Analysis of PDEs

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Contents

Lecture 1

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1 Basics

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set. A *PDE of order k* is an expression of the following form

$$F(x, u, Du, \dots, D^k u) = 0 \tag{1}$$

where $u : \mathcal{U} \to \mathbb{R}$ is the unknown, $F : \mathcal{U} \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^k}$ is a general function. We say that u is a *classical solution of* eq. (1) if u satisfies eq. (1) in $\mathcal{U}$.

We may also consider the case where $u(x) \in \mathbb{R}^p$ and $F \in \mathbb{R}^q$, and we call this a *system of PDEs*.

1.1 Examples of PDEs

Example 1.1.1 (ODE system)
One example of a model used in mathematical biology is the ODE system

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= f(u, v) \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= g(u, v) \end{aligned}$$

Example 1.1.2 (Laplace's equation)

$$\Delta u = \sum_{i=1}^{n} \partial_i^2 u = 0$$

The Laplacian is an *averaging* operator.

Example 1.1.3 (Heat equation)

$$u_t = D\Delta u$$

This is also called the diffusion equation, D is called the diffusion constant.

Example 1.1.4 (Navier-Stokes)
The Navier-Stokes equations in fluid dynamics is

$$\begin{aligned} u_t &= \nu \Delta u - u \cdot \mathrm{grad} u - \mathrm{grad} \rho + f \\ \mathrm{div}(u) &= 0 \end{aligned}$$

Example 1.1.5 (Transport equation)
The transport equation is

$$u_t + v u_x = 0$$

where v is a constant, corresponding to the velocity. A modification is the *advection-diffusion* equation,

$$u_t + v \cdot \mathrm{grad} u = D\Delta u + f$$

Example 1.1.6 (Poisson equation)

$$\Delta u = f$$

Describes electric field due to some charge, or Newtonian gravity.

Example 1.1.7 (Wave equation)

$$\Box u = -u_t t + c^2 \Delta u = 0$$

This models sound waves, seismic waves, ...

Example 1.1.8 (KdV equation)
This equation admits *soliton* solution.

$$u_t + \partial_x^3 u - 6u\partial_x u = 0$$

Example 1.1.9 (Maxwell equations)

$$\begin{aligned} \operatorname{div}(E) &= \rho \\ \operatorname{div}(B) &= 0 \\ \partial_t E &= \nabla \times B = J \\ \partial_t B &= -\nabla \times F \end{aligned}$$

Example 1.1.10 (Einstein's equations)

$$\operatorname{Ric}(g) - \frac{1}{2} g R(g) = 0$$

1.2 Data and Well-Posedness

All of the examples from above need additional information to solve, which we call the *data*. For example, we might need $u|_{\partial\mathcal{U}}$ and so on. A guiding principle to this process is called *well-posedness (in the sense of Hadamard)*.

We say that a PDE problem (equation and the data) is well-posed if we have

1. A solution exists (in some function space).
2. Given some data, the solution should be unique (depends on the function space of choice).
3. The solution depends continuously on the data.

The aim is to find the largest space for which a solution exists, but small enough so that it is unique. (For example, strong cosmic censorship in GR?)

Notation 1.2.1 (Multi-index notation)**.** We will use multi-index notation,

$$\mathbb{N} = \{0, 1, \dots\}$$

$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is called a *multi-index*, and we define the *order* of α

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

and the α-th derivative is

$$D^\alpha f(x) = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$$

If $x = (x_1, \ldots, x_n)$, then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and
$$\alpha! = \alpha_1! \cdots \alpha_n!$$

1.3 Classifying PDEs (of order k)

We say eq. (1) is *linear* if F is a linear function of x, u and its derivative. That is, we can write it as

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x)$$

Moreover, we say that a linear PDE is *homogeneous* if $f = 0$. We say eq. (1) is *semilinear* if the highest order derivatives appear linearly with coefficients depending only on x. That is, we have

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u(x) + F(x, u, Du, \ldots, D^{k-1}u) = 0$$

One example would be $\Delta u = u_x^2$. Finally, we say eq. (1) is *quasilinear* if the highest order derivatives appear linearly, but the coefficients depending on lower order derivatives. That is, we have

$$\sum_{|\alpha|=k} a_\alpha(x, u, \ldots, D^{k-1}u) D^k u + F(x, u, \ldots, D^{k-1}u) = 0$$

For example, we can have $uu_{xx} + u_{yy} - u_x^2 = 0$. Finally, we say eq. (1) is *fully non-linear* if it is non of the above.

2 Cauchy-Kovalevskaya theorem

2.1 ODE theory

Fix $\mