikzcd} A \cap B \arrow[r, "i_A"] \arrow[d, "i_B"'] & A \arrow[d, "j_A"] \\ B \arrow[r, "j_B"'] & X \end{tikzcd}$$

Similarly, we have a homomorphism

$$\partial^*_{MV} : H^i(A \cap B) \to H^{i+1}(X)$$

which makes the sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}(A \cap B) \xrightarrow{\partial^*_{MV}} \mathrm{H}^i(X) \xrightarrow{j^*_A \oplus -j^*_B} \mathrm{H}^i(A) \oplus \mathrm{H}^i(B) \xrightarrow{i^*_A + i^*_B} \mathrm{H}^i(A \cap B) \xrightarrow{\partial^*_{MV}} \mathrm{H}^{i-1}(X) \longrightarrow \cdots$$

exact.

Remark 2.3.8. (i) The Mayer-Vietoris maps are not induced by maps of spaces, but they are constructed algebraically. Suppose $\sigma \in C_{i+1}(X)$ is a cycle, and suppose we can write $\sigma = \sigma_A + \sigma_B$, where $\sigma_A \in C_{i+1}(A)$ and $\sigma_B \in C_{i+1}(B)$ are chains, and in general, not cycles. Since $\partial\sigma = 0, \partial\sigma_A + \partial\sigma_B = 0$. We define

$$\partial_{MV}(\sigma) = [\partial\sigma_A] \in \mathrm{H}_i(A \cap B)$$

Since $\partial\sigma_A \in C_i(A)$, we must have that $\partial\sigma_A = -\partial\sigma_B \in C_i(A \cap B)$. As $\partial^2 = 0$, $\partial\sigma_A$ is closed and so it represents a class in homology.

This is (correct) intuition, but not a proof.

(ii) The Mayer-Vietoris sequence is natural for maps of pairs. That is, if $X = A \cup B$, $Y = C \cup D$, and $f : X \to Y$ has $f(A) \subseteq C$, $f(B) \subseteq D$, then we have a map of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_{i+1}(X) \arrow[r] \arrow[d, dashed, "f_*"] & \mathrm{H}_i(A \cap B) \arrow[r] \arrow[d, dashed, "f_*"] & \mathrm{H}_i(A) \oplus \mathrm{H}_i(B) \arrow[r] \arrow[d, dashed, "f_* \oplus f_*"] & \mathrm{H}_i(X) \arrow[r] \arrow[d, dashed, "f_*"] & \mathrm{H}_{i-1}(A \cap B) \arrow[r] \arrow[d, dashed, "f_*"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_{i+1}(Y) \arrow[r] & \mathrm{H}_i(C \cap D) \arrow[r] & \mathrm{H}_i(C) \oplus \mathrm{H}_i(D) \arrow[r] & \mathrm{H}_i(Y) \arrow[r] & \mathrm{H}_{i-1}(C \cap D) \arrow[r] & \cdots
\end{tikzcd}$$

where all the squares commute.

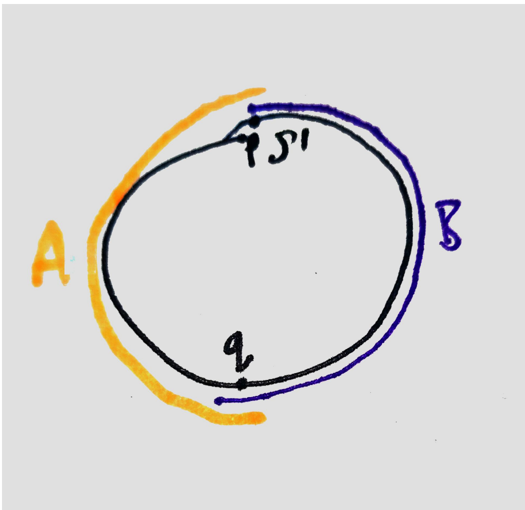
2.4 Examples of Homology calculations

We will prove these later when we use them.

Example 2.4.1

$$\mathrm{H}_*(S^1) = \begin{cases} \mathbb{Z} & * = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Here, $S^1 = A \cup B$, where $A, B \simeq \{*\}$ and $A \cap B \simeq \{p, q\}$.



Using this, we know $\mathrm{H}_*(A), \mathrm{H}_*(B), \mathrm{H}_*(A \cap B)$. For $i \geq 2$, we have

$$\cdots \longrightarrow \underbrace{\mathrm{H}_i(A) \oplus \mathrm{H}_i(B)}_{=0} \longrightarrow \mathrm{H}_i(X) \longrightarrow \underbrace{\mathrm{H}_{i-1}(A \cap B)}_{=0} \longrightarrow \cdots$$

and so $\mathrm{H}_i(X) = 0$ for all $i \geq 2$. For lowe degrees, we have

$$\begin{tikzcd}
\underbrace{\mathrm{H}_1(A \cap B)}_{=0} \arrow[r] & \underbrace{\mathrm{H}_1(A) \oplus \mathrm{H}_1(B)}_{=0} \arrow[r] & \mathrm{H}_1(X) \arrow[dll, "\partial_{MV}"'] & \\
\underbrace{\mathrm{H}_0(A \cap B)}_{\langle p \rangle \oplus \langle q \rangle} \arrow[r, "\alpha"'] & \underbrace{\mathrm{H}_0(A) \oplus \mathrm{H}_0(B)}_{=\mathbb{Z} \oplus \mathbb{Z}} \arrow[r] & \underbrace{\mathrm{H}_0(X)}_{=\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

where $\alpha(n, m) = (n + m, n + m)$. Moreover, we use that S^1 is path connected, and H_0 is generated by any point. Therefore, $\mathrm{H}_1(X)$ sits in an exact sequence

$$0 \longrightarrow \mathrm{H}_1(X) \xrightarrow{\partial_{MV}} \mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^2 \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0$$

and so,

$$\mathrm{H}_1(X) \simeq \ker(\alpha) = \langle (1, -1) \rangle = \mathbb{Z}(p - q)$$

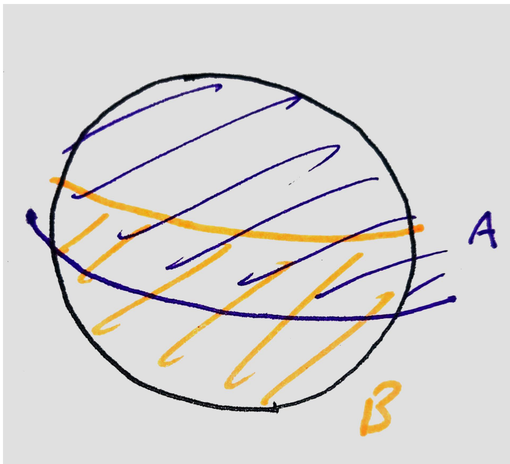
□

Example 2.4.2

$$\mathrm{H}^*(S^n) = \mathrm{H}_*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Lecture 5

Proof. We will compute the cohomology. Define $A, B \subseteq S^n$ by



Then A, B are contractible, and $A \cap B$ is homotopic to S^{n-1}. We work inductively in n, using the Mayer-

Vietoris sequence. The sequence gives

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{H}^{i-1}(S^n) \arrow[dll, "\partial^*_{MV}"] \\
\mathrm{H}^i(S^n) \arrow[r] & \mathrm{H}^i(*) \oplus \mathrm{H}^i(*) \arrow[r] & \mathrm{H}^{i-1}(S^{n-1}) \arrow[dll, "\partial^*_{MV}"] \\
\mathrm{H}^{i+1}(S^n) \arrow[r] & \mathrm{H}^{i+1}(*) \oplus \mathrm{H}^i(*) \arrow[r] & \cdots
\end{tikzcd}$$

If $i > 0$, we have

$$0 \longrightarrow \mathrm{H}^i(S^{n-1}) \xrightarrow{\partial^*_{MV}} \mathrm{H}^{i+1}(S^n) \longrightarrow 0$$

and so

$$\mathrm{H}^i(S^{n-1}) = \mathrm{H}^{i+1}(S^n)$$

This gives us almost everything inductively. We can assume $n \geq 2$. At the bottom of the sequence, we have

$$\underbrace{\mathrm{H}^0(S^n)}_{=\mathbb{Z}} \longrightarrow \underbrace{\mathrm{H}^0(*) \oplus \mathrm{H}^0(*)}_{=\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\alpha} \underbrace{\mathrm{H}^0(S^{n-1})}_{=\mathbb{Z}} \longrightarrow \mathrm{H}^1(S^n) \longrightarrow 0$$

We have shown (on examples sheet 1) that $\mathrm{H}^0(X) = 1$, generated by the 0-dimensional cochain, sending $p \mapsto 1$. As

$$\alpha(p, q) = p + q$$

α is surjective. Thus, we have

$$0 \longrightarrow \mathrm{H}^1(S^n) \longrightarrow 0$$

and so $\mathrm{H}^1(S^n) = 0$. □

Corollary 2.4.3. $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $m = n$.

Proof. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism. This gives us a homeomorphism

$$f : \mathbb{R}^n \setminus 0 \to \mathbb{R}^m \setminus \{f(0)\}$$

But $\mathbb{R}^n \setminus \{*\}$ is homotopy equivalent to S^{n-1}. Thus f induces an isomorphism in homology

$$\mathrm{H}_i(S^{n-1}) = \mathrm{H}_i(S^{m-1})$$

But by our homology computations, this implies that $n = m$. □

Note in the below we assume $n > 0$.

Definition 2.4.4 (degree)
Suppose $f : S^n \to S^n$ is a continuous map, then f induces a map

$$f_* : \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$$

which is multiplication by some $d \in \mathbb{Z}$. We call $\deg(f) = d$ the *degree* of f.

Remark 2.4.5. Since $\mathrm{H}_n(S^n) = \mathbb{Z}$ which is free, therefore f_* is determined by $f_*(1)$. Moreover, we will need to use the *same* identification of $\mathrm{H}_n(S^n) = \mathbb{Z}$, and so $\deg(f)$ is well defined. Note that if we use different isomorphisms, then $\deg(f)$ is only defined up to a sign.

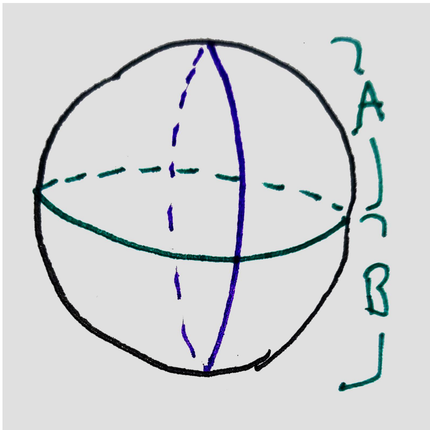
Note that

- if $f \simeq g$, then $\deg(f) = \deg(g)$,
- $\deg(\mathrm{id}) = 1$.
- $\deg(\mathrm{const}) = 0$, since the constant map factors through $\{*\}$, and so the induced map on homology factors through $\mathrm{H}^n(*) = 0$.

Lemma 2.4.6. If $A \in O(n+1)$, then A acts on S^n, and we have

$$\deg(A) = \det(A) \in \{\pm 1\}$$

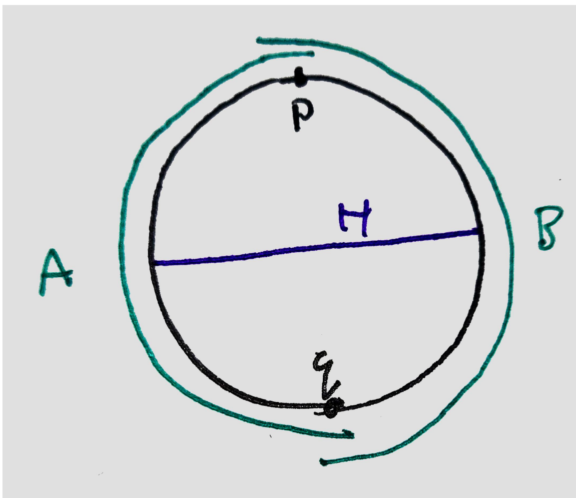
Proof. The group $O(n+1)$ has two path components. By homotopy invariance of degree, any A is homotopic to id_{S^n}, or a reflection in an equatorial hyperplane H.



The reflection preserves $A, B, A \cap B$, and H intersects $A \cap B = S^{n-1}$ at two points. The Mayer-Vietoris sequence gives

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}_n(S^n) \arrow[r, "\cong"] \arrow[d, "A_*"'] & \mathrm{H}_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "A_*"] & 0 \\
0 \arrow[r] & \mathrm{H}_n(S^n) \arrow[r, "\cong"'] & \mathrm{H}_{n-1}(S^{n-1}) \arrow[r] & 0
\end{tikzcd}$$

The two vertical maps are the same. Therefore it suffices to consider the result on S^1.



Recall our computation of $H_1(S^1)$ has

$$0 \longrightarrow \mathrm{H}_1(S^1) \longrightarrow \mathrm{H}_0(p \sqcup q) \xrightarrow{\alpha} \mathrm{H}_0(A) \oplus \mathrm{H}_0(B) \longrightarrow \cdots$$

where $\alpha(u, v) = (u + v, u + v)$. Hence $H_1(S^1)$ is generated by $(1, -1)$, which we can think of as $p - q$. But in this case, reflection in H swaps p and q, and so it acts on $H^1(S^1)$ by multiplication by -1. $\square$

Corollary 2.4.7. The antipodal map $a_n : S^n \to S^n$ has degree $(-1)^{n+1}$.

Proof. a_n is orthogonal. Or equivalently, it is a composition of $n + 1$ reflections, and we have that

$$\deg(fg) = \deg(f)\deg(g)$$

$\square$

Corollary 2.4.8. If $f : S^n \to S^n$ has no fixed point, then f is homotopic to the antipodal map.

Proof. In fact, we will see that $f(x) \neq g(x)$ for all $x \in S^n$, then $f \simeq a_n \circ g$. Taking $g = \mathrm{id}$ gives the result. Consider the map

$$x \mapsto \frac{tf(x) - (1 - t)g(x)}{\left\|tf(x) - (1 - t)g(x)\right\|}$$

for $0 \leq t \leq 1$. Note that the denominator never vanishes. This then defines a homotopy from f to $-g$. $\square$

Using this, the degree of any map without fixed points is $(-1)^{n+1}$. Another corollary is that if $f(x) \neq -x$ for all $x \in S^n$, then $f \simeq a \circ a = \mathrm{id}$.

Definition 2.4.9 (vector field on S^n)
A *vector field on* S^n is a map $v : S^n \to \mathbb{R}^{n+1}$, such that for all $x \in S^n$,

$$\langle x, v(x)\rangle = 0$$

That is, $v(x)$ is a tangent vector to S^n at x.

Proposition 2.4.10 (Hairy ball theorem). S^n has a nowhere vanishing vector field if and only if n is odd.

Proof. If $n = 2k - 1$,

$$v(x_1, y_1, \ldots, x_k, y_k) = (-y_1, x_2, \ldots, -y_k, x_k)$$

works.

Lecture 6

Now suppose a nowhere vanishing vector field $v : S^n \to \mathbb{R}^{n+1}$ exists. By normalising, i.e. considering

$$\frac{v(x)}{\left\|v(x)\right\|}$$

we can consider $v : S^n \to S^n \subseteq \mathbb{R}^{n+1}$. Now consider the family

$$v_t(x) = \cos(t)x + \sin(t)v(x)$$

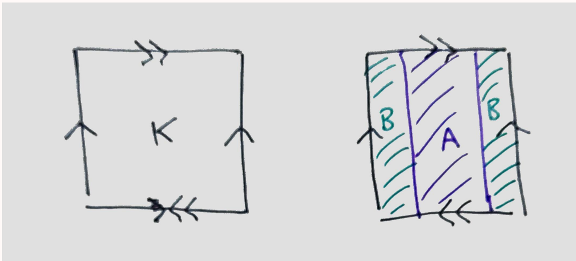
for $0 \leq t \leq \pi$, which has unit length as we assumed $v(x)$ and x are orthogonal. This family has $v_0 = \mathrm{id}$, and $v_\pi = a_n$. Thus, we have that

$$1 = \deg(\mathrm{id}) = \deg(a_n) = (-1)^{n+1}$$

$\square$

Example 2.4.11 (Klein bottle)

Recall the Klein bottle is the result if we glue two Möbius strips together along their common boundary S^1. That is, we define K by the gluing pattern



In this case, we can write $K = A \cup B$, where A, B are Möbius bands, i.e. $A \simeq S^1 \simeq B$. Moreover, $A \cap B \simeq S^1$ as well. The interesting part of the Mayer-Vietoris sequence for homology is as follows:

$$0 \longrightarrow \mathrm{H}_2(K) \longrightarrow \mathrm{H}_1(A \cap B) \xrightarrow{\psi} \mathrm{H}_1(A) \oplus \mathrm{H}_1(B) \longrightarrow \mathrm{H}_1(K) \longrightarrow \mathrm{H}_0(A \cap B) \xrightarrow{\alpha} \mathrm{H}_0(A) \oplus \mathrm{H}_0(B)$$

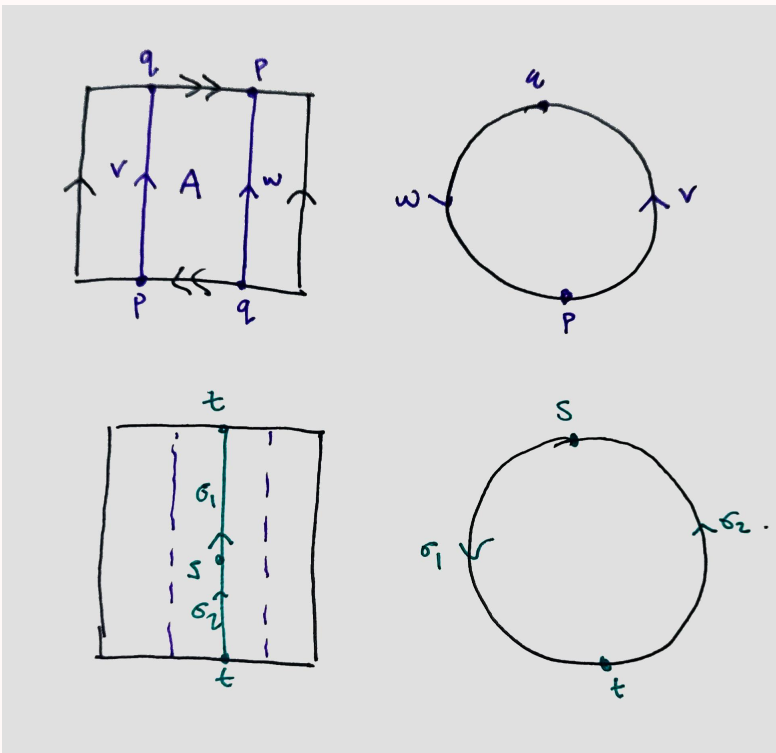
Recall $\alpha(1) = (1, 1)$, and so α is injective, and the sequence becomes

$$0 \longrightarrow \mathrm{H}_2(K) \longrightarrow \mathrm{H}_1(A \cap B) \xrightarrow{\psi} \mathrm{H}_1(A) \oplus \mathrm{H}_1(B) \longrightarrow \mathrm{H}_1(K) \longrightarrow 0$$

By exactness, $\mathrm{H}_1(K) \cong (\mathbb{Z} \oplus \mathbb{Z})/\operatorname{im}(\psi) = \operatorname{coker}(\psi)$ and $\mathrm{H}_2(K) \cong \ker(\psi)$.

Claim 2.4.12. $\psi(1) = (2, 2)$, and so $\mathrm{H}_2(K) = 0$, and $\mathrm{H}_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2$.

Proof of claim. We will use the decomposition of K into two Möbius bands.



In this case, $\mathrm{H}_1(A \cap B) = \mathbb{Z}\langle v + w \rangle$, and $\mathrm{H}_1(A) = \mathbb{Z}\langle \sigma_1 + \sigma_2 \rangle$. At chain level, there exists a collection of 2-simplices in A, with boundary $c - (\sigma_1 + \sigma_2)$. Therefore, $v \mapsto \sigma_1 + \sigma_2$ and $w \mapsto \sigma_1 + \sigma_2$. With this, we have that (by symmetry),

$$\psi(1) = (2, 2)$$

□

Remark 2.4.13. (i) In cohomology, $\mathrm{H}^2(K) = \mathbb{Z}/2$ and $\mathrm{H}^1(K) = \mathbb{Z}$. That is, the homology and the cohomology are not the same.

(ii) We defined $C_i(X) = \left\{\sum_{\text{finite}} a_j\sigma_j \mid a_j \in \mathbb{Z}, \sigma_j \text{ is an } i\text{-simplex}\right\}$. But we can also define

$$C_i(X; G) = \left\{\sum_{\text{finite}} a_j\sigma_j \;\middle|\; a_j \in G, \sigma_j \text{ is an } i\text{-simplex}\right\}$$

for any abelian group G. ∂ is defined as before, and we get a chain complex $C_*(X; G)$ and homology $\mathrm{H}_*(X; G)$. If we instead computed

$$\mathrm{H}_*(K; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & * = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & * = 1 \\ \mathbb{Z}/2 & * = 2 \end{cases}$$

since $\psi = 0$ if we used $\mathbb{Z}/2$ coefficients.

Interlude on relative homology

Some of the questions on sheet 1 concern *relative homology*. If $A \subseteq X$ is a subspace, then any simplex $\sigma : \Delta^1 \to A$ is a simplex in X. Moreover, if σ is a simplex in A, so are all of its boundary faces. Thus, the natural map induced by the inclusion map,

$$C_i(A) \hookrightarrow C_i(X)$$

defines a chain map. Using this, $C_*(A)$ is a *subcomplex* of $C_*(X)$. In particular, we can define the *quotient complex*,

$$C_i(X, A) = \frac{C_i(X)}{C_i(A)}$$

and this is a chain complex, with ∂ induced by the boundary operator on $C_i(X)$. Another way of thinking about $C_i(X, A)$ is that it is the free abelian group on i-simplices in X not wholly contained in A. The homology of $C_*(X, A)$, denoted $\mathrm{H}_*(X, A)$ is called *relative homology*.

We have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_i(A) \longrightarrow \mathrm{H}_i(X) \longrightarrow \mathrm{H}_i(X, A) \xrightarrow[\partial]{} \mathrm{H}_{i-1}(A) \longrightarrow \cdots$$

If A and X are 'well behaved', then $\mathrm{H}_i(X, A) \cong \mathrm{H}_i(X/A)$ for $i > 0$, where X/A is the quotient space where we quotient A into a point.

2.5 Homotopy invariance

Recall if $f, g : X \to Y$ are homotopic, we would like to show that the induced maps on homology and cohomology are the same. Recall also that the maps are induced by (co)chain maps on (co)chain complexes.

Definition 2.5.1 (chain homotopic)

Let C_*, D_* be chain complexes. Let $f_*, g_* : C_* \to D_*$ be chain maps. We say that they are *chain homotopic* if there exists maps

$$P_n : C_n \to D_{n+1}$$

for all n, such that

$$P_{n-1} \circ \partial + \partial \circ P_n = f_n - g_n$$

for all n.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] \arrow[d, "f_n - g_n"] \arrow[dl, "P_n"'] & C_{n-1} \arrow[dl, "P_{n-1}"'] \\
D_{n+1} \arrow[r, "\partial"'] & D_n \arrow[r, "\partial"'] & D_{n-1}
\end{tikzcd}$$

Lemma 2.5.2. Suppose $f_*, g_* : C_* \to D_*$ are chain homotopic. Then they induce the same map on homology.

Proof. Let $\alpha \in C_i$ be a cycle, that is, $\partial\alpha = 0$. By definition, $f_*[\alpha] = [f(\alpha)]$. Consider

$$f_i(\alpha) - g_i(\alpha) = (f_i - g_i)(\alpha) = (\partial P + P\partial)(\alpha) = \partial(P\alpha)$$

But this is in the image of ∂, and so the corresponding homology classes are the same. □

Lecture 7

Exercise: Chain homotopy is an equivalence relation on chain maps $C_* \to D_*$.

Theorem 2.5.3 (homotopy invariance of homology)**.** If $f, g : X \to Y$ are homotopic, then the induced maps on homology are the same.

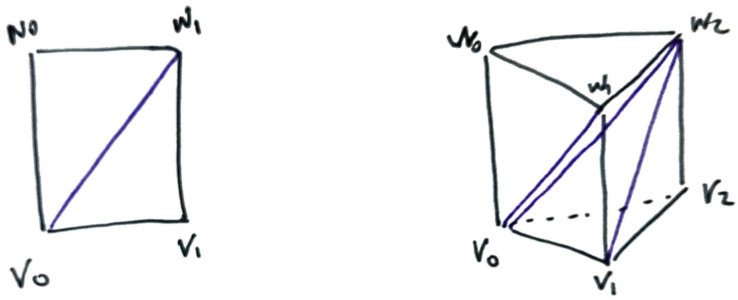
Proof. Let $i_j : X \hookrightarrow X \times \{j\}$ be the natural map, $F : X \times I \to Y$ the homotopy between f, g. But then

$$f_* = (F \circ i_0)_* = F_* \circ i_{0*}$$

Therefore, it suffices to show that i_{0*} and i_{1*} are chain homotopic.
Thus, we want

$$P : C_n(X) \to C_{n+1}(X \times [0,1])$$

such that $\partial P + P\partial = i_{0*} - i_{1*}$. P is a *prism operator* from a universal way of decomposing $\Delta^n \times [0,1]$ into a finite collection of $n+1$-simplices.



Consider the ordered collections$[v_0, \ldots, v_i, w_i, \ldots, w_n]$, where $[v_0, \ldots, v_n] = \Delta^n \times \{0\}$ and $[w_0, \ldots, w_n] = \Delta^n \times \{1\}$. We define P by

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i (\sigma \times I)|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$$

Claim 2.5.4. $\partial P + P\partial = i_{i*} - i_{0*}$. That is,

$$\partial P = i_{1*} - i_{0*} - P\partial$$

i.e. "boundary of prism is the top and bottom faces, as well as the prism of the boundary".

Proof.

$$\partial P(\sigma) = \sum_{j\leq i}(-1)^i(-1)^j(\sigma \times I)|_{[v_0,\dots,\widehat{v_j},\dots,v_i,w_i,\dots,w_n]} + \sum_{j\geq i}(-1)^i(-1)^{j+1}(\sigma \times I)|_{[v_0,\dots,v_i,w_i,\dots,\widehat{w_j},\dots,w_n]}$$

The terms when $j = i$ cancel, except for

$$(\sigma \times I)|_{[w_0,\dots,w_n]} - (\sigma \times I)|_{[v_0,\dots,v_n]}$$

which are the top and bottom faces. The terms with $i > j$ and $i < j$ comprise, up to a sign, $P(\partial\sigma)$. Just compute (in a dark room with some gin). ☐

☐

Remark 2.5.5. If C^*, D^* are cochain complexes, cochain maps f^*, g^* are cochain homotopic if we have $P^i : C^i \to D^{i-1}$, such that

$$\partial P + P\partial = f^* - g^*$$

It is easy to check that in this case, f^* and g^* induce the saem map $\mathrm{H}^*(C^*) \to \mathrm{H}^*(D^*)$.

With this, our prism operator $P : C_n(X) \to C_{n+1}(X \times [0,1])$ has a dual

$$\begin{aligned} P^* : C^{n+1}(X \times [0,1]) &\to C^n(X) \\ P^*(f) &= f \circ P \end{aligned}$$

Dualising everything, the relation $\partial P + P\partial = i_{1*} - i_{0*}$ becomes

$$\partial P^* + P^*\partial = i_1^* - i_0^*$$

and thus homotopic maps induce the same map on cohomology.

2.6 Snake lemma

Definition 2.6.1

A *short exact sequence of chain complexes*

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0$$

is a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\alpha"] \arrow[d] & B_{n+1} \arrow[r, "\beta"] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\alpha"] \arrow[d] & B_n \arrow[r, "\beta"] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha"] \arrow[d] & B_{n-1} \arrow[r, "\beta"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

such that the squares commute (i.e. α, β are chain maps) and the rows are exact.

Proposition 2.6.2 (Snake lemma)**.** Suppose we have a SES $0 \to A_* \to B_* \to C_* \to 0$ of chain complexes, then we have a long exact sequence of homology

$$\cdots \longrightarrow \mathrm{H}_{n+1}(C_*) \xrightarrow{\delta} \mathrm{H}_n(A_*) \xrightarrow{\alpha_*} \mathrm{H}_n(B_*) \xrightarrow{\beta_*} \mathrm{H}_n(C_*) \xrightarrow{\delta} \mathrm{H}_{n-1}(A_*) \longrightarrow \cdots$$

Proof. We will construct δ, but leave all other details. Suppose $c_n \in C_n$ is a cycle, representing a homology class $[c_n] \in H_n(C^*)$. Horizontal exactness means $c_n = \beta(b_n)$ for some $b_n \in B_n$. But

$$\beta(\partial b_n) = \partial\beta(b_n) = \partial c_n = 0$$

Hence $\partial b_n \in \ker(\beta) = \operatorname{im}(\alpha)$, by horizontal exactness. Thus, we have $a_{n-1} \in A_{n-1}$ such that $\alpha(a_{n-1}) = \partial b_n$. In this case,

$$\alpha(\partial a_{n-1}) = \partial\alpha(a_{n-1}) = \partial^2 b_n = 0$$

But α is injective, and so $\partial a_{n-1} = 0$, and this represents a cycle.

$$\begin{tikzcd}
A_n & B_n \arrow[r, two heads, "\beta"] \arrow[d, "\partial"'] & C_n \arrow[d, "\partial"] \\
A_{n-1} \arrow[r, hook, "\alpha"] \arrow[d, "\partial"'] & B_{n-1} \arrow[r, two heads, "\beta"] \arrow[d, "\partial"] & C_{n-1} \\
A_{n-2} \arrow[r, hook, "\alpha"] & B_{n-2} & C_{n-2}
\end{tikzcd}$$

We define $\delta([c_n]) = [a_{n-1}]$.

To complete the proof:

1. Check that δ is independent of the choices of c_n, b_n, a_{n-1}.
2. δ is a homomorphism.
3. Check that the sequence is exact.

□

Example 2.6.3

Recall if G is an abelian group, we introducted $C_*(X; G)$. If we started with an SES of abelian groups

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

then there is an associated SES of chain complexes

$$0 \longrightarrow C_*(X; G_1) \longrightarrow C_*(X; G_2) \longrightarrow C_*(X; G_3) \longrightarrow 0$$

The homomorphism

$$\mathrm{H}_n(X; G_3) \to \mathrm{H}_{n-1}(X; G_1)$$

are often called *Bockstein homomorphisms*.

Example 2.6.4

If $A \subseteq X$ is a subspace, then we defined relative homology, using the SES

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X, A) \longrightarrow 0$$

The associated long exact sequence includes a boundary map

$$\delta : \mathrm{H}_n(X, A) \to \mathrm{H}_{n-1}(A)$$

and we have a LES of the pair (X, A).

2.7 Excision

Theorem 2.7.1 (Excision). Let X be a topological space, $A \subseteq X$ a subspace, $Z \subseteq X$ a subspace with $\mathrm{Cl}(Z) \subseteq \mathrm{Int}(A)$. Then the inclusion

$$i : (X \setminus Z, A \setminus Z) \to (X, A)$$

induces an isomorphism on relative homology

$$i_* : \mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$$

Intuitively, relative homology measures the homology of X where we ignore what happens within A. Therefore, excision means that when computing relative homology, we can ignore what happens within Z.

Lemma 2.7.2 (five lemma). Suppose we have a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

where the rows are exact, $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is δ.

Corollary 2.7.3. If $f : (X, A) \to (Y, B)$ is a map of pairs, and the maps $f_* : \mathrm{H}_*(X) \to \mathrm{H}_*(Y)$ and $f_* : \mathrm{H}_*(A) \to \mathrm{H}_*(B)$ are isomorphisms, then so are the induced maps on relative homology

$$f_* : \mathrm{H}_*(X, A) \to \mathrm{H}_*(Y, B)$$

Proof. Follows from the five lemma and the LES of relative homology. □

Lecture 8

2.8 Relative homology

Definition 2.8.1 (Reduced homology)

If X is a topological space, $x_0 \in X$ is a base point, then we define the *reduced homology of X* as

$$\tilde{\mathrm{H}}_*(X) = \mathrm{H}_*(X, x_0)$$

It is easy to see that $\tilde{\mathrm{H}}_i(X) = \mathrm{H}_i(X)$ for $i > 0$, and $\tilde{\mathrm{H}}_i(X) \oplus \mathbb{Z} = \mathrm{H}_i(X)$.

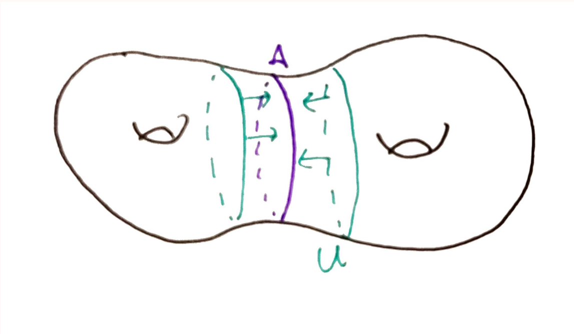
Definition 2.8.2 (good pair)

A pair (X, A) is *good* if $A \subseteq X$ is closed, and A a neighbourhood deformation retract. That is, there exists an open neighbourhood $A \subseteq U \subseteq X$ of A, and a homotopy $H : U \times [0, 1] \to U$, with

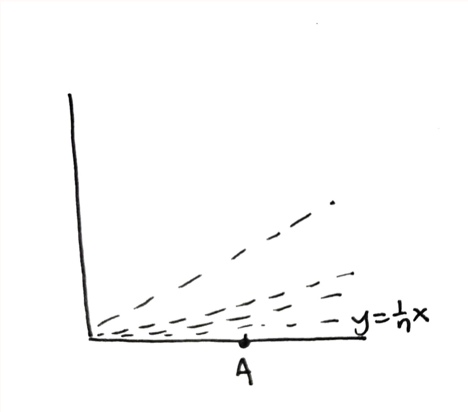
- $H(u, 0) = u$,
- $H(u, 1) \in A$ for all $u \in U$,
- $H(a, t) = a$ for all $a \in A, t \in [0, 1]$.

Example 2.8.3

One example would be a loop on Σ_2, where we can choose an annular neighbourhood.



On the other hand, consider



There is no neighbourhood retract fixing A.

Proposition 2.8.4. If (X, A) is a good pair, the natural map $(X, A) \to (X/A, A/A = \{\bullet\})$, induces isomorphism

$$\mathrm{H}_n(X, A) \to \mathrm{H}_n(X/A, A/A) = \tilde{\mathrm{H}}_*(X/A)$$

In this case, the intuition is that we can in fact collapse A down to a point.

Proof. Let $A \subseteq U \subseteq X$ be as in the definition of a good pair, and note that $\mathrm{H}_*(A) = \mathrm{H}_*(U)$, and as such, we have an isomorphism on relative homology

$$\mathrm{H}_*(X, A) \cong \mathrm{H}_*(X, U)$$

Since the homotopy $H : U \times [0, 1] \to U$ is fixed on A, it induces a homotopy on U/A, and so

$$\frac{A}{A} = \{\bullet\} \hookrightarrow \frac{U}{A}$$

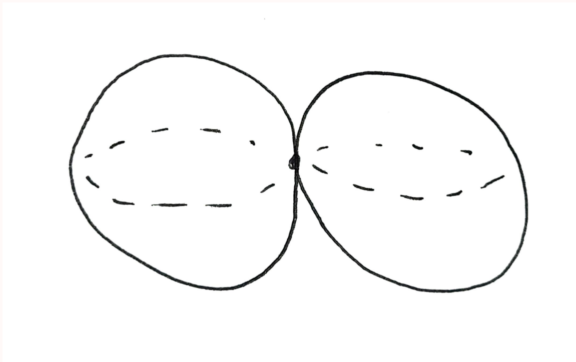
is also a neighbourhood deformation retract. With this,

$$\begin{tikzcd}
H_*(X, A) \arrow[r, "\cong"] \arrow[d] & H_*(X, U) & H_*(X/A, U/A) \arrow[l, "\sim"'] \arrow[d, "\sim\text{homeomorphism of pairs}"] \\
H_*(X/A, \bullet) \arrow[r, "\cong"] & H_*(X/A, U/A) & H_*((X/A)/\bullet, (U/A)/\bullet) \arrow[l, "\sim"']
\end{tikzcd}$$

and so the left map is an isomorphism. Note $\sim$ is an isomorphism by excision, and $\cong$ is an isomorphism by the above, from the homotopy. $\square$

Example 2.8.5 (i) $\mathrm{H}_j(D^n, \partial D^n) = \tilde{\mathrm{H}}_j(D^n/\partial D^n) = \tilde{\mathrm{H}}_j(S^n) = \begin{cases} \mathbb{Z} & j = n \\ 0 & \text{otherwise} \end{cases}$

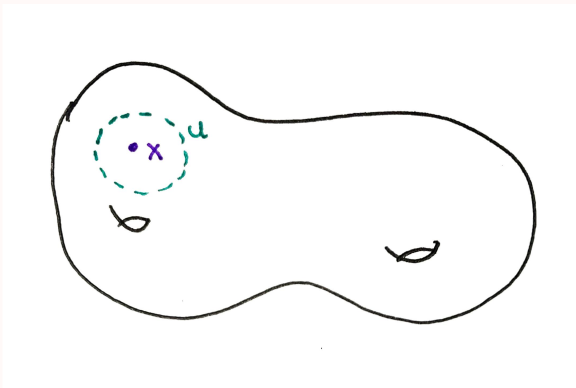
(ii) $\mathrm{H}_j(S^2, S^1_{\text{eq}}) = \tilde{\mathrm{H}}_j(S^2 \vee S^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & j = 2 \\ 0 & \text{otherwise} \end{cases}$ Since we can think of S^2/S^1 as $S^2 \vee S^2$ via



(iii) A *manifold* of dimension n is a Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^n$. If M^n is a manifold, $x \in M$, then

$$\mathrm{H}_j(M, M \setminus x) \underbrace{=}_{\text{by excision}} \mathrm{H}_j(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \underbrace{=}_{\text{by homotopy invariance}} \mathrm{H}_j(D^n, \partial D^n) = \mathbb{Z} \text{ when j} = \text{n}$$

For excision, we choose an open set U which is homeomorphic to a ball, and remove the complement of U.



2.9 Small simplices theorem

Both the Mayer-Vietoris property and excision will follow from the *small simplices theorem*.

Definition 2.9.1 (chain complex adapted to a cover)
Let X be a topological space,

$$\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$$

be such that $\{\mathrm{Int}(U_\alpha)\}_{\alpha \in I}$ is an open cover of X. Let

$$C_j^{\mathcal{U}}(X) = \left\{ \sum_{\text{finite}} a_j \sigma_i \;\middle|\; a_i \in \mathbb{Z}, \sigma_i : \Delta^i \to X, \mathrm{im}(\sigma_i) \subseteq U_{\alpha(\sigma_i)} \text{ for some } \alpha(\sigma_i) \in I \right\}$$

This defines a subcomplex of $C_*(X)$ with the usual boundary map.

Theorem 2.9.2 (small simplices)**.** The inclusion $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ induces an isomorphism on homology.

Remark 2.9.3. If X, Y are spaces with covers $\mathcal{U}, \mathcal{V}$, and if $f : X \to Y$ sends each U_α to some $V_{f(\alpha)}$, then f induces a map

$$f_* : C_j^{\mathcal{U}}(X) \to C_*^{\mathcal{V}}(Y)$$

But $(X, \mathcal{U}) \mapsto C_*^{\mathcal{U}}(X)$ is *not* functorial for arbitrary maps.

Intuitively, what we want to do here is to split up the chains, so that it is a sum of chains in the open cover.

We will use small simplices to prove Mayer-Vietoris and excision.

Proof of Mayer-Vietoris, theorem 2.3.7. Let $\mathcal{U} = \{A, B\}$, where $A, B \subseteq X$ open. Then we have an SES of chain complexes

$$0 \longrightarrow C_*(A \cap B) \longrightarrow C_*(A) \oplus C_*(B) \longrightarrow C_*^{\mathcal{U}}(X) \longrightarrow 0$$

The last map is surjective because we have taken the subgroup $C_*^{\mathcal{U}}(X)$. By the Snake lemma and the small simplices theorem, we get the Mayer-Vietoris sequence. □

Remark 2.9.4. (i) The boundary map is what we wrote down in the special case earlier.

(ii) The naturality of Mayer-Vietoris is the naturality of the map

$$C_*^{\mathcal{U}}(X) \to C_*^{\mathcal{V}}(Y)$$

where $\mathcal{U} = \{A, B\}, \mathcal{V} = \{C, D\}$ and $f(A) \subseteq C, f(B) \subseteq D$.

Proof of excision, theorem 2.7.1. Recall we have $Z \subseteq A \subseteq X$, $\mathrm{Cl}(Z) \subseteq \mathrm{Int}(A)$. Let $B = X \setminus Z, \mathcal{U} = \{A, B\}$. Then $\mathrm{Int}(A) \cup \mathrm{Int}(B) = X$, and so we can apply the small simplices theorem.

Consider

$$\frac{C_n^{\mathcal{U}}(X)}{C_n(A)} = \frac{C_n(B)}{C_n(A \cap B)} = \text{free ab. group of simplices in } B \text{ not wholly contained in } A$$

and so we have the short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & C_*(A) \arrow[r] \arrow[d, hook] & C_*^{\mathcal{U}}(X) \arrow[r] \arrow[d, hook] & C_*^{\mathcal{U}}(X)/C_*(A) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(A) \arrow[r] & C_*(X)/C_*(A) \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are induced by the inclusion, and all squares commute. Therefore, we have long exact sequences in homology and natural maps between them,

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_i(A) \arrow[r] \arrow[d, "="] & \mathrm{H}_i(C_*^{\mathcal{U}}(X)) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}_i(C_*^{\mathcal{U}}(X)/C_*(A) \arrow[r] \arrow[d] & \mathrm{H}_{i-1}(A) \arrow[r] \arrow[d, "="] & \mathrm{H}_{i-1}(C_*^{\mathcal{U}}(X)) \arrow[r] \arrow[d, "\sim"] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_i(A) \arrow[r] & \mathrm{H}_i(X) \arrow[r] & \mathrm{H}_i(X, A) \arrow[r] & \mathrm{H}_{i-1}(A) \arrow[r] & \mathrm{H}_{i-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where $=$ are identity maps, $\sim$ is an isomorphism by the small simplices theorem, and so the middle map is an isomorphism by the five lemma. But we have that

$$\mathrm{H}_i(X, A) = \mathrm{H}_i(C_*(X)/C_*(A)) = \mathrm{H}_i(C_*(B)/C_*(A \cap B)) = \mathrm{H}_i(B, A \cap B) = \mathrm{H}_*(X \setminus Z, A \setminus Z)$$

□

Small simplices uses *barycentric subdivision*. It's important to get intuition for this, which is that we can repeatedly subdivide a simplex into smaller simplices, and using the Lebesgue covering lemma, if we subdivide enough, we can ensure that each simplex is contained in an open set in the cover.

However, the technical details are not very interesting...

If Δ^n is the standard simplex, let $b_n = \frac{1}{n+1}(1, \dots, 1)$ be its *barycentre*. Take an i-simplex $\sigma : \Delta^i \to \Delta^n$, we define

$$\begin{aligned} \mathrm{Cone}_i^{\Delta^n}(\sigma) &: \Delta^{i+1} \to \Delta^n \\ (t_0, \dots, t_{i+1}) &\mapsto t_0 b_n + (1-t_0)\sigma\left(\frac{(t_1, \dots, t_{i+1})}{1-t_0}\right) \end{aligned}$$

We can view $\mathrm{Cone}_i^{\Delta^n} : C_i(\Delta^n) \to C_{i+1}(\Delta^n)$ by extending linearly.

Exercise:

$$\partial(\mathrm{Cone}_i^{\Delta^n}(\sigma)) = \begin{cases} \sigma - \mathrm{Cone}_{i-1}^{\Delta^n}(\partial\sigma) & i > 0 \\ \sigma - \varepsilon(\sigma) b_n & i = 0 \end{cases}$$

where

$$\begin{aligned} \varepsilon &: C_0(\Delta^n) \to \mathbb{Z} \\ \sum n_i \sigma_i &\mapsto \sum n_i \end{aligned}$$

If we define $c_\bullet : C_*(\Delta^n) \to C_*(\Delta^n)$ by

$$c_\bullet(\sigma) = \begin{cases} \varepsilon(\sigma) b_n & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\partial \,\mathrm{Cone}^{\Delta^n} + \mathrm{Cone}^{\Delta^n} \partial = \mathrm{id} - c_\bullet$$

Our aim is to introduce a barycentric subdivision operation $\varphi : C_*(X) \to C_*(X)$, which for each simplex divides the boundary and cones off to the barycentre.

Definition 2.9.5 (natural)
A collection of chain maps $\varphi^X : C_*(X) \to C_*(X)$ is *natural* if for $f : X \to Y$, $f_* \circ \varphi^X = \varphi^Y \circ f_*$.

If $\sigma : \Delta^n \to X$ is an n-simplex, and $i_n : \Delta^n \to \Delta^n$ denotes the inclusion map, then $\sigma = \sigma \circ i_n$, and so

$$\varphi^X(\sigma) = \varphi_n^X(\sigma_* i_n) = \sigma_*(\varphi_n^{\Delta^n}(\sigma))$$

Thus, if we can subdivide the n-simplex, then we can use naturality to extend φ^X for all X.

The formula is

$$\varphi_n^X(\sigma) = \sigma_*\left(\mathrm{Cone}_{n-1}^{\Delta^n}\left(\varphi_{n-1}^{\Delta^n}(\partial i_n)\right)\right)$$

Lemma 2.9.6.
(i) If $\sigma = [v_0, \dots, v_n] \subseteq \mathbb{R}^{n+1}$ is an n-simplex, for example any n-simplex in a subdivision of Δ^N for $N \geq n$, then for any simplex τ in its barycentric subdivision,

$$\mathrm{diam}(\tau) \leq \frac{n}{n+1}\mathrm{diam}(\sigma)$$

(ii) If $\sigma \in C_n^{\mathcal{U}}(X)$, then $\varphi_n^X(\sigma) \in C_n^{\mathcal{U}}(X)$.

(iii) If $\sigma \in C_n(X)$, there exists $k > 0$ such that $(\varphi_n^X)^k(\sigma) \in C_n^{\mathcal{U}}(X)$.

Proof. (i) is just Euclidean geometry, (ii) is obvious. For (iii), we use the fact that σ is a finite sum of simplices, and so it suffices to prove this for a single simplex, since we can just take the maximum value of k. Then

$$\left\{\sigma^{-1}(\mathrm{Int}(U_\alpha))\right\}_{\alpha \in A}$$

is an open cover of Δ^n, which is a compact metric space, and so it has a Lebesgue number ε. That is, every open ε-ball in Δ^n lies in $\sigma^{-1}(\mathrm{Int}(U_\alpha))$ for some α. But we can choose k such that

$$\left(\frac{n}{n+1}\right)^k < \varepsilon$$

and so we are done. □

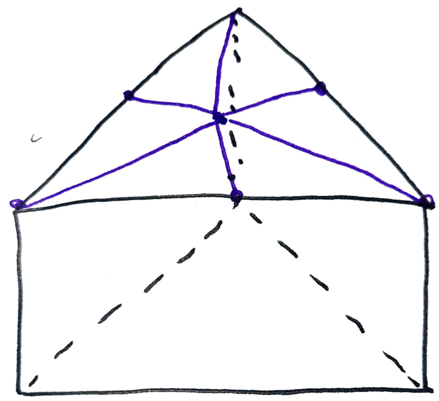
Proposition 2.9.7. There is a natural (with respect to maps of spaces) chain homotopy

$$P_X^* : C_*(X) \to C_{*+1}(X)$$

with

$$\partial P_n^X + P_{n-1}^X \partial = \varphi_n^* - \mathrm{id}_{C_n(X)}$$

Proof. As in homotopy invariance, we construct a prism $\Delta^n \times [0,1]$ by gluing Δ^n to its barycentric subdivision. We can subdivide this into $n+1$-simplices. See Hatcher for a formula.



Let $U : \mathrm{H}_n(C_*^{\mathcal{U}}(X)) \to \mathrm{H}_n(C_*(X))$ denote the induced map. Choose $[\sigma] \in \mathrm{H}_n(X)$, and choose k such that $(\varphi_n^X)^k(c) \in C_n^{\mathcal{U}}(X)$. Since φ_n^X is homotopic to id, so is $(\varphi_n^X)^k$, so there exists F such that

$$\partial F + F\partial = (\varphi^*)^k - \mathrm{id}$$

Thus, $(\varphi^X)^k(c) = c + \partial(\text{stuff})$, and so U is surjective.

On the other hand, if $U([c]) = 0$, then there exists $z \in C_{n+1}(X)$ with $\partial z = c$. There exists k such that

$$(\varphi_{n+1}^X)^k(z) \in C_{n+1}^{\mathcal{U}}(X) \quad \text{and} \quad (\varphi_{n+1}^X)^k(z) - z = (\partial F + F\partial)(c)$$

Then

$$c = \partial z = \partial(\varphi^X)^k(z) - \partial F(\partial z) \in C_{n+1}^{\mathcal{U}}$$

and so $[c] = 0$ in $\mathrm{H}_n(C_*^{\mathcal{U}}(X))$ already. □

3 Cellular homology

Singular homology is most effective on 'nice' spaces. One example would be *cell complexes* or CW-complexes. We will introduce a more manageable chain complex, for computing the homology or cohomology of a cell complex.

3.1 Definitions

Definition 3.1.1 (cell complex)

A *cell complex* is a topological space X obtained inductively as follows:

- X_0 is a discrete set,
- $$X_k = X_{k-1} \cup \bigcup_{i \in I_k} D_i^k$$

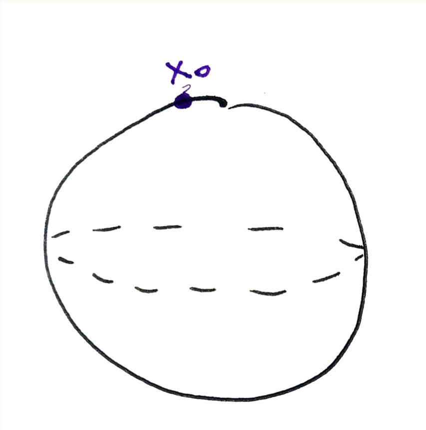
attached via maps $\varphi_i^k : \partial D_i^k = S^{k-1} \to X$. More formally, we can write this as a quotient of

$$X_{k-1} \sqcup \bigsqcup_{i \in I_k} D_i^k$$

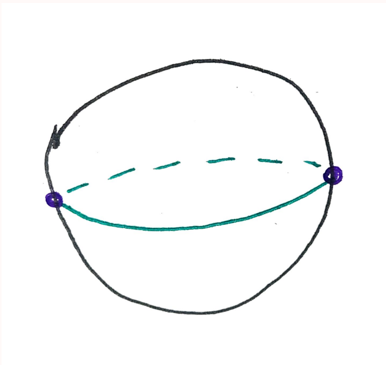
- $X = \bigcup_k X_k$, where we note that $X_{k-1} \subseteq X_k$, equipped with the *weak topology*. That is, $U \subseteq X$ is open if and only if $U \cap X_k$ is open in X_k for all k.

We call the D_i^k the *k-cells of X*, the φ_i^k the *attaching maps*, X_k the *k-skeleton of X*.

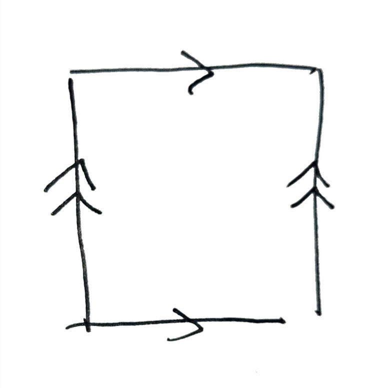
Example 3.1.2 (spheres)
$S^n = \{*\} \cup \{\text{open disc}\}$, i.e. $X_0 = \{*\}$ and a unique n-cell.



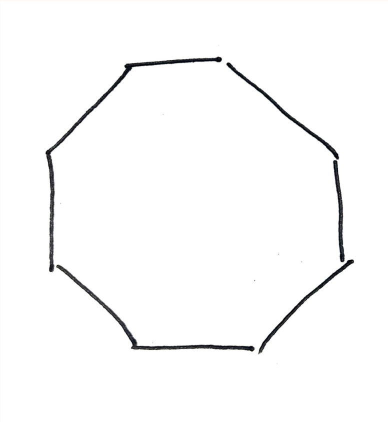
On the other hand, we can also construct S^2 as 2 0-cells, 2 1-cells and 2 2-cells.



Example 3.1.3 (torus)
The standard gluing diagram of T^2 gives a cell structure with 1 0-cells, 2 1-cells and 1 2-cell.



More generally, the surface Σ_g of genus g has 1 0-cell, $2g$ 1-cells and 1 2-cells.



The gluing pattern is $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.

Example 3.1.4

If X, Y are cell complexes, $x_0 \in X_0$, $y_0 \in Y_0$, then

$$X \vee Y = \frac{X \sqcup Y}{x_0 = y_0}$$

is naturally a cell complex.

Remark 3.1.5. If X is a cell complex, then X is a disjoint union of the open cells $\mathrm{Int}(D_i^k)$.

Definition 3.1.6 (finite dimensional, finite cell complex)

If $X = X_n$ for some n, we say X is *finite dimensional*. If $X = X_n$ and for all $k \leq n$, I_k is finite (i.e. X has finitely many cells), we say that X is an *finite cell complex*.

Remark 3.1.7. A finite cell complex is compact.

In fact, the converse is also true. If X is an infinite cell complex, we have an infinite, closed, discrete subset of X by choosing a point in each cell which is not in the boundary, and so X cannot be compact. Lecture 10

Definition 3.1.8 (subcomplex)
A *subcomplex of* X is a closed subspace of X which is a union of cells of X.

3.2 Homology of cell complexes

Lemma 3.2.1. (i) $A \subseteq X$ is open (resp. closed) if and only if its preimage in any cell is open (resp. closed), through the composition

$$\varphi_\alpha : D_\alpha \hookrightarrow X_{n+1} \sqcup \bigsqcup_\beta D_\beta \twoheadrightarrow X_n \hookrightarrow X$$

We call φ_α the *characteristic map of the cell* D_α. Note its restriction to the boundary is the attaching map.

(ii) Cell complexes are Hausdorff and locally contractible. In particular, connected and path connected are equivalent.

(iii) If $Z \subseteq X$ is compact, then $Z \subseteq X_N$ for some N.

(iv) If $A \subseteq X$ is a subcomplex, then the pair (X, A) is good.

Proof. Exercises, or see Hatcher. □

Corollary 3.2.2. If $A \subseteq X$ is a subcomplex, then $\mathrm{H}_*(X, A) \cong \widetilde{H}_*(X/A)$. In particular,

$$\mathrm{H}_i(X_k, X_{k-1}) = \begin{cases} \mathbb{Z}^{\oplus I_k} & i = k \\ 0 & \text{otherwise} \end{cases}$$

Proof.

$$\frac{X_k}{X_{k-1}} = \bigvee_{I_k} S^k$$

Result follows by Mayer-Vietoris, we note that

$$\left(\bigsqcup_{\alpha \in I_k} S^k_\alpha, \bigsqcup_{\alpha \in I_k} \{x_\alpha\} \right)$$

is a good pair, and the quotient is the wedge sum. Thus, the reduced homology of the wedge sum is given by the relative homology

$$\mathrm{H}_* \left(\bigsqcup S^k, \bigsqcup \{x_k\} \right) = \bigoplus \mathrm{H}_*(S^k, x_k)$$

□

Proposition 3.2.3. Let X be a cell complex.

(i) $\mathrm{H}_k(X_n) = 0$ for all $k > n$,

(ii) The inclusion $X_n \hookrightarrow X$ induces an isomorphism of $\mathrm{H}_k(X_n) \cong \mathrm{H}_k(X)$ for $k < n$.

Proof. For (i), consider the long exact sequence given by (X_n, X_{n-1}),

$$\cdots \longrightarrow \mathrm{H}_{k+1}(X_n, X_{n-1}) \longrightarrow \mathrm{H}_k(X_{n-1}) \longrightarrow \mathrm{H}_k(X_n) \longrightarrow \mathrm{H}_k(X_n, X_{n-1}) \longrightarrow \cdots$$

if $k > n$, then $\mathrm{H}_k(X_n, X_{n-1}) = \mathrm{H}_{k+1}(X_n, X_{n-1})$ as it is a wedge of spheres, and so

$$\mathrm{H}_k(X_{n-1}) \cong \mathrm{H}_k(X_n)$$

But we can iterate this, since $k > n > n-1$, and so

$$\mathrm{H}_k(X_n) \cong \mathrm{H}_k(X_{n-1}) \cong \cdots \cong \mathrm{H}_k(X_0) = 0$$

since X_0 is a discrete set.

For (ii), consider the same sequence

$$\cdots \longrightarrow \mathrm{H}_{k+1}(X_n, X_{n-1}) \longrightarrow \mathrm{H}_k(X_{n-1}) \longrightarrow \mathrm{H}_k(X_n) \longrightarrow \mathrm{H}_k(X_n, X_{n-1}) \longrightarrow \cdots$$

as above, with $k < n-1$. In this case, $k+1 < n$ and $k < n$, and so the same logic shows

$$\mathrm{H}_k(X_{n-1}) \cong \mathrm{H}_k(X_n) \cong \mathrm{H}_k(X_N)$$

for all $N \geq n$. If X is finite dimensional, then we are done. In general, if $\alpha \in \mathrm{H}_k(X)$, then it is represented by a finite collection of k-simplices. But this is a compact space, and so it lies within X_N for some N, and so α lies in the image of the map $\mathrm{H}_k(X_N) \to \mathrm{H}_k(X)$. Conversely, if we have $\alpha \in \mathrm{H}_k(X)$ bounding a $k+1$-chain, then that union of simplices lies in $X_{N'}$ for some $N' \geq N$, and so $\alpha = 0$ in $\mathrm{H}_k(X_{N'})$. $\square$

Corollary 3.2.4. If X is a finite cell complex of dimension N, then $\mathrm{H}_j(X) = 0$ for all $j > N$.

Definition 3.2.5 (cellular chain complex)
We define the *cellular chain complex* $C_*^{\text{cell}}(X)$ of a cell complex X (with its cell structure) via

$$C_k^{\text{cell}}(X) := \mathrm{H}_*(X_k, X_{k-1}) = \text{free abelian group on } k\text{-cells}$$

with the differential defined as:

$$\begin{tikzcd}
 & \mathrm{H}_k(X_k) \arrow[dr] & & & \\
\mathrm{H}_{k+1}(X_{k+1}, X_k) \arrow[ur] \arrow[rr, "\partial_{k+1}^{\text{cell}}"'] & & \mathrm{H}_k(X_k, X_{k-1}) \arrow[dr] \arrow[rr, "\partial_k^{\text{cell}}"'] & & \mathrm{H}_{k-1}(X_{k-1}, X_{k-2}) \\
 & & & \mathrm{H}_{k-1}(X_{k-1}) \arrow[ur] &
\end{tikzcd}$$

where the diagonal maps are defined using the long exact sequence of a pair. We will write $\mathrm{H}_*^{\text{cell}}(X)$ for the homology of the chain complex $C_*^{\text{cell}}(X)$.

Claim 3.2.6. $\partial_k^{\text{cell}} \circ \partial_{k+1}^{\text{cell}} = 0$.

Proof. The composition $\partial_k^{\text{cell}} \circ \partial_{k+1}^{\text{cell}}$ includes two consecutive maps (the orange ones in the above diagram) in the long exact sequence of the pair (X_k, X_{k-1}), which compose to zero. As such, $\partial_k^{\text{cell}} \circ \partial_{k+1}^{\text{cell}} = 0$. $\square$

Note $C_*^{\text{cell}}(X)$ depends on the choice of cell structure on X.

Proposition 3.2.7. $\mathrm{H}_*(X) \cong \mathrm{H}_*^{\text{cell}}(X)$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & & & 0 & \\
0 = \mathrm{H}_k(X_{k-1}) \arrow[dr] & & \mathrm{H}_k(X_{k+1}) = \mathrm{H}_k(X) \arrow[ur] & & \\
 & \mathrm{H}_k(X_k) \arrow[ur] \arrow[dr, "i_k"] & & & \\
\mathrm{H}_{k+1}(X_{k+1}, X_k) \arrow[ur, "\partial_{k+1}"] \arrow[rr, "\partial^{\mathrm{cell}}_{k+1}"'] & & \mathrm{H}_k(X_k, X_{k-1}) \arrow[rr, "\partial^{\mathrm{cell}}_k"'] \arrow[dr, "\partial_k"'] & & \mathrm{H}_{k-1}(X_{k-1}, X_{k-2}) \\
 & & & \mathrm{H}_{k-1}(X_{k-1}) \arrow[ur, "i_{k-1}"] & \\
 & & 0 = \mathrm{H}_{k-1}(X_{k-2}) \arrow[ur] & &
\end{tikzcd}$$

From this, we have that

$$\mathrm{H}_k(X) = \frac{\mathrm{H}_k(X_k)}{\mathrm{im}(\partial_{k+1})} = \frac{i_k(\mathrm{H}_k(X_k))}{\mathrm{im}(i_k \circ \partial_{k+1})} = \frac{\ker(\partial_k)}{\mathrm{im}(\partial^{\mathrm{cell}}_{k+1})} = \frac{\ker(i_{k-1} \circ \partial_k)}{\mathrm{im}(\partial^{\mathrm{cell}}_{k+1})} = \frac{\ker(\partial^{\mathrm{cell}}_k)}{\mathrm{im}(\partial^{\mathrm{cell}}_{k+1})} = \mathrm{H}^{\mathrm{cell}}_k(X)$$

□

Corollary 3.2.8. Let X be a finite cell complex. Then

(i) $\mathrm{H}_k(X)$ is a finitely generated abelian group of rank at most $n_k = |I_k|$,

(ii) if $\mathrm{H}_k(X) \neq 0$, every cell structure must contain k-cells.

(iii) if X admist a cell structure with cells in only even dimension, then $\mathrm{H}_*(X) \cong C^{\mathrm{cell}}_*(X)$.

(iv) if F is a field, then $\mathrm{H}_*(X; F)$ is a finite dimensional F-vector space.

For (iii), we saw that $\mathbb{CP}^n$ satisfies the requirements on examples sheet 1, and the same is true for the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^n)$ of k-dimensional linear subspaces of $\mathbb{C}^n$ (where $\mathrm{Gr}(1, \mathbb{C}^n) = \mathbb{CP}^{n-1}$), and various other spaces in complex algebraic geometry.

The same sort of argument as for $\mathbb{CP}^n$ shows that $\mathbb{RP}^n$ has a single cell in each degree $0 \leq i \leq n$, that is, the cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

But we need new tools to compute the cellular boundary.

Lecture 11

That is, if e_α is a k-cell, then

$$\partial^{\mathrm{cell}}_k(e_\alpha) = \sum_\beta d_{\alpha\beta} e_\beta$$

of $k-1$-cells. We would like to find the $d_{\alpha\beta} \in \mathbb{Z}$.

Lemma 3.2.9. $d_{\alpha\beta}$ is the degree of the following map on spheres:

$$S^{k-1}_\alpha \xrightarrow{\varphi_\alpha} X_{k-1} \twoheadrightarrow X_{k-1}/X_{k-2} \xrightarrow{\sim} \bigvee_\beta S^{k-1}_\beta \xrightarrow{\mathrm{proj}} \mathrel{\mkern-14mu}\twoheadrightarrow S^{k-1}_\beta$$

Remark 3.2.10. For this to be well defined (and not just defined up to a sign), we need to fix isomorphisms $H_{k-1}(S^{k-1}) \cong \mathbb{Z}$.

Proof. Consider

$$\begin{tikzcd}
H_k(D^k_\alpha, \partial D^k_\alpha) \arrow[r, "\cong", "\partial_{LES}"'] \arrow[d, "\varphi_\alpha"] & H_{k-1}(\partial D^k_\alpha) \arrow[r] \arrow[d, "\varphi_\alpha|_{\partial D^k_\alpha}"'] & H_{k-1}(S^{k-1}_\beta) \\
H_k(X_k, X_{k-1}) \arrow[r, "\partial_{LES}"] \arrow[dr, "\partial_k^{\mathrm{cell}}"'] & H_{k-1}(X_{k-1}) \arrow[d] & \\
& H_{k-1}(X_{k-1}, X_{k-2}) \arrow[r, "\cong"] & \tilde{H}_{k-1}(X_{k-1}/X_{k-2}) \arrow[uu, "\text{collapse}"']
\end{tikzcd}$$

Chasing a generator around this diagram:

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[d, mapsto] & 1 \arrow[r, mapsto] & \deg(f_{\alpha\beta}) \\
e_\alpha \arrow[dr, mapsto] & & \\
& \sum d_{\alpha\beta} e_\beta \arrow[r, mapsto] & d_{\alpha\beta} e_\beta \arrow[uu, mapsto]
\end{tikzcd}$$

The result then follows. $\square$

This is useful if we can compute the degree of maps between spheres. Suppose $f : S^n \to S^n$ and $y \in S^n$ has finitely many preimages

$$f^{-1}(y) = \{x_1, \ldots, x_m\}$$

In this case, we can choose pairwise disjoint discs U_i containing x_i, and a disc V containing y such that $f(U_i) \subseteq V$. Then f defines a map

$$(U_i, x_i) \to (V, y)$$

This then defines a map

$$\mathbb{Z} \cong H_n(U_i, U_i \setminus x) \to H_n(V, V \setminus y) = \mathbb{Z}$$

Moreover, recall that the isomorphisms above are from excision, and so the maps above are between the *same* copy of $\mathbb{Z}$. With this, we have a *local degree of f at x*. We will write

$$\deg_{x_i}(f) \text{ or } \deg_f(x_i) \in \mathbb{Z}$$

for this. Another way (once we have a bit more machinery) is that S^n is orientable, and so U_i, V inherit an orientation. Once we fix this orientation, the local degree is well defined.

Lemma 3.2.11. Under the assumption that such a y exists,

$$\deg(f) = \sum_{i=1}^{m} \deg_{x_i}(f)$$

Proof. We have the diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[rr, "\deg(f)"] \arrow[d] & & H_n(S^n) \arrow[d] \\
H_n(S^n, S^n \setminus \{x_1, \ldots, x_n\}) \arrow[rr, "f_*"'] & & H_n(S^n, S^n \setminus \{y\}) \\
H_n(\bigsqcup_i U_i, \bigsqcup_i (U_i \setminus \{x_i\})) \arrow[u, "\cong"', "\text{excision}"] \arrow[rr] \arrow[d, "\cong"'] & & H_n(V, V \setminus \{y\}) \arrow[u, "\text{excision}", "\cong"'] \\
\bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i) \arrow[urr, "\bigoplus \deg_{x_i}(f)"'] & &
\end{tikzcd}$$

and the result follows from the fact that the diagram commutes. $\square$

Example 3.2.12

By the same argument as for $\mathbb{CP}^n$, $\mathbb{RP}^n$ has a cell structure of the form

$$e^n \cup \mathbb{RP}^{n-1}$$

That is, there is an i-cell for $0 \leq i \leq n$. The attaching map $\partial e^n = S^{n-1} \to \mathbb{RP}^{n-1}$ is the canonical $2:1$ map. With this, the cellular complex is:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \longrightarrow 0$$

Consider

$$\partial_k^{\text{cell}} : C_k^{\text{cell}} \to C_{k-1}^{\text{cell}}$$

This is induced by

$$\partial D^k = S^{k-1} \longrightarrow \mathbb{RP}^{k-1} \longrightarrow \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}$$

Let η be the composition. At a general point p of the image S^{k-1}, p has two preimages under η. Moreover, near each of the preimages, η is a homeomorphism. Fix $V \subseteq \mathbb{RP}^{k-1}$ an open disc, under the map $S^{K-1} \to \mathbb{RP}^{k-1}$, V has two (disjoint) preimages, U_1, U_2, with $\eta|_{U_2} = \eta|_{U_1} \circ (\text{antipodal})$, and so

$$\deg_{x_1}(f) = (-1)^k \deg_{x_2}(f)$$

With this, ∂_k^{cell} is multiplication by $1 + (-1)^k$ (possibly up to a sign). The complex then becomes

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \xrightarrow[2]{} \mathbb{Z} \xrightarrow[0]{} \mathbb{Z} \longrightarrow 0$$

and so the homology is

$$H_*(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2 & 0 < * < n, * \text{ odd} \\ \mathbb{Z} & * = n, n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Example 3.2.13 (Exercise)

If $p(z)$ is a complex polynomial, then p extends to a continuous map $\hat{p} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of degree $\deg(p)$, and if $x \in \mathbb{C}$ is a root of p, the local degree $\deg_x(\hat{p})$ is the multiplicity of the root.

c.f. the fundamental theorem of algebra. This is essentially local degree in complex analysis or Riemann surfaces.

Remark 3.2.14. If $f : S^n \to S^n$ is a *smooth map*, then $f^{-1}(y)$ is finite if y is a *regular value*, and by Sard's theorem the set of critical values has measure zero, and as such, the set of regular values is dense.

Moreover, every continuous map $S^n \to S^n$ is homotopic to a smooth map.

3.3 Digression on cohomology

Set $C^*_{\text{cell}}(X) = \mathrm{Hom}(C^{\text{cell}}_*(X), \mathbb{Z})$, and ∂^*_{cell} for the adjoint of ∂^{cell}_*. Consider

$$\begin{tikzcd} \mathrm{H}^i(X_i, X_{i-1}) \arrow[r] \arrow[rr, bend right, "\text{candidate for } \partial^*_{\text{cell}}"'] & \mathrm{H}^i(X_i) \arrow[r] & \mathrm{H}^{i+1}(X_{i+1}, X_i) \end{tikzcd}$$

But on examples sheet 2, we show that the diagram below

$$\begin{tikzcd} \mathrm{H}^i(X_i, X_{i-1}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}^i(X_i) \arrow[r] \arrow[d] & \mathrm{H}^{i+1}(X_{i+1}, X_i) \arrow[d, "\cong"] \\ \mathrm{Hom}(\mathrm{H}_i(X_i, X_{i-1}), \mathbb{Z}) \arrow[r] \arrow[rr, bend right, "\text{adjoint to } \partial^{\text{cell}}_k"'] & \mathrm{Hom}(\mathrm{H}_i(X_i), \mathbb{Z}) \arrow[r] & \mathrm{Hom}(\mathrm{H}_{i+1}(X_{i+1}, X_i), \mathbb{Z}) \end{tikzcd}$$

commutes. Thus, we could have defined

$$C^i_{\text{cell}}(X) = \mathrm{H}^i(X_i, X_{i-1})$$

and the boundary operators will have given the same cochain complex. Note however in the diagram above, the middle map need not be an isomorphism.

Proposition 3.3.1. Let X be a finite cell complex. Then we have a (non-canonical) isomorphism

$$\mathrm{H}^i(X) \cong \frac{\mathrm{H}_i(X)}{\mathrm{Tor}(\mathrm{H}_i(X))} \oplus \mathrm{Tor}(\mathrm{H}_{i-1}(X))$$

where for an abelian group A, $\mathrm{Tor}(A)$ is the subgroup of torsion elements.

Lecture 12

Proof. This is pure algebra. Let C_* be a chain complex, such that the chain groups C_i are finitely generated and free. Let C^* be the corresponding cochain complex, and then corresponding relation holds. That is,

$$\mathrm{H}^j(X) \cong \frac{\mathrm{H}_j(X)}{\mathrm{Tor}(\mathrm{H}_j(X))} \oplus \mathrm{Tor}(\mathrm{H}_{j-1}(X))$$

and so, all we are using is that for a finite cell complex, C^{cell}_* is finitely generated and free,

Break the chain complex into a sequence of short exact seqeuences

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow \mathrm{H}_n(C_*) \longrightarrow 0$$

where $Z_n = \ker(\partial : C_n \to C_{n-1})$, $B_{n-1} = \operatorname{im}(\partial : C_n \to C_{n-1})$. In the top sequence, all of the groups are free. So we can split the top sequence. That is, there exists $\alpha_n : B_{n-1} \to C_n$ such that $\partial_n \alpha_n = \mathrm{id}$. Then we have a (non-canonical) isomorphism

$$C_n \overset{\sim}{=} Z_n \oplus B_{n-1}$$

With this, the chain complex looks like

$$\cdots \longrightarrow Z_{n+1} \oplus B_n \longrightarrow Z_n \oplus B_{n-1} \longrightarrow \mathbb{Z}_{n-1} \oplus B_{n-2} \longrightarrow \cdots$$

In terms of this decomposition, $\partial(B_n) \subseteq Z_n$. That is, C_* breaks down as a direct sum of length two complexes.

We can simplify this further. By choosing helpful bases for B_n and Z_{n-1} using the Smith normal form, which says that using a $\mathbb{Z}$-change of basis, we can write

$$\partial_n = \begin{pmatrix} d_1 & & & & & \\ & \ddots & & & & \\ & & d_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

where $d_1 \mid d_2, d_2 \mid d_3$ and so on. Then C_* is broken into a direct sum of two kinds of complex:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d} \mathbb{Z} \longrightarrow 0$$

for $d \neq 0$, and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

For these two very simple complexes, the claim in the proposition is clear. □

With this, we can see that the natural map $\mathrm{H}^*(X) \to \mathrm{H}_*(X)$ is an isomorphism provided X is a finite cell complex, with $\mathrm{H}_i(X)$ free abelian for all i.

Remark 3.3.2. For abelian groups H, G, let

$$\mathrm{Ext}^1(H, G) = \frac{\{0 \to G \to J \to H\}}{\sim}$$

be extensions of H by G, modulo an equivalence relation where there exists $\phi : J_1 \to J_2$ making

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d, dashed, "="] & J_1 \arrow[r] \arrow[d, dashed, "\phi"] & H \arrow[r] \arrow[d, dashed, "="] & 0 \\ 0 \arrow[r] & G \arrow[r] & J_2 \arrow[r] & H \arrow[r] & 0 \end{tikzcd}$$

commute. Note by the five lemma, ϕ is an isomorphism.

Theorem 3.3.3 (universal coefficient)**.** There are exact sequences (for *any* topological space X),

$$0 \longrightarrow \mathrm{Ext}^1(\mathrm{H}_{n-1}(X), G) \longrightarrow \mathrm{H}^n(X; G) \longrightarrow \mathrm{Hom}(\mathrm{H}_n(X); G) \longrightarrow 0$$

which are (non-canonically) split.

Proof. Omitted. □

Recall if X is a finite cell complex, then $\mathrm{H}_i(X)$ is a finitely generated abelian group, and $\mathrm{H}_i(X; F)$ is a finite dimensional F-vector space.

Definition 3.3.4 (Euler characteristic)
If X is a finite cell complex, we define its *Euler characteristic*

$$\chi(X) = \sum_{i\geq 0} (-1)^i \operatorname{rank}_{\mathbb{Z}}(\mathrm{H}_i(X))$$

and more generally, if F is a field,

$$\chi(X, F) = \sum_i (-1)^i \dim(\mathrm{H}_i(X; F))$$

Lemma 3.3.5. If X is a finite cell complex, then

$$\chi(X) = \sum_i (-1)^i N_i$$

where X has N_i i-cells in its cell structure.

Proof. Recall our short exact sequences

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow \mathrm{H}_n(X) \longrightarrow 0$$

where $C_* = C_*^{\mathrm{cell}}(X)$. Say $N_n = \operatorname{rank}(C_n)$, $z_n = \operatorname{rank}(Z_n)$ and $b_n = \operatorname{rank}(B_n)$. Then

$$N_n = z_n + b_{n-1} \qquad \text{and} \qquad \operatorname{rank}(\mathrm{H}_n(X)) = z_n - b_n$$

Substituting,

$$\begin{aligned}
\sum_k (-1)^k \operatorname{rank}_{\mathbb{Z}}(\mathrm{H}_k(X)) &= \sum_{k\geq 0} (-1)^k (z_k - b_k) \\
&= \sum_{k\ ge0} (z_k - (N_{k+1} - z_{k+1})) \\
&= \sum_{k\geq 0} (-1)^{k+1} N_{k+1} + z_0 \\
&= \sum_{k\geq 0} (-1)^k N_k
\end{aligned}$$

since $z_0 = N_0$. ☐

The same computation shows that $\chi(X; F) = \sum (-1)^k N_k$, and so the Euler characteristic is independent of our choice of field.

Example 3.3.6
$\chi(S^4) = 2, \chi(\mathbb{CP}^3) = 3$ and so S^4 is not homotopy equivalent to $\mathbb{CP}^3$.

Example 3.3.7
If X is a union of two subcomplexes A, B, then

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

Example 3.3.8
If X, Y are finite cell complexes, then $X \times Y$ admits a cell structure, such that the open cells in $X \times Y$ are products of open cells in X and open cells in Y. Then

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

3.4 Generalised homology theories

Definition 3.4.1 (generalised homology theory)
A *generalised homology theory* is an assignment

$$(X, A) \mapsto h_*(X, A) = \bigoplus_{i \in \mathbb{Z}} h_i(X, A)$$

of a graded abelian group to a pair (X, A), where X is a topological space and $A \subseteq X$ a subspace. This needs to satisfy:

1. (functoriality) A map $f : (X, A) \to (Y, B)$ induces $f_* : h_*(X, A) \to h_*(Y, B)$, with $\mathrm{id}_* = \mathrm{id}$ and $(f \circ g)_* = f_* \circ g_*$,
2. (homotopy invariance) if $f \simeq g$ as maps of pairs, $f_* = g_*$,
3. (long exact sequence) writing $h_i(X) = h_i(X, \varnothing)$, we have a long exact sequence

   $$\cdots \longrightarrow h_i(A) \longrightarrow h_i(X) \longrightarrow h_i(X, A) \longrightarrow h_{i-1}(A) \longrightarrow \cdots$$

   which is natural. The maps $h_i(A) \to h_i(X)$ and $h_i(X) \to h_i(X, A)$ are induced by inclusion.
4. (excision) If $\mathrm{Cl}(Z) \subseteq \mathrm{Int}(A)$, then

   $$h_i(X \setminus Z, A \setminus Z) \cong h_*(X, A)$$

   via the inclusion map.
5. (unions)

   $$\bigoplus_\alpha h_*(X_\alpha) \cong h_*\left(\bigsqcup_\alpha X_\alpha\right)$$

   via the sum of the inclusion maps.

These are called the *Eilenberg-Steenrod axioms*, and $h_*(\{\mathrm{pt}\})$ are called the *coefficients* of the theory.

Example 3.4.2
We can build examples of the form

$$(X, A) \mapsto \mathrm{H}_*(X, A) \otimes_{\mathbb{Z}} R$$

where R is an abelian group. A 'meta-theorem' says that interesting generalised homology theories are not from chain complexes.

Proposition 3.4.3. If h_*, k_* are generalised homology theories on the set of pairs (X, A) of a cell complex and a subcomplex, and if $\Phi : h_* \to k_*$ is a natural transformation, then Φ being an isomorphism on a point implies it is an isomorphism on all such pairs.

Lecture 13

Outline proof where X is finite dimensional. We induct on $\dim(X)$. If $X = X_0$, then X is a discrete set and the unions axiom implies the result.

Suppose inductively $\Phi : h_*(X, A) \to k_*(X, A)$ is an isomorphism whenever $\dim(X) \leq n - 1$. Let X be an n-dimensional cell complex. In this case, we have long exact sequences

$$\begin{tikzcd}
h_{i+1}(X, X_{n-1}) \arrow[r] \arrow[d, "\Phi"] & h_i(X_{n-1}) \arrow[r] \arrow[d, "\Phi"', "\cong"] & h_i(X) \arrow[r] \arrow[d, "\Phi"] & h_i(X, X_{n-1}) \arrow[r] \arrow[d, "\Phi"] & h_{i-1}(X_{n-1}) \arrow[d, "\Phi"', "\cong"] \\
k_{i+1}(X, X_{n-1}) \arrow[r] & k_i(X_{n-1}) \arrow[r] & k_i(X) \arrow[r] & k_i(X, X_{n-1}) \arrow[r] & k_{i-1}(X_{n-1})
\end{tikzcd}$$

By the five lemma, if Φ is an isomorphism on $h_i(X, X_{n-1})$ for all i, then it is an isomorphism on $h_i(X)$. But by excision,

$$h_*(X, X_{n-1}) = h_*(X_n, X_{n-1}) \cong h_* \left(\bigsqcup_\alpha D_\alpha^n, \bigsqcup_\alpha \partial D_\alpha^n \right)$$

Here, we are using the fact that X_{n-1} has a neighbourhood $N_\varepsilon(X_{n-1}) \subseteq X_n$ for all ε sufficiently small, whcih are constructed cell by cell, and retract onto the boundary on each cell. By the unions axiom,

$$h_* \left(\bigsqcup_\alpha D_\alpha^n, \bigsqcup_\alpha \partial D_\alpha^n \right) = \bigoplus_\alpha h_*(D_\alpha^n, \partial D_\alpha^n)$$

Now consider the LES of the pair

$$\begin{tikzcd}
h_i(\partial D^n) \arrow[r] \arrow[d, "\Phi"] & h_i(D^n) \arrow[r] \arrow[d, "\Phi"] & h_i(D^n, \partial D^n) \arrow[r] \arrow[d, "\Phi"] & h_{i-1}(\partial D^n) \arrow[r] \arrow[d, "\Phi"] & h_{i-1}(D^n) \arrow[d, "\Phi"] \\
k_i(\partial D^n) \arrow[r] & k_i(D^n) \arrow[r] & k_i(D^n, \partial D^n) \arrow[r] & k_{i-1}(\partial D^n) \arrow[r] & k_{i-1}(D^n)
\end{tikzcd}$$

Using the fact that ∂D^n is $(n-1)$-dimensional, and D^n is contractible (and using homotopy invariance), by the five lemma the middle map is an isomorphism. Hence by induction, we have that

$$\Phi_{(X, \varnothing)} : h_*(X) \to k_*(X)$$

is an isomorphism when $\dim(X) = n$. Now the same argument with the LES of the pair and the five lemma shows that $\Phi_{(X,A)}$ is an isomorphism if X is a cell complex and A is a subcomplex. $\square$

> **Remark 3.4.4.** The result is true for infinite dimensional cell pairs, but it uses the "telescope construction", see Hatcher.

> **Remark 3.4.5.** There is a corresponding notion of a generalised cohomology theory. We replace the covariant functor h_* with a contravariant functor h^*, reverse the long exact sequence, and replace the direct sum with the direct product in the unions axiom.
>
> The analogue of the previous proposition holds with the same proof.

4 Cohomology

4.1 Cup product

> **Definition 4.1.1** (cup product)
> Let X be any topological space, $\phi \in C^k(X), \psi \in C^\ell(X)$, then their *cup product* $\phi \smile \psi \in C^{k+\ell}(X)$ is
>
> $$(\phi \smile \psi)(\sigma : [v_0, \dots, v_{k+\ell}] \to X) = \phi \left(\sigma|_{[v_0,\dots,v_k]} \right) \psi \left(\sigma|_{[v_k,\dots,v_{k+\ell}]} \right)$$

Notation 4.1.2. We will write $\phi \cdot \psi = \phi \smile \psi$.

For the cup product, it's useful to have the de Rham theory in mind. In this case, $\alpha \in \Omega^k(M), \beta \in \Omega^\ell(M)$, and the cup product is the wedge product $\alpha \wedge \beta \in \Omega^{k+\ell}(M)$. The lemma below is then the same formula as the exterior derivative of a wedge product.

Lemma 4.1.3. If $\partial^* : C^*(X) \to C^{*+!}(X)$ is the boundary operator, then

$$\partial^*(\phi \cdot \psi) = (\partial^* \phi) \cdot \psi + (-1)^k \phi \cdot \partial^* \psi$$

Proof. Let $[v_0, \ldots, v_{k+\ell+1}]$ be a $(k+\ell+1)$-simples of X. Then

$$(\partial^* \phi)\psi([v_0, \ldots, v_{k+\ell+1}]) = \sum_{i=0}^{k+1} (-1)^i \phi([v_0, \ldots, \widehat{v_i}, \ldots, v_{k+1}])\psi([v_{k+1}, \ldots, v_{k+\ell+1}])$$

On the other hand,

$$(-1)^k \phi(\partial^*)([v_0, \ldots, v_{k+\ell+1}]) = \phi([v_0, \ldots, v_k]) \sum_{i=k}^{k+\ell+1} (-1)^i \psi([v_k, \ldots, \widehat{v_i}, \ldots, v_{k+\ell+1}])$$

where the indexing on the right hand side absorbed the sign $(-1)^k$.

In the two expressions, there is only one term which appears twice, which are

$$(-1)^{k+1}\phi([v_0, \ldots, v_k])\psi([v_{k+1}, \ldots, v_{k+\ell+1}]) + (-1)^k \phi([v_0, \ldots, v_k])\psi([v_{k+1}, \ldots, v_{k+\ell+1}]) = 0$$

and cancel. The remaining terms give

$$(\phi\psi)\left(\sum_{i=0}^{k+\ell+1} (-1)^i [v_0, \ldots, \widehat{v_i}, \ldots, v_{k+\ell+1}]\right) = (\phi\psi)(\partial\sigma) = \partial^*(\phi\psi)(\sigma)$$

□

Corollary 4.1.4. Cup product descends to cohomology, and so it induces a map

$$\mathrm{H}^k(X) \times \mathrm{H}^\ell(X) \to \mathrm{H}^{k+\ell}(X)$$

which makes $\mathrm{H}^*(X)$ into a graded unital ring.

Proof. Let $\phi \in C^k(X)$, $\psi \in C^\ell(X)$ be cocycles, The lemma gives that $\partial^*(\phi\psi) = 0$, and so $[\phi\psi]$ represents a cohomology class in $\mathrm{H}^{k+\ell}(X)$. Next, we need to show that this is independent of choices.

A general representative of $[\phi]$ is $\phi + \partial^*\alpha$, then $\phi\psi$ becomes

$$(\phi + \partial^*\alpha)\psi = \phi\psi + (\partial^*\alpha)\psi = \phi\psi + \partial^*(\alpha\psi)$$

With this, the element

$$[\phi][\psi] = [\phi\psi]$$

is well defined.

Recall we have an element $1 \in C^0(X)$, given by $1(p) = 1$ for all $p \in X$. With this,

$$\partial^* 1(\sigma) = 1(v_0) - 1(v_1) = 1 - 1 = 0$$

and so $\partial^* 1 = 0$. With this, we have an associated element $1 = [1] \in \mathrm{H}^0(X)$. It is easy to see that $1\psi = \psi$ and $\psi 1 = \psi$, and so this is a unit. □

Remark 4.1.5 (on coefficients)**.** Recall for any abelian group G, we have $C_j(X; G)$ for chains with coefficients in G, and a corresponding cochain group

$$C^j(X; G) = \mathrm{Hom}_{\mathbb{Z}}(C_j(X; \mathbb{Z}), G)$$

If G is a commutative (not necessarily unital) ring, then we can define cup product on $C^*(X; G)$, which induces a cup product on $\mathrm{H}^*(X; G)$, making it into a graded (not necessarily unital) ring. If G is a unital ring, then $\mathrm{H}^*(X; G)$ is a unital ring.

Proposition 4.1.6 (Properties of the cup product)**.**

1. (associativity) $(\phi \cdot \psi) \cdot \tau = \phi \cdot (\psi \cdot \tau)$,
2. if $f : X \to Y$ is a continuous map, then the induced map $f^* : \mathrm{H}^*(Y) \to \mathrm{H}^*(X)$ is a ring homomorphism. In fact, this is already true at the cochain level.
3. The *cross product* is

$$\begin{aligned} \mathrm{H}^i(Y) \times \mathrm{H}^j(Z) &\to \mathrm{H}^{i+j}(Y \times Z) \\ (\phi, \psi) &\mapsto \mathrm{pr}_Y^* \phi \cdot \mathrm{pr}_Z^* \psi \end{aligned}$$

Lecture 14

Example 4.1.7
If $X = \{*\}$, then

$$\mathrm{H}^*(X) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

In this case, $\mathrm{H}^*(X) = \mathbb{Z}$ has its usual ring structure.

Example 4.1.8
Now consider $X = S^n$, where $n > 1$. Then

$$\mathrm{H}^*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{otherwise} \end{cases}$$

If x is a generator of $\mathrm{H}^n(S^n)$, then $x \cdot x \in \mathrm{H}^{2n} S^n = 0$, and so

$$\mathrm{H}^*(S^n) = \frac{\mathbb{Z}[x]}{\langle x^2 \rangle}$$

where $|x| = n$.

Notation 4.1.9. We write $|x| = k$ for the *degree* of $x \in \mathrm{H}^k(X)$.

Example 4.1.10
If X, Y are cell complexes, $x_0 \in X$, $y_0 \in Y$, then we have maps

$$\begin{tikzcd} & X \vee Y \arrow[dl, shift left, "p_X"] \arrow[dr, shift left, "p_Y"] & \\ X \arrow[ur, shift left, "i_X"] & & Y \arrow[ul, shift left, "i_Y"] \end{tikzcd}$$

where p_X is given by collapsing Y and i_X is the inclusion map. This gives us ring homomorphisms

$$p_X^* : \mathrm{H}^*(X) \to \mathrm{H}^*(X \vee Y)$$

Since x_0, y_0 are neighbourhood deformation retracts, we have an open cover of $X \vee Y$ and using Mayer-Vietoris, we have that

$$\widetilde{\mathrm{H}}^*(X) \oplus \widetilde{\mathrm{H}}^*(Y) \to \widetilde{\mathrm{H}}^*(X \vee Y)$$

is an isomorphism of abelian groups. Hence we know $\mathrm{H}^*(X \vee Y)$ in terms of $\mathrm{H}^*(X)$ and $\mathrm{H}^*(Y)$. With this, if $\alpha \in \mathrm{H}^i(X), \beta \in \mathrm{H}^j(Y)$, with $i, j > 0$, then $\alpha \cdot \beta = 0$.

More simply,

$$\mathrm{H}^*(X \sqcup Y) = \mathrm{H}^*(X) \oplus \mathrm{H}^*(Y)$$

as rings.

Proposition 4.1.11. $\mathrm{H}^*(X)$ is a *graded commutative ring*, that is, if $\phi \in \mathrm{H}^k(X), \psi \in \mathrm{H}^\ell(X)$, then

$$\phi \cdot \psi = (-1)^{k\ell} \psi \cdot \phi$$

Remark 4.1.12. If R is a commutative ring, then $\mathrm{H}^*(X; R)$ is a graded-commutative ring. Moreover, unlike associativity, this is *not* true at the cochain level.

Example 4.1.13

Suppose X has

$$\mathrm{H}^*(X) = \begin{cases} \mathbb{Z} & * = 0, 3, 6 \\ 0 & \text{otherwise} \end{cases}$$

For degree reasons, the only possible non-trivial cup product is

$$\mathrm{H}^3(X) \times \mathrm{H}^3(X) \to \mathrm{H}^6(X)$$

But if $\theta \in \mathrm{H}^3(X)$ is a generator, then by graded commutativity,

$$\theta \cdot \theta = -\theta \cdot \theta$$

and so $\theta \cdot \theta = 0$ since $\mathrm{H}^6(X)$ is torsion free.

Theorem 4.1.14 (Künneth)**.** Let Y be a space be such that $\mathrm{H}^i(Y)$ is free and finitely generated for all i. Then the cross product

$$\mathrm{H}^k(X) \times \mathrm{H}^\ell(Y) \to \mathrm{H}^{k+\ell}(X \times Y)$$

induces an isomorphism of *graded rings*

$$\bigoplus_{k+\ell=n} \mathrm{H}^k(X) \otimes \mathrm{H}^\ell(Y) \to \mathrm{H}^{k+\ell}(X \times Y)$$

whenever X is a finite cell complex.

Remark 4.1.15. In general, cross product induces a homomorphism

$$\mathrm{H}^*(X; R) \otimes_R \mathrm{H}^*(Y; R) \to \mathrm{H}^*(X \times Y; R)$$

of abelian groups, and if we declare the left hand side is a (graded) ring via

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac) \otimes (bd)$$

and so the map from the cross product is a ring homomorphism. In the case of the Künneth formula, we then obtain an isomorphism of rings.

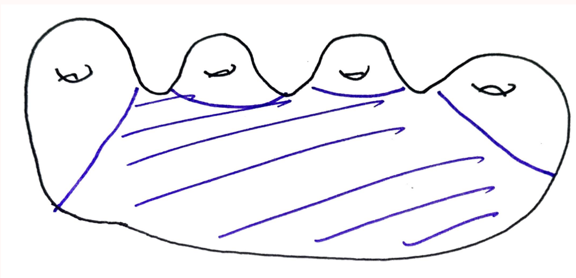
Example 4.1.16
Recall
$$\mathrm{H}^*(S^1) = \frac{\mathbb{Z}[x]}{x^2}$$
with $|x| = 1$. Equivalently, this is $\Lambda(x)$, exterior algebra on one generator (i.e $\Lambda^*\mathbb{Z}$).

Recall additively,
$$\mathrm{H}^*(T^2) = \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^2 & * = 1 \\ 0 & \text{otherwise} \end{cases}$$
In terms of the Künneth formula,
$$\begin{aligned} \mathrm{H}^0(T^2) &= \mathrm{H}^0(S^1) \otimes \mathrm{H}^0(S^1) \\ \mathrm{H}^1(T^2) &= \mathrm{H}^0(S^1) \otimes \mathrm{H}^1(S^1) \oplus \mathrm{H}^1(S^1) \otimes \mathrm{H}^0(S^1) \\ \mathrm{H}^2(T^2) &= \mathrm{H}^1(S^1) \otimes \mathrm{H}^1(S^1) \end{aligned}$$
if $x_i \in \mathrm{H}^1(S^1)$ is a generator for the i-th factor, then the map $\mathrm{H}^1(T^2) \times \mathrm{H}^1(T^2) \to \mathrm{H}^2(T^2)$ will be
$$(1 \otimes x_2)(x_1 \otimes 1) = -x_2 x_1$$
This is isomorphic to the exterior algebra $\Lambda(x_1, x_2)$. Iteratively,
$$\mathrm{H}^*(T^n) = \Lambda(x_1, \ldots, x_n)$$
where $\mathrm{H}^1(\mathrm{T}^n) \cong \mathbb{Z}^n$ has generators $x_1, \ldots, x_n$.

Example 4.1.17
By Mayer-Vietoris, we know that
$$\mathrm{H}^*(\Sigma_g) = \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^{2g} & * = 1 \end{cases}$$
Let $1 \in \mathrm{H}^0(\Sigma_g)$ and $u \in \mathrm{H}^2(\Sigma_g)$ be generators.

Claim 4.1.18.
$$\mathrm{H}^*(\Sigma_g) = \frac{\mathbb{Z}\left\langle x_1, y_1, \ldots, x_g, y_g \right\rangle}{\left\langle x_i x_j = y_i y_j = 0, x_i y_j = \delta_{ij} u \right\rangle}$$
(and with the relations from skew-commutativity).

Proof. By collapsing the purple subspace



we have a projection map $\pi : \Sigma_g \to \bigvee_{i=1}^{g} T_i^2$. We also have a natural inclusion map $p : \bigsqcup_i T_i^2 \to \bigvee_{i=1}^{g} T_i^2$.

The maps π^* and p^* are ring homomorphisms on cohomology, and they define isomorphism on H^1. In particular,

$$\mathrm{H}^i(\bigsqcup_i T^2) \cong \bigoplus_{i=1}^n \Lambda(x_i, y_i)$$

where $|x_i| = |y_i| = 1$, and generate $\mathrm{H}^1(T_i^2)$. Since p^*, π^* are ring homomorphisms on H^1, we can use these to define classes in $\mathrm{H}^1(\Sigma_g)$.

On H^2, we have

$$\begin{tikzcd} \mathbb{Z} = \mathrm{H}^i(\Sigma_g) & \mathrm{H}^2(\bigvee_{i=1}^g T_i^2) = \mathbb{Z}^g \arrow[l, "\pi^*"'] \arrow[d, "p^*"] \\ & \mathrm{H}^2(\bigsqcup_i T_i^2) = \mathbb{Z}^g \end{tikzcd}$$

and if $u_i = x_i y_i \in \mathrm{H}^2(T_i^2)$ are generators, we need to show that the map $\mathbb{Z}^g \to \mathbb{Z}$ sends u_i to u. To see this, we need to consider the map

$$\mathrm{H}^2(T^2) \to \mathrm{H}^2(\Sigma_g)$$

when $\Sigma_g \to T^2$ is the map to one factor. Recall from out computation of degree of maps between spheres that if we have a point with finite preimage, we can express

$$\deg(f) = \sum_i \deg_{x_i}(f)$$

as a sum of the local degree. The same argument works in this case, and so the degree is well defined.

Thus, up to changing the sign of y_j, we get the result we want, since the degree of the map $\Sigma_g \to T^2$ is 1. □

Corollary 4.1.19. Let $f : S^n \to T^n$ be any map, and $n > 1$. Then f has degree zero, where $\deg(f) : \mathrm{H}^n(T^n) \to \mathrm{H}^n(S^n)$.

Proof. f induces a ring homomorphism from

$$\mathrm{H}^*(T^n) = \Lambda(x_1, \ldots, x_n)$$

to

$$\mathrm{H}^*(S^n) = \frac{\mathbb{Z}[u]}{u^2}$$

Since $n > 1$, $\mathrm{H}^1(S^n) = 0$, and so $f^*(x_i) = 0$. Hence

$$f^*(x_1 \cdots x_n) = f^*(x_1) \cdots f^*(x_n) = 0$$

□

Lecture 15

Proof of theorem 4.1.14. Recall that

$$C^*(X, A) = \{f \in C^*(X) \mid C_*(A) \subseteq \ker(f)\}$$

If $\varphi \in C^k(X, A), \psi \in C^\ell(X)$, then for $\sigma : \Delta k + \ell \to A$ a simplex in A,

$$\varphi \cdot \psi(\sigma) = \varphi(\text{front of } \sigma)\psi(\text{back of } sigma)$$

and so $\varphi \cdot \psi \in C^{k+\ell}(X, A)$. From this, we have a relative cup product

$$\mathrm{H}^k(X, A) \times \mathrm{H}^\ell(X) \to \mathrm{H}^{k+\ell}(X, A)$$

In particular, $\mathrm{H}^*(X, A)$ is a graded ring, but it is typically not unital. We also have a relative cross product,

$$C^k(X, A) \otimes C^\ell(Y) \to C^{k+\ell}(X \times Y, A \times Y)$$

and this induces a map on cohomoloogy

$$\mathrm{H}^k(X, A) \otimes \mathrm{H}^\ell(Y) \to \mathrm{H}^{k+\ell}(X \times Y, A \times Y)$$

Now consider the associations for fixed Y, with cellular pairs (X, A),

$$\begin{aligned} h^*(X, A) &:= \mathrm{H}^*(X, A) \otimes \mathrm{H}^*(Y) \\ k^*(X, A) &:= \mathrm{H}^*(X \times Y, A \times Y) \end{aligned}$$

and the relative cup product defines a map $\Phi : h^*(X, A) \to k^*(X, A)$. If $(X, A) = (\mathrm{pt}, \varnothing)$, then

$$\Phi : h^*(\mathrm{pt}) = \mathbb{Z} \otimes \mathrm{H}^*(Y) \xrightarrow{\cong} \mathrm{H}^*(\mathrm{pt} \times Y) = \mathrm{H}^*(Y)$$

By our discussion of generalised cohomology theories, if Φ is a natural transformation, and if h^*, k^* are generalised cohomology theories, then Φ will be an isomorphism for all cellular pairs (X, A).

h^*, k^* **are generalised cohomology theories:** For k^*, all the axioms follow by our known properties of singular cohomology. For h^*, naturality, homotopy invariance and excision are immediate. The long exact sequence and unions axioms hold as we are assuming that $\mathrm{H}^i(Y)$ is finitely generated and free. That is, if M is finitely generated and free, then

(i) the functor $\mathrm{T}_M(N) = M \otimes N$ is exact (i.e. it preserves exact sequences),

(ii) $M \otimes \prod_\alpha N_\alpha = \prod_\alpha (M \otimes N_\alpha)$,

Φ **is a natural transformation**

We know the cup product and cross products are natural for maps of spaces. So naturality, homotopy invariance and excision axioms are fine. Consider

$$\begin{tikzcd} \mathrm{H}^k(A) \otimes \mathrm{H}^\ell(Y) \arrow[r, "\partial_{\mathrm{LES}} \otimes \mathrm{id}"] \arrow[d, "\Phi"] & \mathrm{H}^{k+1}(X, A) \otimes \mathrm{H}^\ell(Y) \arrow[d, "\Phi"] \\ \mathrm{H}^{k+\ell}(A \times Y) \arrow[r, "\partial_{\mathrm{LES}}"] & \mathrm{H}^{k+\ell+1}(X \times Y, A \times Y) \end{tikzcd}$$

We need the square to commute and then Φ will be compatible with the long exact sequences.

Recall given $\phi \in C^k(A)$ with $\partial^* \phi = 0$, we extend it to $\widehat{\phi} \in C^k(X)$, and set

$$\partial_{\mathrm{LES}}[\phi] = \partial^*[\widehat{\phi}]$$

Thus, if $\phi \in C^\ell(Y)$ is any cocyle, then $\widehat{\phi} \times \psi$ is an extension of $\phi \times \psi$ from $A \times Y$ to $X \times Y$. Hence the square commutes. ☐

Sketch proof of proposition 4.1.11. Let $\varepsilon_n = (-1)^{n(n+1)/2}$, and

$$\begin{aligned} \rho : C_n(X) &\to C_n(X) \\ [v_0, \ldots, v_n] &\mapsto \varepsilon_n [v_0, \ldots, v_n] \end{aligned}$$

Claim 4.1.20. ρ is a chain map, which is chain homotopic to the identity.

Given the claim,

$$(\rho^* \phi) \cdot (\rho^*(\psi))([v_0, \ldots, v_{k+\ell}]) = \phi(\varepsilon_k [v_k, \ldots, v_n]) \psi(\varepsilon_\ell [v_{k+\ell}, \ldots, v_k])$$

whereas

$$\rho^*(\psi \cdot \phi) = \varepsilon_{k+\ell} \psi([v_{k+\ell}, \ldots, v_k]) \phi([v_k, \ldots, v_0])$$

and we also have that

$$\varepsilon_k \varepsilon_\ell = (-1)^{k\ell} \varepsilon_{k+\ell}$$

and ρ being chain homotopic to id shows that $\rho^* = \mathrm{id}$ on H^*, and the result then follows.

To see that ρ is a chain map, we can just compute. We would like the diagram to commute

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] & C_{n-1}(X) \\ C_n(X) \arrow[u, "\rho"] \arrow[r, "\partial"] & C_{n-1}(X) \arrow[u, "\rho"] \end{tikzcd}$$

But

$$\partial(\rho\sigma) = \varepsilon_n \sum (-1)^i \sigma|_{[v_n,\dots,\widehat{v_{n-i}},\dots,v_0]}$$

and

$$\rho(\partial\sigma) = \varepsilon_{n-1} \sum (-1)^{n-i} \sigma|_{[v_n,\dots,\widehat{v_{n-i}},\dots,v_0]}$$

Since $\varepsilon_n = (-1)^n \varepsilon_{n-1}$, we are done.

For the homotopy, we will need a *twisted prism operator*. We want $P : C_n(X) \to C_{n+1}(X)$ such that

$$\partial P + P\partial = \rho - \mathrm{id}$$

If $\pi : \Delta^n \times [0,1]^n \to \Delta^n$ is the projection map, define

$$P\sigma = \sum_i (-1)^i \varepsilon_{n-i} \pi([v_0, \dots, v_i, w_n, \dots, w_i])$$

□

Lecture 16

4.2 Projective space

We will go through an extended example. Recall $\mathbb{CP}^n$ has a cell structure with one cell in each even dimension $0, 2, \dots, 2n$. In particular, $\partial^*_{\text{cell}} = 0$. Hence $\mathrm{H}^*(\mathbb{CP}^n) \cong C^*_{\text{cell}}(\mathbb{CP}^n)$.

As a ring, we will show that

$$\mathrm{H}^*(\mathbb{CP}^n) = \frac{\mathbb{Z}[x]}{x^{n+1}}$$

where $|x| = 2$. Thus, $x^i \neq 0$ and generates $\mathrm{H}^i(\mathbb{CP}^n) \cong \mathbb{Z}$ for $0 \leq i \leq n$.

Lemma 4.2.1. There is a natural map

$$\pi : \underbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}_{n \text{ copies}} \to \mathbb{CP}^n$$

which is invariant under permutation, and so it induces a homeomorphism

$$\overline{\pi} : \frac{(\mathbb{CP}^1)^n}{S_n} \cong \mathbb{CP}^n$$

Proof. For $(a : b) \in \mathbb{CP}^1$, we associate the linear homogeneous polynomial $(bx - ay)$ which vanishes at $(a : b)$. If we have $(a_1 : b_1), \dots, (a_n, b_n) \in \mathbb{P}^1$, consider

$$\prod_{i=1}^{n} (b_i x - a_i y) = \alpha_0 x^n + \alpha_1 x^{n-1} y + \cdots + \alpha_{n-1} x y^{n-1} + \alpha_n y^n$$

and define

$$\pi((a_1 : b_1), \dots, (a_n : b_n)) = (\alpha_0 : \cdots : \alpha_n)$$

Clearly

(i) π is continuous,

(ii) it descends to a map $(\mathbb{CP}^1)^n / S_n \to \mathbb{CP}^n$,

(iii) from the fundamental theorem of algebra, the map is surjective (consider roots),

(iv) the induced map $\overline{\pi}$ is a bijection, as a polynomial, up to scaling, is determined by its roots. So $\overline{\pi}$ is a homeomorphism by the topological inverse function theorem.

$\square$

We've seen $\mathrm{H}^{2n}(\mathbb{CP}^n) = \mathbb{Z}$, and that

$$\mathrm{H}^*(\mathbb{CP}^1) = \frac{\mathbb{Z}[x]}{x^2}$$

where $|x| = 2$. Since $\mathrm{H}^*(\mathbb{CP}^1)$ is finitely generated and free in each degree, and so we can use Künneth to get that

$$\begin{aligned} \mathrm{H}^*(\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1) &\cong \bigotimes_{i=1}^{n} \mathrm{H}^i(\mathbb{CP}^n) \\ &= \frac{\mathbb{Z}[u_1, \ldots, u_n]}{\langle u_1^2, \ldots, u_n^2 \rangle} \end{aligned}$$

where $|u_i| = 2$. In particular,

$$\mathrm{H}^{2n}((\mathbb{CP}^1)^n) \cong \mathbb{Z}$$

generated by $u_1 \cdots u_n$. Hence it makes sense to compute the degree of π. That is, what is the induced map

$$\pi^* : \mathrm{H}^*(\mathbb{CP}^n) \to \mathrm{H}^*((\mathbb{CP}^1)^n)$$

Choose a generic point $q = (\alpha_0 : \cdots : \alpha_n)$ representing a polynomial with distinct roots, then naturally

$$\pi^{-1}(q) = \{p_\sigma \mid \sigma \in S_n\}$$

is a finite set of $n!$ elements. By our considerations of local degree, we can fix a small disc $q \subseteq V$, V homeomorphic to $\mathbb{R}^{2n}$, with q not intersecting the locus of polynomials with repeated roots, such that

$$\pi^{-1}(V) = \bigsqcup_{\sigma \in S_n} U_\sigma$$

where $\pi : U_\sigma \to V$ is a homeomorphism. That is, away from the locus of polynomials with repeated roots, the S_n action is free. With this, by local degree computations,

$$\deg(\sigma) = \sum_{\sigma \in S_n} \deg_{p_\sigma}(\pi)$$

where each $\deg_{p_\sigma}(\pi) = \pm 1$ since π is a local homeomorphism. But the maps

$$\pi_\sigma : U_\sigma \to V \quad \text{and} \quad \pi_\tau : U_\tau \to V$$

differ by the homeomorphism of $(\mathbb{CP}^1)^n$ given by the elmeent $\sigma\tau^{-1} \in S_n$.

But if we fix an isomorphism $\mathrm{H}^2(\mathbb{P}^1) \cong \mathbb{Z}$, where $u_i = 1$, then the S_n action on $(\mathbb{CP}^1)^n$ induces an action on $\mathrm{H}^2((\mathbb{CP}^1)^n) = \mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_n$, by permuting the u_i (for example, thinking about cellular maps). Hence the action preserves $u_1 \cdots u_n \in \mathrm{H}^{2n}((\mathbb{CP}^1)^n)$. Thus, all of the local degrees are the same. Hence (up to a sign) $\deg(\pi) = n!$. Now consider the pullback map

$$\mathrm{H}^*(\mathbb{CP}^n) \to \mathrm{H}^*((\mathbb{CP}^1)^n) = \frac{\mathbb{Z}[u_1, \ldots, u_n]}{\langle u_1^2, \ldots, u_n^2 \rangle}$$

Let x be a generator of $\mathrm{H}^2(\mathbb{CP}^n)$. In fact, from the cell structure of $\mathbb{CP}^n$, the 2-skeleton is a copy of $\mathbb{CP}^1$. The inclusion $\mathbb{CP}^1 \to \mathbb{CP}^n$ induces an isomorphism

$$\mathrm{H}^2(\mathbb{CP}^n) \to \mathrm{H}^2(\mathbb{CP}^1)$$

Now consider

$$\mathbb{CP}^1 \times \text{ some generic, i.e. pairwise distinct points } \subseteq (\mathbb{CP}^1)^n$$

The image under π is a line in $\mathbb{CP}^n$. Hence we can choose x so that it restricts to u_1. By symmetry (i.e. S_n-equivariance), $\pi^*(x) = u_1 + \cdots + u_n$.

In this case, $\pi^*(x^n) = \deg(\pi) u_1 \cdots u_n$. But then the right hand side is non-zero, and so $x^n \neq 0$. In fact, $(u_1 + \cdots + u_n)^n = n! u_1 \cdots u_n$. Hence $x^i \neq 0$ for all $1 \leq i \leq n$. Moreover, x^i is the generator of $\mathrm{H}^{2i}(\mathbb{CP}^n)$[1].

We can think about this in terms of algebraic geometry (assuming some more machinery). The generator in $\mathrm{H}^2(\mathbb{CP}^n)$ is the Poincaré dual to the fundamental class of a hyperplane $[H] \in \mathrm{H}^{2n-2}(\mathbb{CP}^n)$. That is, it is the class of a linear form. What the homeomorphism π represents is that a point in $\mathbb{CP}^n$ can be represented by the intersection of n (generic) hyperplanes, and cup product is Poincaré dual to intersection.

Corollary 4.2.2. A map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ cannot have degree -1.

Proof. $\deg(f)$ is defined by $f^*(x^2) = (\deg f)x^2$ where $|x| = 2$, but $f^*(x) = \lambda x$, for some $\lambda \in \mathbb{Z}$, and so $f^*(x^2) = \lambda^2 x^2$, and $\lambda^2 \neq -1$. ☐

In fact, we see that the degree of f has to be a square.

A consequence of our computations is that for $M = (\mathbb{CP}^1)^n, \mathbb{CP}^n$, and a non-zero class $\alpha \in \mathrm{H}^*(M)$, there exists $\beta \in \mathrm{H}^*(M)$ such that $|\alpha \cdot \beta| = \dim(M)$[2]. In fact, this is a general fact of the cohomology classes of compact oriented manifolds. This is called Poincaré duality, and it is our next goal. But we would like to understand the cohomology of a manifold.

Locally, M is a disc, and the cohomology of a disc is not very interesting.

4.3 Cohomology with compact support

Let X be any space, $K_1, K_2 \subseteq X$ be compact subsets. If $K_1 \subseteq K_2$, then $X \setminus K_1 \supseteq X \setminus K_2$, and so we have an inclusion of pairs $(X, X \setminus K_2) \subseteq (X, X \setminus K_1)$. This defines a pullback map on cohomology

$$\mathrm{H}^*(X, X \setminus K_1) \to \mathrm{H}^*(X, X \setminus K_2)$$

Definition 4.3.1 (cohomology with compact support)

The *cohomology of X with compact supports* is $\mathrm{H}^*_{\mathrm{ct}}(X)$, given by

$$\varinjlim_{K \subseteq X \text{ compact}} \mathrm{H}^*(X, X \setminus K)$$

4.3.1 Crash course on direct limits

Let A be a poset, such that for any $a, b \in A$, there exists $c \in A$ with $a \leq c, b \leq c$.

A *direct limit of abelian groups on A*: Given the data

- abelian groups $\{G_a\}_{a \in A}$,
- a homomorphism $\rho_{ab} : G_a \to G_b$ if $a \leq b$, such that
  1. $\rho_{aa} = \mathrm{id}_{G_a}$,
  2. $\rho_{bc}\rho_{ab} = \rho_{ac}$ if $a \leq b \leq c$.

The direct limit is

$$\varinjlim_{a \in A} G_a = \frac{\bigoplus_{a \in A} G_a}{\langle x - \rho_{ab}(x) \rangle}$$

If $x \in G_a, y \in G_b$, choose c with $a \leq c, b \leq c$, then $x \sim \rho_{ac}(x) \in G_c$, $y \sim \rho_{bc}(x) \in G_c$, and so we can set

$$[x] + [y] = [\rho_{ac}(x) + \rho_{bc}(y)]$$

[1] To see this, note that the inclusion $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ induces isomorphisms

$$\mathrm{H}^i(\mathbb{CP}^{n-1}) \cong \mathrm{H}^i(\mathbb{CP}^n)$$

for $0 \leq i \leq 2n - 2$, and so if we show that x^n generates $\mathrm{H}^{2n}(\mathbb{CP}^n)$, then under the above isomorphism, x^n also generates $\mathrm{H}^{2n}(\mathbb{CP}^{n+k})$ for any $k \geq 0$. The result then follows by induction.

[2] Dimension of M as a real manifold.

This makes $\varinjlim_a G_a$ into an abelian group.

If $\Gamma \subseteq A$ is *cofinal*, that is, for all $a \in A$, there exists $\gamma \in \Gamma$, with $a \leq \gamma$, then

$$\varinjlim_{a \in A} G_a = \varinjlim_{\gamma \in \Gamma} G_\gamma$$

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Example 4.3.2

Let $A = \mathbb{N}$ with its usual order, $G_a = \mathbb{Z}/p^a$ for a fixed prime p. The maps are

$$\begin{aligned} \frac{\mathbb{Z}}{p^a} &\to \frac{\mathbb{Z}}{p^{a+1}} \\ x &\mapsto px \end{aligned}$$

The direct limit is

$$\varinjlim_A G_a = \mathbb{Z}(p^\infty) = \left\{ z \in S^1 \mid z \text{ is a } p^n \text{ root of unity for some } n \in \mathbb{N} \right\}$$

The result is called the *Prüfer group*

Example 4.3.3

Let $A = \mathbb{N}$ again, with $n \leq m \iff m \mid n$, and groups $G_a = \mathbb{Z}$, and ρ_{ab} is multiplication by b/a. In this case,

$$\varinjlim_A G_a = \mathbb{Q}$$

Note that the elements $(n!)$ form a cofinal family, and so we have that the limit is

$$\mathbb{Z} \xrightarrow{\cdot 1} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \longrightarrow \cdots$$

Equivalently,

$$\mathbb{Z} \xrightarrow{\mathrm{id}} \tfrac{1}{2!}\mathbb{Z} \xrightarrow{\mathrm{id}} \tfrac{1}{3!}\mathbb{Z} \xrightarrow{\mathrm{id}} \cdots$$

and so the limit is

$$\bigcup_{n \in \mathbb{N}} \frac{1}{n!}\mathbb{Z} = \mathbb{Q}$$

Example 4.3.4

If X is a compact set, then the poset $\mathcal{K}$ of compact subsets of X ordered by inclusion has a final element, namely X, and so

$$\mathrm{H}^*_{\mathrm{ct}}(X) = \varinjlim_{\mathcal{K}} \mathrm{H}^*(X, X \setminus K) = \mathrm{H}^*(X, X \setminus X) = \mathrm{H}^*(X)$$

Example 4.3.5

$$\mathrm{H}^*_{\mathrm{ct}}(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise} \end{cases}$$

By Heine-Borel, every compact $K \subseteq \mathbb{R}^n$ lies in $\overline{B}(0, N)$ for some N, and so

$$\varinjlim_K \mathrm{H}^*(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \varinjlim_N \mathrm{H}^*(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}(0, 1))$$

But via the long exact sequence of the pair, and homotopy invariance,

$$\mathrm{H}^*(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}(0, N)) \to \mathrm{H}^{i-1}(S^{n-1})$$

Moreover, this is compatible with the inclusion $\overline{B}(0, N) \subseteq \overline{B}(0, N+1)$, and so the direct limit is just

$$\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \longrightarrow \cdots$$

which is $\mathbb{Z}$, where the only non-zero degree is when $* = n$.

Note that

$$\mathrm{H}^*_{\mathrm{ct}}(\{\mathrm{pt}\}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{H}^*_{\mathrm{ct}}(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise} \end{cases}$$

and so it is *not* homotopy invariant. Moreover, it is *not* functorial either. On the other hand, if $f : X \to Y$ is continuous, closed and proper, then it induces a map

$$\mathrm{H}^*_{\mathrm{ct}}(Y) \to \mathrm{H}^*_{\mathrm{ct}}(X)$$

Again in this case it is helpful to keep in mind the case of de Rham theory. The corresponding idea is to consider the space of compactly supported differential forms, i.e. $\alpha \in \Omega^k(M)$ which is zero outside of a compact set $K \subseteq M$.

On the other hand, if $i : U \to X$ is the inclusion of an open set in a Hausdorff space X, then we have an *extension by zero* map $i_* : \mathrm{H}^*_{\mathrm{ct}}(U) \to \mathrm{H}^*_{\mathrm{ct}}(X)$, using the fact that if $K \subseteq U$ is compact, then $K \subseteq X$ is compact. This gives a map from compact sets on U to compact sets on X, which gives an induced map on cohomology. In particular, we are interested in the case when X is a manifold and $U \subseteq X$ is a disc.

5 Cohomology of manifolds

Recall in this course, an n-manifold is a Hausdorff space locally homeomorphic to $\mathbb{R}^n$.

5.1 Orientation

Definition 5.1.1 (local orientation)

Let R be a unital commuative ring (most importantly $\mathbb{Z}$ and $\mathbb{Z}/2$). A *local R-orientation* for a manifold M at $x \in M$ is a choice of generator

$$\varepsilon_x \in \mathrm{H}_n(M, M \setminus x; R) \cong R$$

Recall by excision, $\mathrm{H}_n(M, M \setminus x; R) \cong \mathrm{H}_n(U, U \setminus x)$, where U is an open ball.

Definition 5.1.2 (oriented)

A manifold M is *R-oriented* if we have chosen local orientations ε_x for all $x \in M$, such that if $\varphi : U \to \mathbb{R}^n$ is a chart on M in a preferred atlas, such that for all $p_i \in U \subseteq M$,

$$\mathrm{H}_n(M, M \setminus p_i) \cong \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi(p_i))$$

by excision above, and we can define a map of pairs $(\mathbb{R}^n, \mathbb{R}^n \setminus \varphi(p_1)) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \varphi(p_2))$ by translation, which induces an isomorphism on the right hand side. This induces an isomorphism

$$\psi : \mathrm{H}_n(M, M \setminus p_1) \to \mathrm{H}_n(M, M \setminus p_2)$$

we require $\psi(\varepsilon_{p_1}) = \varepsilon_{p_2}$.

Diagramatically, ψ is given by

$$\begin{tikzcd}
H_n(M, M\setminus p) \arrow[r, "\text{excision}"] \arrow[d, "\psi"'] & H_n(U, U\setminus p) \arrow[r, "\text{chart}"] & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\varphi(p)) \arrow[d, "\text{translation}"] \\
H_n(M, M\setminus q) \arrow[r, "\text{excision}"'] & H_n(U, U\setminus p) \arrow[r, "\text{chart}"'] & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\varphi(q))
\end{tikzcd}$$

Remark 5.1.3. If $U, V \subseteq \mathbb{R}^n$ are open, $f : U \to V$ is a homeomorphism is *orientation preserving* if for all $x \in U$, $f(x) \in V$,

$$\begin{tikzcd}
H_n(\mathbb{R}^n, \mathbb{R}^n\setminus 0) \arrow[r, "\text{translation}"] \arrow[d, "\text{id}"'] & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus x) \arrow[r, "\text{excision}"] & H_n(U, U\setminus x) \arrow[d, "f_*"] \\
H_n(\mathbb{R}^n, \mathbb{R}^n\setminus 0) \arrow[r, "\text{translation}"'] & H_n(\mathbb{R}^n, \mathbb{R}^n\setminus y) \arrow[r, "\text{excision}"'] & H_n(V, V\setminus x)
\end{tikzcd}$$

commutes. Then M is orientable if it admits an atlas $\{U_\alpha, \varphi_\alpha\}$ of charts where the transition maps are orientation preserving.

Remark 5.1.4. If $R = \mathbb{Z}/2$, then $\mathbb{Z}/2$ has only one generator, and so all manifolds are $\mathbb{Z}/2$ orientable.

Theorem 5.1.5 (Poincaré duality)**.** Let R be an R-oriented manifold of dimension n, then there is a distinguished isomorphism

$$D : H^i_{ct}(M; R) \to H_{n-i}(M; R)$$

In particular, if M is compact, then

$$D : H^i(M; R) \to H_{n-i}(M; R)$$

Remark 5.1.6. By considering $R = \mathbb{Z}$ and M being $\mathbb{RP}^2$ or the Klein bottle, we see that we need the orientability requirement.

The isomorphism D is obtained from the *cap product*.

5.2 Cap product

Definition 5.2.1 (cap product)

Let X be any topological space, the *cap product* is

$$\begin{aligned}\frown : C_k(X) \otimes C^\ell(X) &\to C_{k-\ell}(X)\\ [v_0, \dots, v_k] \otimes \psi &\mapsto \psi([v_0, \dots, v_\ell])[v_\ell, \dots, v_k]\end{aligned}$$

for $\ell \le k$, and 0 if $\ell > k$.

Lemma 5.2.2. For any space X,

(i) $\partial(\sigma \frown \phi) = (-1)^\ell(\partial\sigma \frown \phi - \sigma \frown \partial^*\phi)$ for $\sigma \in C_k(X), \phi \in C^\ell(X)$. Indeed, $\frown$ induces a pairing (for $\ell \le k$)

$$H_k(X) \otimes H^\ell(X) \to H_{k-\ell}(X)$$

(ii) Given a map $f : X \to Y$,

$$f_*(\alpha) \frown \psi = f_*(\alpha \frown f^*(\psi))$$

for $\alpha \in H_k(X), \psi \in H^\ell(Y)$.

(iii)
$$\psi(\sigma \frown \phi) = (\phi \smile \psi)(\sigma) \in \mathbb{Z}$$
For $\sigma \in C_{k+\ell}(X)$, $\phi \in C^k(X)$, $\psi \in C^\ell(X)$,

(iv) for a pair (X, A), there is a *relative cap product*
$$C_k(X, A) \otimes C^\ell(X, A) \to C_{k-\ell}(X)$$
which descends to coohomology.

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Proof. For (i),
$$\begin{aligned}
\partial\sigma \frown \phi &= \sum_{i=0}^{\ell} \phi([v_0, \dots, \widehat{v_i}, \dots, v_{\ell+1}])[v_{\ell+1}, \dots, v_k] + \sum_{i=\ell+1}^{k} (-1)^i \phi([v_0, \dots, v_\ell])[v_\ell, \dots, \widehat{v_i}, \dots, v_k] \\
\sigma \frown \partial^*\phi &= \sum_{i=0}^{\ell+1} (-1)^i \phi([v_0, \dots, \widehat{v_i}, \dots, v_{\ell+1}])[v_{\ell+1}, \dots, v_k] \\
\partial(\sigma \frown \phi) &= \sum_{i=\ell}^{k} (-1)^{i+\ell} \phi([v_0, \dots, v_\ell])[v_\ell, \dots, \widehat{v_i}, \dots, v_k]
\end{aligned}$$
Rearrange/compare terms to get the result.

For (ii), and (iii), they hold at chain lebel from definitions. Say at the level of (co)homology we have for (ii)
$$\begin{tikzcd}
\mathrm{H}_k(X) \otimes \mathrm{H}^\ell(Y) \arrow[r, "f_* \otimes \mathrm{id}"] \arrow[d, equal] & \mathrm{H}_k(Y) \otimes \mathrm{H}^\ell(Y) \arrow[r, "\frown"] & \mathrm{H}_{k-\ell}(Y) \\
\mathrm{H}_k(X) \otimes \mathrm{H}^\ell(Y) \arrow[r, "\mathrm{id} \otimes f^*"'] & \mathrm{H}_k(X) \otimes \mathrm{H}^\ell(X) \arrow[r, "\frown"'] & \mathrm{H}_{k-\ell}(X) \arrow[u, "f_*"']
\end{tikzcd}$$
and for (iii)
$$\begin{tikzcd}
\mathrm{H}^\ell(X) \arrow[r] \arrow[d, "\phi \smile \cdot"'] & \mathrm{Hom}(\mathrm{H}_\ell(X), \mathbb{Z}) \arrow[d, "(\cdot \frown \phi)^*"] \\
\mathrm{H}^{k+\ell}(X) \arrow[r] & \mathrm{Hom}(\mathrm{H}_{k+\ell}(X), \mathbb{Z})
\end{tikzcd}$$
Note that the horizontal maps in (iii) don't need to be isomorphism, but if we worked over a field, then they would be.

Ivan lost (iv) somewhere... ☐

We want to define the map D from the statment of Poincaré duality.

Proposition 5.2.3. Let M be an oriented n-manifold, and so we have $\omega_x \in \mathrm{H}_n(M, M \setminus x)$ which are coherent. Then for each $K \subseteq M$ compact, there exists a unique $\omega_K \in \mathrm{H}_n(M, M \setminus K)$ such that the map on pairs
$$(M, M \setminus K) \to (M, M \setminus x)$$
sends ω_K to ω_x for all $x \in K$. Note that $\mathrm{H}_i(M, M \setminus K) = 0$ for $i > n$.

Given the proposition, then M be oriented, $K \subseteq L \subseteq M$ be compact. We can consider
$$\begin{tikzcd}
\mathrm{H}_i(M, M \setminus L) \arrow[d] & \otimes & \mathrm{H}^k(M, M \setminus L) \arrow[r, "\frown"] & \mathrm{H}_{i-k}(M) \arrow[d, equal] \\
\mathrm{H}_i(M, M \setminus K) & \otimes & \mathrm{H}^k(M, M \setminus K) \arrow[u] \arrow[r, "\frown"'] & \mathrm{H}_{i-k}(M)
\end{tikzcd}$$

where the vertical maps are induced from the inclusion i (of pairs). Now

$$\omega_K \frown \phi = i_*\omega_L \frown \phi = \omega_L \frown i^*\phi$$

since by uniqueness, $i_*\omega_L = \omega_K$.

The map $\phi \mapsto \omega_k \frown \phi$ is compatible with the maps in the directed system defininining $\mathrm{H}^*_{\mathrm{ct}}$ via invlusions $K \hookrightarrow L$. So there exists an induced map $\mathrm{H}^k_{\mathrm{ct}}(M) \to \mathrm{H}_{n-k}(M)$.

Moreover, if M is compact, then we have $\omega_M \in \mathrm{H}_n(M)$, and $D(\phi) = \omega_M \frown \phi$ is called the *fundamental class*, denoted by $[M]$.

Proof of proposition 5.2.3. We will prove this for more and more general classes of K. We say that $K \subseteq M$ compact is *good* if it satisfies the conclusions of the proposition.

Step 1: If $A, B, A \cap B$ are good, then so is $A \cup B$. In this case, we have

$$\cdots \to \mathrm{H}_{n+1}(M, M \setminus A \cap B) \to \mathrm{H}_n(M, M \setminus A \cup B) \to \mathrm{H}_n(M, M \setminus A) \oplus \mathrm{H}_n(M, M \setminus B) \to \mathrm{H}_n(M, M \setminus A \cap B) \to \cdots$$

By uniqueness, $\omega_A \mapsto \omega_{A\cap B}$ and $\omega_B \mapsto \omega_{A\cap B}$ under relative inclusions. Hence the map

$$\mathrm{H}_n(M, M \setminus A) \oplus \mathrm{H}_n(M, M \setminus B) \to \mathrm{H}_n(M, M \setminus A \cap B)$$

is zero. Hence we have a class $\omega_{A\cup B}$ mapping to ω_A and ω_B. By assumption, $\mathrm{H}_{n+1}(M, M \setminus A \cap B)$ is zero, and so $\omega_{A\cup B}$ is unique. Moreover, for all $x \in A \cup B$, by construction $\omega_{A\cup B} \mapsto \omega_x$. Finally, $\mathrm{H}_i(M, M \setminus A \cup B) = 0$ for $i > n$ by exactness of the Mayer-Vietoris sequence.

Step 2: If $K \subseteq \mathbb{R}^n$ is convex, then K is good. If so, then $\mathrm{H}_*(\mathbb{R}^n, \mathbb{R}^n \setminus K) = \mathrm{H}_*(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$. But this case is easy, since we can use translations. Using step 1, finite unions of convex sets is good.

Step 3: Every compact set $K \subseteq \mathbb{R}^n$ is good. If $K \subseteq \mathbb{R}^n$ is compact, then $K \subseteq B = \overline{B}(0, R)$ for some $R > 0$. Define ω_K to be the restriction of ω_B to K, via

$$(\mathbb{R}^n, \mathbb{R}^n \setminus B) \to (\mathbb{R}^n, \mathbb{R}^n \setminus K)$$

We will write $\omega_K = \omega_B|_K$. Since $\omega_B \mapsto \omega_x$ under restriction, so does ω_K. So what we need is uniqueness. That is, we need to know that no other element of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus K)$ also satisfies the requirement that it pushes forward to ω_x for all $x \in K$.

That is, we want to know if that if $\lambda \in \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus K)$, with $\lambda|_x = 0$ for all $x \in K$, then $\lambda = 0$. Suppose we have such a λ. So λ is represented by a chain, and $\partial\lambda$ is a finite union of simplices in $\mathbb{R}^n \setminus K$. Thus there exists a finite union of balls B_j such that

- $K \subseteq \widetilde{K} = \bigcup_j B_j$
- $\partial\lambda \cap \tilde{K} = \varnothing$

That is,

$$\lambda \in \mathrm{im}(\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \widetilde{K}) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus K))$$

But $\tilde{K}$ is a finite union of convex sets, so it is good. Hence $\lambda = 0$ by the uniqueness for $\tilde{K}$.

Step 4: If $K \subseteq M$ is compact, it is good. We can write K as a finite union of K_i, where each K_i is a compact subset of the coordinate neighbourhoods. Each K_i is good, and so their union is good by step 1. ☐

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5.3 Consequences of Poincaré duality

Take coefficients in a field F. Recall that

$$\psi(\sigma \frown \phi) = (\phi\psi)(\sigma)$$

and we have an isomorphism

$$\mathrm{H}^k(M; F) \to \mathrm{Hom}(\mathrm{H}_k(X; F), F) = \mathrm{H}_k(X; F)^*$$

a and so we get a pairing

$$\begin{aligned} \mathrm{H}^k_{\mathrm{ct}}(M; F) \otimes \mathrm{H}^{n-k}(M; F) &\to F \\ (\phi, \psi) &= \psi(D\phi) \end{aligned}$$

and Poincaré dueality says that this pairing is non-degenerate. In particular, if M is compact, then the map is

$$\begin{aligned} \mathrm{H}^k(M;F) \otimes \mathrm{H}^{n-k}(M;F) &\to F \\ (\phi, \psi) &\mapsto \langle \phi \cdot \psi, [M] \rangle \end{aligned}$$

In the de Rham theory, the pairing is given by

$$\begin{aligned} \mathrm{H}^k_{\mathrm{ct}}(M) \otimes \mathrm{H}^{n-k}(M) &\to \mathbb{R} \\ (\alpha, \beta) &\mapsto \int_M \alpha \wedge \beta \end{aligned}$$

Corollary 5.3.1. Let M, N be oriented connected compact n-manifolds, and so $\mathrm{H}^n(M) \cong \mathrm{H}_0(M) \cong \mathbb{Z}$, and the orientation gives us a preferred generator. Thus, a map $f : M \to N$ has a degree $\deg(f) \in \mathbb{Z}$. If this degree is non-zero, then over a field, the pullback map

$$f^* : \mathrm{H}^*(N;F) \to \mathrm{H}^*(M;F)$$

is injective in each degree.

Proof. If $\alpha \in \mathrm{H}^i(N;F)$ is a non-zero class, the non-degeneracy of cup product as a pairing implies there exists $\beta \in \mathrm{H}^{n-i}(N;F)$ such that $\alpha\beta \neq 0 \in \mathrm{H}^n(N;F)$. Since $\deg(f) \neq 0$, $f^* : \mathrm{H}^n(N;F) \to \mathrm{H}^n(M;F)$ is non-zero, and so it is an isomorphism. Hence $f^*(\alpha\beta) \neq 0$. But then $f^*(\alpha)f^*(\beta) \neq 0$, and so $f^*(\alpha) \neq 0$. □

Corollary 5.3.2. Let M be a compact manifold of odd dimension $n = 2k+1$. Then the Euler characteristic of M is zero.

Proof. First of all, note that in this case, $\mathrm{H}^i(M;\mathbb{Z})$ is finitely generated for all i, and non-zero only if $i \leq n$. Hence the alternating sum

$$\chi(M) = \sum_i (-1)^i \operatorname{rank}(\mathrm{H}^i(M;\mathbb{Z}))$$

is well defined. Moreover, we saw that we could compute $\chi(M)$ by working over a field F. Suppose M is oriented. Then

$$\chi(M) = \sum_{i=0}^{2k+1} (-1)^i \dim_F(\mathrm{H}^i(M;F))$$

Let $b^i = \dim_F(\mathrm{H}^i(M;F))$ be the Betti numbers. Then

$$\chi(M) = b_0 - b_1 + \cdots + b_{2k} - b_{2k+1}$$

But the non-degenerate pairing implies that $b_i = b_{n-1}$, and so $b_0 = b_{2k+1}$ and so on. Hence $\chi(M) = 0$. Now if we take $F = \mathbb{Z}/2$, then M is F-oriented. □

An alternative proof is to use the existence of an oriented double cover. For this, we first note that the universal coefficient theorem and Poincaré duality proves the result for any compact orientable manifold (here we can use $\mathbb{Z}$-coefficients). Now if $\widetilde{M} \to M$ is a double cover, then $\chi(\widetilde{M}) = 2\chi(M)$, which gives the result.

Definition 5.3.3 (manifold with boundary)
A *manifold with boundary* is a Hausdorff space M, locally homeomorphic to

$$\mathbb{R}^n_{\geq 0} \{(x_1, \ldots, x_n) \mid x_1 \geq 0\}$$

We define the boundary ∂M of M to be the points x which under some chart, $\phi(x) \in \{x_1 = 0\}$.

∂M is well defined by some point set topology. Now given a compact manifold M^n, is there a compact manifold with boundary W^{n+1}, with $\partial W \cong M$? If so, we say M is *null-cobordant*.

Lemma 5.3.4. If $M = \partial W$ as above, then $\chi(M)$ is even. In particular, $\mathbb{CP}^2$ is not the boundary of any compact 5-manifold.

Proof. Without loss of generality, $\dim(M)$ is even. Suppose $M = \partial W$, and we form the *double* Z of W, which is two copies of W glued along their common boundary M. That is,

$$Z = W_{\text{left}} \cup_M W_{\text{right}}$$

In this case, Z is a compact manifold without boundary, with $\dim(Z)$ odd, and so $\chi(Z) = 0$.

But we can compute $\chi(Z)$ using Mayer-Vietors. $Z = U \cup V$, with U, V homotopy equivalent to W, and $U \cap V$ homotopy equivalent to M. Note we use the fact that a neighbourhood of $\partial W \subseteq W$ is homeomorphic to $\partial W \times [0, \varepsilon)$. That is, we have a collar neighbourhood. Using this, we have an exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(Z) \longrightarrow \mathrm{H}^i(W) \oplus \mathrm{H}^i(W) \longrightarrow \mathrm{H}^i(M) \longrightarrow \mathrm{H}^{i+1}(Z) \longrightarrow \cdots$$

We know all the groups in the above are finitely generated. But the Mayer-Vietoris sequence is a chain complex with homology zero, and so the Euler characteristic of the chain complex is zero. But taking the alternating sum of the ranks, we get

$$\chi(Z) - 2\chi(W) + \chi(M)$$

But $\chi(Z) = 0$, and so $\chi(M) = 2\chi(W)$. □

5.4 Proof of Poincaré duality

Lemma 5.4.1. Let X be a locally compact Hausdorff space. If $X = U \cup V$, with $U, V \subseteq X$ open, then there exists a Mayer-Vietoris type sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}_{\mathrm{ct}}(X) \longrightarrow \mathrm{H}^i_{\mathrm{ct}}(U \cap V) \longrightarrow \mathrm{H}^i_{\mathrm{ct}}(U) \oplus \mathrm{H}^i_{\mathrm{ct}}(V) \longrightarrow \mathrm{H}^i_{\mathrm{ct}}(X) \longrightarrow \cdots$$

Lemma 5.4.2. Suppose M is a oriented n-manifold. We will say $U \subseteq M$ open is *good* if Poincaré duality holds on U. That is, $D_U : \mathrm{H}^k_{\mathrm{ct}}(U) \to \mathrm{H}_{n-k}(U)$ is an isomorphism. Note in this case, U is also an oriented manifold.

Suppose $U, V, U \cap V$ are good. Then $U \cup V$ is good.

Proof. Assume without loss of generality that $M = U \cup V$. From lemma 5.4.1, we have

$$\begin{tikzcd}
\mathrm{H}^k_{\mathrm{ct}}(U \cap V) \arrow[r] \arrow[d, "D"] & \mathrm{H}^k_{\mathrm{ct}}(U) \oplus \mathrm{H}^k_{\mathrm{ct}}(V) \arrow[r] \arrow[d, "D"] & \mathrm{H}^k_{\mathrm{ct}}(M) \arrow[r] \arrow[d, "D"] & \mathrm{H}^{k+1}_{\mathrm{ct}}(U \cap V) \arrow[r] \arrow[d, "D"] & \mathrm{H}^{k+1}_{\mathrm{ct}}(U) \oplus \mathrm{H}^{k+1}(V) \arrow[d, "D"] \\
\mathrm{H}_{n-k}(U \cap V) \arrow[r] & \mathrm{H}_{n-k}(U) \oplus \mathrm{H}_{n-k}(V) \arrow[r] & \mathrm{H}_k(M) \arrow[r] & \mathrm{H}_{n-(k+1)}(U \cap V) \arrow[r] & \mathrm{H}_{n-(k+1)}(U) \oplus \mathrm{H}_{n-(k+1)}(U \cap V)
\end{tikzcd}$$

The horizontal sequences are exact, and the vertical maps are isomorphisms except the centre one. We would like to apply the five lemma. The fact that the squares commute when the square does not involve the boundary maps is clear. For the square

$$\begin{tikzcd}
\mathrm{H}^k_{\mathrm{ct}}(M) \arrow[r] \arrow[d, "D"] & \mathrm{H}^{k+1}_{\mathrm{ct}}(U \cap V) \arrow[d, "D"] \\
\mathrm{H}_k(M) \arrow[r] & \mathrm{H}_{n-(k+1)}(U \cap V)
\end{tikzcd}$$

it does not commute as is. Hpwever it does commute up to a sign, and we can use the five lemma in this case. We will omit the proof of this fact. See Hatcher/Spanier for details.

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Now suppose $M = \bigcup_i U_i$, with $U_1 \subseteq U_2 \subseteq \cdots$ are all good, then so is M since any compact subset is contained in some U_i, and so

$$\varinjlim \mathrm{H}^*\mathrm{ct}(U_i) \to \mathrm{H}^*_{\mathrm{ct}}(M)$$

is an isomorphism. On the other hand, any homology class is represented by a finite union of simplices, which has compact image, we have natural maps

$$\mathrm{H}_*(U_i) \to \mathrm{H}_*(M)$$

and taking the direct limit,

$$\varinjlim \mathrm{H}_*(U_i) \to \mathrm{H}_*(M)$$

is also an isomorphism.

Using this, and that any open subset of $\mathbb{R}^n$ is a counable union of open balls, we see that any open subset of $\mathbb{R}^n$ is good. Thus, if the manifold M is *second countable*, then it is covered by countably many discs, and we are done.

In general, use Zorn's lemma for the collection of all good open subsets of M, □

Proof of lemma 5.4.1. If $(X, Y) = (A \cup B, C \cup D)$ is a union of pairs, then we have a relative Mayer-Vietoris seqeuence

$$\mathrm{H}^i(X, Y) \longrightarrow \mathrm{H}^i(A, C) \oplus \mathrm{H}^i(B, D) \longrightarrow \mathrm{H}^i(A \cap B, C \cap D) \longrightarrow \mathrm{H}^{i+1}(X, Y)$$

using the small-simplices theorem (see Sheet 4). If $X = U \cup V$, $K \subseteq U, L \subseteq V$ compact, set $A = B = X$, $C = X \setminus K, D = X \setminus Y$, $Y = X \setminus (K \cap L)$, $C \cap D = X \setminus (K \cup L)$. We then get an exact sequence

$$\mathrm{H}^i(X, X \setminus (K \cap L)) \longrightarrow \mathrm{H}^i(X, X \setminus K) \oplus \mathrm{H}^i(X, X \setminus L) \longrightarrow \mathrm{H}^i(X, X \setminus (K \cup L)) \longrightarrow \mathrm{H}^{i+1}(X, X \setminus (K \cap L))$$

Excise $X \setminus U \cap V, X \setminus U, X \setminus V$ from the first three terms, we get

$$\mathrm{H}^i(U \cap V, (U \cap V) \setminus (K \cap L)) \to \mathrm{H}^i(U, U \setminus K) \oplus \mathrm{H}^i(V, V \setminus L) \to \mathrm{H}^i(X, X \setminus (K \cup L)) \to \mathrm{H}^{i+1}(U \cap V, (U \cap V) \setminus (K \cap L))$$

But every compact subset $Q \subseteq U \cap V$ is of the form $K \cap L$, where $K \subseteq U$ compact, $L \subseteq V$ compact. For example, $U = V = Q$.

Thus, if we range over all K, L, in the first three terms we get all possible compact subsets. Since X is locally compact, any compact subset of X is contained in $K \cup L$, for some compact $K \subseteq U, L \subseteq V$. Thus, the compact subsets of the form $K \cup L$ form a cofinal family. Taking the direct limit over $K \subseteq U, L \subseteq V$ compact, and using that the direct limit of exact sequences is exact, we get the Mayer-Vietoris seqeunce for $\mathrm{H}^*_{\mathrm{ct}}$. □

6 Vector bundles

Definition 6.0.1 (vector bundle)

Let B be a topological space. A *vector bundle of rank d $E \to B$* is:

- A family of d-dimensional vector spaces $\{E_b\}_{b \in B}$,
- with a topology on $E = \bigsqcup E_b$, such that
  - (i) the natural projection $p : E \to B$ is continuous
  - (ii) and locally trivial. That is, for all $b \in B$, there exists a neighbourhood U of b, and a *local trivialisation*, which is a homeomorphism ψ making the diagram

$$\begin{tikzcd} p^{-1}(U) = E|_U \arrow[rr, "\psi"] \arrow[dr, "p"'] & & U \times \mathbb{R}^d \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commute, and such that the map

$$\psi : E_b \to \{b\} \times \mathbb{R}^d$$

is a linear isomorphism for all $y \in U$.

(iii) The subspace topology on E_b from E is the same as the topology on E_b as a Euclidean space.

We call E the total space, B the base space, E_b the fibres.

Definition 6.0.2 (section)
A map $s : B \to E$ is called a *section* of E if $p \circ s = \mathrm{id}_B$.

The map

$$\begin{aligned} B &\to E \\ b &\mapsto 0 \in E_b \end{aligned}$$

is called the *zero section*.

Example 6.0.3
The *trivial vector bundle of rank d* is $E = B \times \mathbb{R}^d$ with the product topology and the projection map.

Definition 6.0.4 (pullback)
If $p : E \to X$ is a vector bundle, $f : Y \to X$ is any map, we define the *pullback* as $f^*E \to Y$, with

$$f^*E = \{(e, y) \in E \times Y \mid p(e) = f(y)\}$$

This has a natural projection map to Y. Then $f^*E_y = E_{f(y)}$.

Definition 6.0.5 (Whitney sum)
Let $p : E \to X, q : F \to X$ be vector bundles, then we define their *Whitney sum* has

$$E \oplus F = \{(x, y) \in E \times F \mid p(e) = q(f)\}$$

This has a natural map to $X \times X$, which lands in the diagonal, that we identify with X.

Remark 6.0.6. Both of these operations have

1. it takes trivial bundles to trivial bundles,
2. they commute with retriction to open subsets $U \subseteq B$. In other words,

$$f^*(E|_U) = (f^*E)|_{f^{-1}U} \quad \text{and} \quad (E \oplus F)|_U = E|_U \oplus F|_U$$

and so they are locally trivial.

Other operations include tensor product, dual, exterior powers etc.

If $p : E \to X$ is a vector bundle, we say $F \subseteq E$ is a *subbundle* if for all $x \in X$, $F_x = p^{-1}(E_x)$ is a linear subspace, and we have an open neighbourhood of U with a trivalisation of F making

$$\begin{tikzcd} E|_U \arrow[r, "\psi"] & U \times \mathbb{R}^d \\ F|_U \arrow[u, hook] \arrow[r, "\tilde{\psi}"'] & U \times \mathbb{R}^k \arrow[u, hook] \end{tikzcd}$$

commute.

If $F \subseteq E$ is a subbundle, then we can define $E/F \to X$, with fibre E_x/F_x.

Definition 6.0.7 (isomorphism)
We say that vector bundles $E \to X, F \to X$ are *isomorphic* if there exists homeomorphisms $\alpha : E \to F, g : X \to X$, making the diagram

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[d] & F \arrow[d] \\ X \arrow[r, "g"'] & X \end{tikzcd}$$

commute, such that $\alpha : E_x \to F_{g(x)}$ is a linear isomorphism.

The most important case is when $g = \mathrm{id}$.

Lecture 21

Define
$$\mathrm{Gr}(k, \mathbb{R}^n) = \{k\text{-dimensional linear subspaces in } \mathbb{R}^n\} = \frac{O(n)}{O(k) \times O(n-k)}$$
for the *Grassmannian*. The *tautological bundle*
$$E \to \mathrm{Gr}(k, \mathbb{R}^n)$$
has fibre at $x \in \mathrm{Gr}(k, \mathbb{R}^n)$ the linear subspace $\langle x \rangle$ corresponding to x. Concretely, we can define
$$E = \{(x, e) \in \mathrm{Gr}(k, \mathbb{R}^n) \times \mathbb{R}^n \mid e \in \langle x \rangle \subseteq \mathbb{R}^n\}$$
This is a vector bundle, with the natural projection map $p : E \to \mathrm{Gr}(k, \mathbb{R}^n)$.

Proof. Choose an inner product on $\mathbb{R}^n$, given $x \in \mathrm{Gr}(k, \mathbb{R}^n)$, let
$$U = \left\{y \in \mathrm{Gr}(k, \mathbb{R}^n) \mid \langle y \rangle \cap \langle x \rangle^\perp = 0\right\}$$
On this, we have a trivialisation
$$\begin{aligned} \Psi : E|_U &\to U \times \langle x \rangle \\ (y, \xi) &\mapsto (y, \mathrm{pr}_{\langle x \rangle}(\xi)) \end{aligned}$$
where
$$\mathrm{pr}_{\langle x \rangle} : \mathbb{R}^n \to \langle x \rangle$$
is the orthogonal projection. The definition for ψ shows that we have a local trivialisation. $\square$

Remark 6.0.8. There is an obvious notion of a *complex vector bundle*, with fibres being complex vector bundles, and we have an associated tautological bundle $E \to \mathrm{Gr}(k, \mathbb{C}^n)$ as well.

Example 6.0.9
$\mathrm{Gr}(1, \mathbb{R}^n) = \mathbb{RP}^{n-1}$, has a tautological (real) line bundle, whereas the $\mathrm{Gr}(1, \mathbb{C}^n) = \mathbb{CP}^{n-1}$, we have a tautological (complex) line bundle.
Note a complex line bundle as real dimension 2.

Lemma 6.0.10 (partition of unity). Let X be (para)compact Hausdorff, if $\{U_\alpha\}$ is an open cover of X, then we have a subordinate partition of unity. That is, we have maps $\{\lambda_\alpha : X \to [0, 1]\}$, such that

- $\mathrm{supp}(\lambda_\alpha) \subseteq U_\alpha$,
- at each $x \in X$, the number of non-zero λ_α is finite,
- $\sum \lambda_\alpha(x) = 1$ for all $x \in X$.

Proof. Omitted. □

Definition 6.0.11 (inner product)
An *inner product* on a vector bundle E is a continuous map $\langle \cdot, \cdot \rangle : E \otimes E \to \mathbb{R}$, such that on each fibre the map is an inner product.

Lemma 6.0.12. A vector bundle $p : E \to X$ over a (para)compact Hausdorff space has an inner product. Moreover, E is *globally generated.* For all $x \in X$, $\xi \in E_x$, there exists $s \in \Gamma(E)$ such that $s(x) = \xi$.

Proof. Fix a trivialising open cover $\{U_\alpha\}$ for E, and an inner product on $\mathbb{R}^d$, where $d = \mathrm{rank}_{\mathbb{R}}(E)$. If $\Psi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^d$ is a trivialisation, then we can use Ψ_α to define an inner product $\langle \cdot, \cdot \rangle_\alpha$ on $E|_{U_\alpha}$.

Let $\{\lambda_\alpha\}$ is a subordinate partition of unity, then for $u \otimes v \in E \otimes E$, define

$$\langle u, v \rangle = \sum_\alpha \lambda_\alpha \langle u, v \rangle_\alpha$$

Note that $\langle u, v \rangle_\alpha$ is only defined if $p(u \otimes v) \in U_\alpha$, but if this wasn't true, then $\lambda_\alpha(p(u \otimes v)) = 0$. By definition, this is a finite sum, and that this defines an inner product.

Now if $x \in U_\alpha$, $\xi \in E_x$, choose a section $s_\alpha \in \Gamma(E|_{U_\alpha})$ with $s_\alpha(x) = \xi$. We can do this as we can take something which is constant in a trivialisation. Let $s = \sum_\alpha \lambda_\alpha s_\alpha$. Then $s(x) = \xi$. □

Corollary 6.0.13. If X is compact Hausdorff, $E \to X$ is a rank d vector bundle, then there exists $N \in \mathbb{N}$ and $f : X \to \mathrm{Gr}(d, \mathbb{R}^N)$ such that $E \cong f^* E_{\mathrm{taut}}$.

Proof. Since X is compact, we have a finite set $\{s_1, \dots, s_N\} \subseteq \Gamma(E)$, such that on each E_x the $s_i(x)$ span. Fix an inner product $\langle \cdot, \cdot \rangle$ on E, and now consider the map

$$\begin{aligned} \alpha : E &\to X \times \mathbb{R}^N \\ (x, \xi) &\mapsto (x, \langle s_1(x), \xi \rangle, \dots, \langle s_N(x), \xi \rangle) \end{aligned}$$

Since the sections s_i span, we see that α embeds E_x into $\mathbb{R}^N$ for all x. That is, it embeds E as a sub-bundle of a trivial bundle. But then we can just define

$$\begin{aligned} f : X &\to \mathrm{Gr}(d, \mathbb{R}^N) \\ x &\mapsto \alpha(E_x) \subseteq \mathbb{R}^N \end{aligned}$$

By construction, the pullback of the tautological bundle is E. □

Remark 6.0.14. Our proof actually shows that if E is a vector bundle over a compact Hausdorff space, then there exists another vector bundle $F \to X$ such that $E \oplus F$ is the trivial bundle, since we can just take $F_x = \alpha(E_x)^\perp$.

Remark 6.0.15. In fact, for this class of X,

$$\frac{\{\text{vector bundles of rank } d\}}{\text{isomorphism}} \leftrightarrow \text{homotopy classes of maps} X \to \mathrm{Gr}(k, \mathbb{R}^\infty) = [X, \mathrm{Gr}(k, \mathbb{R}^\infty)]$$

where

$$\mathrm{Gr}(k, \mathbb{R}^\infty) = \bigcup_{n \geq 0} \mathrm{Gr}(k, \mathbb{R}^n)$$

is the infinite Grassmannian.

6.1 Cohomology

First note that if E has rank d, then $\mathrm{H}^d(E_x, E_x \setminus 0) = \mathbb{Z}$.

Definition 6.1.1 (oriented)

We say that a rank d vector bundle E is *oriented* if for all $x \in X$, we have a generator ε_x of $\mathrm{H}^d(E_x, E_x \setminus 0)$, which vary locally trivially. That is, if $x \in U$ and E is trivialised over U, say

$$\Psi : E|_U \cong U \times \mathbb{R}^d$$

is a trivialisation, then this induces an isomorphism $E_y \to \{y\} \times \mathbb{R}^d$. Using this, we have an isomorphism $E_y \to E_x$, which should send ε_y to ε_x.

Notation 6.1.2. We will write $E^\sharp$ for the complement of the zero section in E.

Remark 6.1.3. If we have a coefficient ring R, then we have a natural definition of R-orientation. In particular, every vector bundle is $\mathbb{Z}/2$-orientable.

Theorem 6.1.4 (Thom isomorphism)**.** Let $\pi : E \to X$ be an oriented vector bundle of rank n, then

(i) $\mathrm{H}^k(E, E^\sharp) = 0$ for $k < n$,

(ii) there exists a unique element $u_E \in \mathrm{H}^n(E, E^\sharp)$ such that restricting, $u_E|_x = \varepsilon_x \in \mathrm{H}^n(E_x, E_x \setminus 0)$.

(iii) The map

$$\begin{aligned} \mathrm{H}^k(X) &\to \mathrm{H}^{k+n}(E, E^\#) \\ \alpha &\mapsto \pi^*\alpha \smile u_E \end{aligned}$$

is an isomorphism.

The class u_E is called the *Thom class* of E.

In the case of a smooth oriented manifold M^n, $TM \to M$ is a vector bundle of rank n, and so what this is saying is that the cohomology of TM relative to the zero section is just the cohomology of M, shifted by n. Moreover, the isomorphism is given by wedge product with a fixed n-form.

Moreover, the zero section is a smooth manifold, of dimension n, and so it has a canonical class $[E^0]$. We claim that the Poincaré dual of this class is the Thom class (Bott-Tu 6.24 (b)).

Finally, since a vector bundle is locally trivial, we can study the topology of it by studying the gluing of local trivialisations, c.f. cocycle condition. The zero section has to be glued to the zero section, and so we study the (co)homology relative to it.

Definition 6.1.5 (Euler class)

Consider the long exact seqeunce of the pair $(E, E^\sharp)$. Then we have a natural map $\mathrm{H}^n(E, E^\#) \to \mathrm{H}^n(E)$. Now $\mathrm{H}^n(E)$ is homotopy equivalent to $\mathrm{H}^n(X)$. The image of the Thom class under this map is the *Euler class*

$$e_E \in \mathrm{H}^n(X)$$

Remark 6.1.6. A *characteristic class* for vector bundles (perhaps satisfying some conditions, such as orientability) is an assignment

$$E \mapsto c(E) \in \mathrm{H}^*(X)$$

such that for all $f : Y \to X$,

$$c(f^*E) = f^*c(E)$$

The uniqueness in the Thom isomorphism implies that the Euler class e_E is a characteristic class.

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Definition 6.1.7 (sphere bundle)

Let $E \to X$ be a vector bundle, with an inner product on E. We define the *sphere bundle*

$$\mathrm{S}(E) = \{e \in E \mid \langle e, e\rangle = 1\}$$

Up to homotopy, this is independent of the choice of $\langle \cdot, \cdot \rangle$, as the inclusion into $E^\sharp$ is a homotopy equivalence.

Assume E is oriented and of rank d, then we have the long exact sequence of the pair $(E, E^\sharp)$ and using the Thom isomorphism, we have

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^i(E, E^\sharp) \arrow[r] & \mathrm{H}^i(E) \arrow[r] & \mathrm{H}^i(E^\sharp) \arrow[r] & \mathrm{H}^{i+1}(E, E^\sharp) \arrow[r] & \cdots \\ \cdots \arrow[r] & \mathrm{H}^{i-d}(X) \arrow[r] \arrow[u, leftrightarrow, "\sim"] & \mathrm{H}^i(X) \arrow[r] \arrow[u, leftrightarrow, "\sim"] & \mathrm{H}^i(\mathrm{S}(E)) \arrow[r] \arrow[u, leftrightarrow, "\sim"] & \mathrm{H}^{i-d+1}(X) \arrow[r] \arrow[u, leftrightarrow, "\sim"] & \cdots \end{tikzcd}$$

and so we obtain the *Gysin sequence*

$$\cdots \longrightarrow \mathrm{H}^i(X) \xrightarrow{\phi} \mathrm{H}^{i+d}(X) \longrightarrow \mathrm{H}^{i+d}(\mathrm{S}(E)) \longrightarrow \mathrm{H}^{i+1}(X) \longrightarrow \cdots$$

The map ϕ is the cup product with the Euler class e_E, basically by definition. In de Rham theory, the map $\mathrm{H}^{i+d}(\mathrm{S}(E)) \to \mathrm{H}^{i+1}(X)$ is given by integration over the S^{d-1}-fibre.

Example 6.1.8

Let $L \to \mathbb{CP}^n$ be the tautological complex line bundle. Recall that a complex vector bundle is canonically oriented, since $\mathrm{GL}(d, \mathbb{C}) \subseteq \mathrm{GL}_+(2d, \mathbb{R})$. Hence L has a Thom class and an Euler class. But

$$L = \left\{(x, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} \mid v \in \langle x\rangle\right\}$$

using the usual inner product on $\mathbb{C}^{n+1}$, we see that

$$\mathrm{S}(L) = S^{2n+1} \subseteq \mathbb{C}^{n+1}$$

since any $v \in \mathbb{C}^{n+1}$ with $||v|| = 1$ is in a unique line through the origin, and the intersection of a line with S^{2n+1} is an S^1. The Gysin sequence becomes

$$\cdots \longrightarrow \mathrm{H}^{i+1}(S^{2n+1}) \longrightarrow \mathrm{H}^i(\mathbb{CP}^n) \longrightarrow \mathrm{H}^{i+2}(\mathbb{CP}^n) \longrightarrow \mathrm{H}^{i+1}(S^{2n+1}) \longrightarrow \cdots$$

If $i \leq 2n - 2$, then we have

$$0 \longrightarrow \mathrm{H}^i(\mathbb{CP}^n) \longrightarrow \mathrm{H}^{i+2}(\mathbb{CP}^n) \longrightarrow 0$$

and so setting $x = e_L \in \mathrm{H}^2(\mathbb{CP}^n)$, we see that $\mathrm{H}^{2k}(\mathbb{CP}^n)$ is generated by x^k, and so we recover the result that

$$\mathrm{H}^*(\mathbb{CP}^n) = \frac{\mathbb{Z}[x]}{\langle x^{n+1}\rangle}$$

Remark 6.1.9. Clearly if E is trivial, then $\mathrm{S}(E) = X \times S^{d-1}$, and so we can compute $\mathrm{H}^*(\mathrm{S}(E))$ using Künneth. So the fact that the Gysin sequence is interesting here is detecting the fact that L is non-trivial.

Lemma 6.1.10. If a(n oriented) vector bundle $E \to X$ has a nowhere vanishing section, then $e_E = 0$.

Proof. Suppose $s \in \Gamma(E)$ is a section, which has image in $E^\sharp$. Then

$$e_E \in \operatorname{Im}(\mathrm{H}^k(E^\sharp) \to \mathrm{H}^k(X))$$

where $k = \operatorname{rank}(E)$. To see this,

$$\begin{tikzcd} \mathrm{H}^k(E, E^\sharp) \arrow[r] & \mathrm{H}^k(E) \arrow[r] \arrow[d, "(\mathrm{zero})^* = s^*"'] & \mathrm{H}^k(E^\sharp) \arrow[dl, "s^*"] \\ & \mathrm{H}^k(X) & \end{tikzcd}$$

note that any two sections are homotopic, and so the Euler class must be zero, as mapping the Thom class to $\mathrm{H}^k(E^\sharp)$ gives zero by exactness. □

Remark 6.1.11. If E is an oriented vector bundle of odd rank, the necessarily

$$2e_E = 0 \in \mathrm{H}^{\operatorname{rank}(E)}(X)$$

and so if $\mathrm{H}^{\operatorname{rank}(E)}(X)$ has no 2-torsion, $e_E = 0$. To see this, consider the map

$$\begin{aligned} \alpha : E &\to E \\ v &\mapsto -v \end{aligned}$$

which reverses orientation, i.e. it acts by -1 on $\mathrm{H}^{\operatorname{rank}(E)}(E_x, E_x \setminus 0)$. Hence $\alpha^* u_E = -u_E$. But on the zero section, $\alpha = \mathrm{id}$. Thus, pulling back to the zero section, $e_E = -e_E$.

Proof of the Thom isomorphism. We will prove the Thom isomorphism by inducting on the number of trivialising neighbourhoods for E. That is, we will assume this number is finite, for example if X is compact. Zorn's lemma shows the general case.

For the base case $E = X \times \mathbb{R}^d$ is a trivial bundle.

Lemma 6.1.12 (Relative Künneth)**.** Suppose $\mathrm{H}^*(Y), \mathrm{H}^*(B), \mathrm{H}^*(Y, B)$ are all finitely generated and free for a good pair (Y, B). Then for X which has the homotopy type of a cell complex, the map

$$\mathrm{H}^*(X) \otimes \mathrm{H}^*(Y, B) \to \mathrm{H}^*(X \times Y, X \times B)$$

given by the cross product is an isomorphism.

Proof. We have the diagram

$$\begin{tikzcd} \mathrm{H}^*(X) \otimes \mathrm{H}^*(Y, B) \arrow[r, "\times"] & \mathrm{H}^*(X \times Y, X \times B) \\ \mathrm{H}^*(X) \otimes \mathrm{H}^*(Y/B, \mathrm{pt}) \arrow[r] \arrow[u, "\mathrm{id} \otimes p^*"] & \mathrm{H}^*(X \times Y/B, X \times \mathrm{pt}) \arrow[u, "q^*"'] \end{tikzcd}$$

which commutes, it suffices to prove this for $B = \mathrm{pt}$. In the above, p, q are the quotient maps, and p^*, q^* are isomorphisms on homology. But

$$\mathrm{H}^*(Y, \mathrm{pt}) \longrightarrow \mathrm{H}^*(Y) \longrightarrow \mathrm{H}^*(\mathrm{pt})$$

splits if we choose a point in Y. We know the result for Y and for the point by the Künneth theorem. Using the five lemma gives the result. □

Thus, we have that $E = X \times \mathbb{R}^d$, $E^\sharp = X \times (\mathbb{R}^d \setminus 0)$, and so by the lemma,

$$\mathrm{H}^*(E, E^\sharp) = \mathrm{H}^*(X) \otimes \mathrm{H}^*(\mathbb{R}^d, \mathbb{R}^d \setminus 0)$$

Fix a generator ε_d for $\mathrm{H}^*(\mathbb{R}^d, \mathbb{R}^d \setminus 0)$, and so we can just define the Thom class to be $1 \otimes \varepsilon_d$. Everything else is clear.

For the inductive step, assume the result is known for all oriented vector bundles with trivialising open covers with at most N open sets. Assume $E \to X$ has a cover by $N + 1$ open sets. In this case, we can write

$$X = A \cup B$$

such that the result holds for $E|_A, E|_B, E|_{A\cap B}$. By Mayer-Vietoris, we have a sequence

$$\cdots \longrightarrow \mathrm{H}^{i-1}(E|_{A\cap B}, E^\sharp|_{A\cap B}) \longrightarrow \mathrm{H}^i(E, E^\sharp) \longrightarrow \mathrm{H}^i(E|_A, E^\sharp|_A) \oplus \mathrm{H}^i(E|_B, E^\sharp|_B) \longrightarrow \mathrm{H}^i(E|_{A\cap B}, E^\sharp|_{A\cap B}) \longrightarrow \cdots$$

By this, if $i < d = \mathrm{rank}(E)$, then $\mathrm{H}^i(E, E^\sharp) = 0$. When $i = d$, we get

$$0 \longrightarrow \mathrm{H}^d(E, E^\sharp) \longrightarrow \mathrm{H}^d(E|_A, E^\sharp|_A) \oplus \mathrm{H}^d(E|_B, E^\sharp|_B) \longrightarrow \mathrm{H}^d(E|_{A\cap B}, E^\sharp|_{A\cap B}) \longrightarrow \cdots$$

In this case, we have Thom classes $u_{E|_A}, u_{E|_B}, u_{E|_{A\cap B}}$. By uniqueness, $u_{E|_A}|_{A\cap B} = u_{E|_{A\cap B}} = u_{E|_B}|_{A\cap B}$, hence $(u_{E|_A}, u_{E|_B}) \mapsto 0$. With this, we must have a (unique by injectivity) class u_E such that $u_E|_A = u_{E|_A}$ and $u_E|_B = u_{E|_B}$.

By construction, $u_E|_{E_x} = \varepsilon_x$ is the orientation generator. Thus, all we need to show is that the map

$$\begin{gathered} \mathrm{H}^k(X) \to \mathrm{H}^{k+d}(E, E^\sharp) \\ \alpha \mapsto \pi^*\alpha \smile u_E \end{gathered}$$

is an isomorphism. The Thom map

$$\alpha \mapsto \pi^*\alpha \smile \cdot$$

maps the Mayer-Vietoris sequence for $X = A \cup B$ to a Mayer-Vietoris map for $E = E|_A \cup E|_B$. The result follows by the five lemma, once we show that the squares commute. The non-obvious case involves the boundary map.

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That is,

$$\begin{tikzcd} \mathrm{H}^i(E|_{A\cap B}, E^\sharp|_{A\cap B}) \arrow[r] & \mathrm{H}^{i+1}(E, E^\sharp) \\ \mathrm{H}^{i-d}(A \cap B) \arrow[u] \arrow[r] & \mathrm{H}^{i-d+1}(X) \arrow[u] \end{tikzcd}$$

Let $\varphi \in C^d(E, E^\sharp)$ be a cocycle representing the Thom class u_E. Then $\varphi|_{E|_A}$ represents $u_{E|_A}$.

If $\alpha \in \mathrm{H}^{i-d}(A \cap B)$ is a class, then we can write $\alpha = \psi_A - \psi_B$, where $\psi_A \in C^{i-d}(A)$ and $\psi_B \in C^{i-d}(B)$. Then

$$\partial^*[\alpha] = [\mathrm{d}^*\psi_A]$$

Hence along the bottom right, we get

$$\alpha \mapsto \pi^*(\partial^*\psi_A) \cdot \varphi$$

For the top left, note

$$\pi^*\psi_A \cdot \varphi_{E|_A} - \pi^*\psi_B \cdot \varphi_{E|_B}$$

is a difference of chains in

$$C^i(E|_A, E^\sharp|_A) \quad \text{and} \quad C^i(E|_B, E^\sharp|_B)$$

and so the top left sends

$$\alpha \mapsto \partial^*(\pi^*\psi_A \cdot \varphi_{E|_A}) = \partial^*(\pi^*\psi_A) \cdot \varphi|_{E|_A}$$

These the agree. $\square$

6.2 Cup products on smooth manifolds

Recall if M is a smooth manifold, it has a tangent bundle $TM \to M$ of rank $\dim(M)$. If $Y \subseteq M$ is a (smooth) submanifold, then $TY \subseteq TM|_Y$ is a subbundle. We define the *normal bundle*

$$\nu_{Y/M} = \frac{TM|_Y}{TY}$$

This is a vector bundle of rank $\dim(M) - \dim(Y)$ on Y. We'll write $\nu_Y = \nu_{Y/M}$ when M is clear from context.

Notation 6.2.1. We say that Y is *co-oriented* in M if $\nu_{Y/M}$ is an oriented vector bundle.

Exercise: If M is a smooth manifold, then an orientation on M as in this course, as

$$\varepsilon_x \in H_n(M, M \setminus x)$$

is equivalent to an orientation of TM as a vector bundle. One way of seeing this is using the exponential map of a Riemannian metric.

Definition 6.2.2 (transverse)
Let M be a smooth manifold, $Y, Z \subseteq M$ are smooth submanifolds, then we say that Y and Z *intersect transversely* if for all $p \in Y \cap Z$,

$$T_pY + T_pZ = T_pM$$

Theorem 6.2.3 (tubular neighbourhood)**.** Let M be a smooth manifold,

1. if $Y \subseteq M$ a compact smooth submanifold, Then there exists an open neighbourhood U_Y of $Y \subseteq M$, and a diffeomorphism $\alpha : U_Y \to \nu_{Y/M}$, taking Y to the zero section. Moreover, both U_Y and α are unique up to isotopy.
2. if $Y, Z \subseteq M$ are compact smooth manifolds which intersect transversally, then $Y \cap Z$ is a smooth submanifold, with
$$\operatorname{codim}(Y \cap Z) = \operatorname{codim}(Y) + \operatorname{codim}(Z)$$
and we have an isomorphism of bundles
$$\nu_{Y\cap Z} \cong \nu_Y|_{Y\cap Z} \oplus \nu_Z|_{Y\cap Z}$$
and there are tubular neighbourhoods U_Y, U_Z of Y, Z respectively, with $U_{Y\cap Z} = U_Y \cap U_Z$ compatible with the above isomorphism.

Proof. Omitted. □

Suppose M^d is a smooth oriented manifold, $Y^k \subseteq M$ a smooth compact submanifold. Note that if $V = V_1 \oplus V_2$ is a direct sum of vector spaces, oriented two of the three gives an orientation on the third. Thus, for Y, orienting Y is the same as co-orienting Y. Assume Y is oriented.

Y is a compact topological manifold, and so it has a fundamental class $[Y] \in H_k(Y)$ under the inclusion map $i : Y \hookrightarrow M$, we have a class

$$i_*[Y] = H_k(M) \cong H^{d-k}_{\mathrm{ct}}(M)$$

by Poincaré duality.

Alternatively, we can take the Thom class

$$\nu_Y \in H^{d-k}(\nu, \nu^\sharp) \overset{\text{tubular neighbourhood}}{\cong} H^{d-k}(U_Y, U_Y \setminus Y) \cong H^{d-k}(M, M \setminus Y) \to H^{d-k}_{\mathrm{ct}}(M) \cong H_k(M)$$

Lemma 6.2.4. These two constructions agree.

Notation 6.2.5. We will write ε_Y for the cohomology class dual to an oriented compact submanifold Y.

Proposition 6.2.6. If Y, Z are oriented compact smooth submanifolds of M, which meet transversely, then

$$\varepsilon_{Y\cap Z} = \varepsilon_Y \cdot \varepsilon_Z$$

Remark 6.2.7. $\varepsilon_Y \cdot \varepsilon_Z = (-1)^{\operatorname{codim}(Y)\operatorname{codim}(Z)} \varepsilon_Z \cdot \varepsilon_Y$. Recall

$$\nu_{Y\cap Z} = \nu_Y \oplus \nu_Z$$

and so co-orientations of Y, Z and the ordering of Y and Z induce an orientation on $Y \cap Z$. This makes the above an oriented isomorphism of vector bundles. Thus the proposition fits with skew-commutativity.

Example 6.2.8

If M is oriented, a point $p \in M$ is co-oriented. Thus, it has a well defined class $\varepsilon_p \in \mathrm{H}^d_{\mathrm{ct}}(M)$ which is the orientation generator.

Thus, if $Y \cap Z$ is a transverse intersection, then $\varepsilon_{Y\cap Z}$ is non-zero. Hence $\varepsilon_Y, \varepsilon_Z$ is non-zero. One example of this for Σ_2,

picture

In particular, we can use the loops to compute the cohomology ring structure.

Proof of proposition 6.2.6. If $E \to X$, $F \to X$ are oriented vector bundles, then the relative cross product defines a map

$$\mathrm{H}^i(E, E^\sharp) \otimes \mathrm{H}^j(F, F^\sharp) \to \mathrm{H}^{i+j}(E \oplus F, (E \oplus F)^\sharp)$$

Noting that $(E \times F)|_{\Delta_X} = E \oplus F$. Moreover, we have that

$$u_{E\oplus F} = u_E \times u_F$$

under this map, since we have an isomorphism

$$\mathrm{H}^i(\mathbb{R}^i, \mathbb{R}^i \setminus 0) \otimes \mathrm{H}^j(\mathbb{R}^j, \mathbb{R}^j \setminus 0) \cong \mathrm{H}^{i+j}(\mathbb{R}^{i+j}, \mathbb{R}^{i+j} \setminus 0)$$

Now

$$\varepsilon_{Y\cap Z} = u_{\nu_{Y\cap Z}} = u_{\nu_Y} \cdot u_{\nu_Z} = \varepsilon_Y \cdot \varepsilon_Z$$

□

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Proof of lemma 6.2.4. We have that the diagram

$$\begin{tikzcd}
\mathrm{H}_{n-k}(Y) \arrow[r, "\text{inclusion}"] \arrow[d, "="] & \mathrm{H}_{n-k}(U_Y) \arrow[r, "\mathrm{PD}"] \arrow[d] & \mathrm{H}^k_{\mathrm{ct}}(U_Y) \arrow[d, "\text{extension by zero}"] \\
\mathrm{H}_{n-k}(Y) \arrow[r, "i"] & \mathrm{H}_{n-k}(M) \arrow[r, "\mathrm{PD}"] & \mathrm{H}^k_{\mathrm{ct}}(M)
\end{tikzcd}$$

commutes by out construction of D. So $i_*([Y])$ has the property that $D(i_*[Y])$ is in the image of $\mathrm{H}^k(U_Y) \cong \mathrm{H}^k(\nu_Y, \nu_Y^\sharp) \cong \mathbb{Z}$. Where for the last isomorphism we use the Thom isomorphism. Thus, $i_*[Y]$ and $D(\varepsilon_Y)$ must agree up to a sign. In fact, using our conventions for orientations, they agree. □

Corollary 6.2.9. If $i : Y^{n-k} \hookrightarrow M^n$ are as above, given $\alpha \in \mathrm{H}^{n-k}(M)$, then we can compute

$$\langle i^*\alpha, [Y]\rangle = \langle \varepsilon_Y \cdot \alpha, [M]\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between homology and cohomology.

So ε_Y behaves like a Dirac delta along Y.

Proof.

$$\left\langle \varepsilon_y \cdot \alpha, [M] \right\rangle = \alpha([M] \cap \varepsilon_Y) = \alpha = \alpha(D(\varepsilon_Y)) = \alpha(i_*[Y]) = i^*\alpha([Y])$$

□

Take coefficients in a field F, which we will omit from the notation. Then Poincaré duality for a compact F-oriented manifold M says that we have a non-degenerate pairing

$$\begin{aligned} \mathrm{H}^k(M) \otimes \mathrm{H}^{n-k}(M) &\to F \\ (\alpha, \beta) &\mapsto \alpha \cdot \beta = \langle \alpha \cdot \beta, [M] \rangle \end{aligned}$$

Let $\{a_i\}$ be a basis of $\mathrm{H}^*(M)$, $\{b_j\}$ the corresponding dual basis, so $a_i \cdot b_j = \delta_{ij}$. Note by Künneth that

$$\mathrm{H}^*(M \times M) \cong \mathrm{H}^*(M) \otimes_F \mathrm{H}^*(M)$$

Let $\Delta \subseteq M \times M$ be the diagonal. Then we have

$$\varepsilon_\Delta \in \mathrm{H}^*(M) \otimes \mathrm{H}^*(M)$$

Lemma 6.2.10.

$$\varepsilon_\Delta = \sum_i (-1)^{|b_i|} a_i \otimes b_i$$

Proof. Note that by non-degeneracy of the cup product, it suffices to show both sides evaluaed agains $b_k \otimes a_\ell$ gives the same result. We can write

$$\varepsilon_\Delta = \sum_{i,j} c_{ij} a_i \otimes b_j$$

for some coefficients c_{ij}. So

$$\begin{aligned} \langle \varepsilon_\Delta \cdot (b_k \otimes a_\ell), [M \times M] \rangle &= \sum c_{ij} \left\langle (a_i \otimes b_j) \cdot (b_k \otimes a_\ell), [M \times M] \right\rangle \\ &= \sum (-1)^{\left|b_j\right||b_k|} c_{ij} \left\langle a_i b_k \otimes b_j a_\ell, [M \times M] \right\rangle \\ &= \sum c_{ik} (-1)^{\left|b_j\right||b_\ell|} \delta_{ik} \delta_{j\ell} (-1)^{|a_\ell|\left|b_j\right|} \\ &= (-1)^{|b_\ell|(|b_k| + |a_\ell|)} c_{k\ell} \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \varepsilon_\Delta \cdot (b_k \otimes a_\ell), [M \times M] \rangle &= \langle b_k \otimes a_\ell, [\Delta] \rangle \\ &= (-1)^{|a_k||b_\ell|} \delta_{k\ell} \end{aligned}$$

Hence

$$c_{k\ell} = (-1)^{|b_k||b_\ell|} \delta_{k\ell} = (-1)^{|b_k|} \delta_{\ell k}$$

as required. □

Warning: Many books (e.g. Milnor Characteristic Classes) gives

$$\varepsilon_\Delta = \sum_i (-1)^{|a_i|} a_i \otimes b_i$$

which is $(-1)^{\dim(M)}$ times what we said. They define cap product so that

$$\langle a, \delta \frown b \rangle = \langle a \cdot b, \delta \rangle$$

That is, the define

$$\sigma \frown \phi$$

by feeding the *back* face of σ into ϕ, whereas we fed the front face. This changes the isomorphism D.

6.3 Fixed points

Let M be a closed smooth manifold, oriented over F.

Definition 6.3.1 (non-degenerate fixed points)
If $f : M \to M$ is smooth, we say f *has nondegenerate fixed points* if Γ_f and Δ intersect transversely in $M \times M$.

In this case, $\Gamma_f \cap \Delta$ is a finite set. The *sign* of a non-degenerate fixed point is

$$\operatorname{sign}(x) = \operatorname{sign}(\det(\mathrm{id} - \mathrm{d}f_x))$$

Then at (x, x),

$$\mathrm{T}_{(x,x)}\Gamma_f \oplus \mathrm{T}_{(x,x)}\Delta = \mathrm{T}_{(x,x)}(M \times M)$$

The sum is direct by dimension counting. Now consider the map

$$F + \Delta : \mathrm{T}M \oplus \mathrm{T}M \to \mathrm{T}M \oplus \mathrm{T}M$$

where $F(x, x) = (x, f(x))$. This has

$$D(F + \Delta) = \begin{pmatrix} I & \mathrm{d}f \\ I & I \end{pmatrix}$$

and so the sign of the fixed point says whether $F + \Delta$ has orientation preserving determinant at (x, x) or not. I'm not sure what the above is even supposed to mean. In any case, the sign represents whether $\mathrm{d}f_x - \mathrm{id} : \mathrm{T}_x M \to \mathrm{T}_x M$ is orientation preserving or not. Intuitively what we are interested in is the intersection number (of submanifolds) of Γ_f and Δ, and so the sign tells us the orientation of the intersection.

In this case,

$$\varepsilon_{\Gamma \cap \Delta} = \sum_{x \in \mathrm{Fix}(f)} \operatorname{sign}(x) \varepsilon_x$$

Definition 6.3.2 (Lefschetz number)
The *Lefschetz number* of $f : M \to M$ is

$$\mathrm{STr}(f) = L(f) = \sum_{k \geq 0} (-1)^k \operatorname{tr}\left(f^* : \mathrm{H}^k(M) \to \mathrm{H}^k(M)\right)$$

Theorem 6.3.3 (Lefschetz fixed point)**.** If f has non-degenerate fixed points, then

$$L(f) = \sum_{x \in \mathrm{Fix}(f)} \operatorname{sign}(x)$$

Proof. We've observed

$$\begin{aligned}
\sum_{x \in \mathrm{Fix}(f)} \operatorname{sign}(x) &= \varepsilon_{\Gamma(f) \cap \Delta} \\
&= \langle \varepsilon_{\Gamma_f} \cdot \varepsilon_{\delta}, [M \times M] \rangle \\
&= \left\langle i_{\Gamma_f}^* \varepsilon_\Delta, [\Gamma_f] \right\rangle \\
&= \langle (\mathrm{id} \times f)^* \varepsilon_\Delta, M \rangle \\
&= \sum_i (-1)^{|b_i|} \langle a_i \otimes f^* b_i, [M] \rangle
\end{aligned}$$

Write $f^* b_i = \sum q_{ij} b_j$. Then $\langle a_i \cdot f^* b_i, [M] \rangle = q_{ii}$ since the a_i, b_i are dual bases. So this q_{ii} is the ii-th entry of f in the b_j basis. ☐

Example 6.3.4

Any map $f : \mathbb{CP}^{2k} \to \mathbb{CP}^{2k}$ has a fixed point. In particular, no non-trivial group can act freely on $\mathbb{CP}^{2k}$.

Suppose f is is smooth with non-degenerate fixed points. Then

$$\mathrm{H}^*(\mathbb{CP}^{2k}) = \frac{\mathbb{Z}[x]}{x^{2k+1}}$$

Suppose $f^*(x) = \ell x$ for some $\ell \in \mathbb{Z}$. But then $f^*(x) = \ell^i x^i$ for all i. These all live in even degree, and so the Lefschetz number is

$$L(f) = 1 + \ell + \ell^2 + \cdots + \ell^{2k}$$

which is non-zero for any ℓ.

Now suppose $f : \mathbb{CP}^{2k} \to \mathbb{CP}^{2k}$ is continuous and has no fixed points. Then we have a nearby smooth $\tilde{f}$ which still has no fixed points. Contradiction.

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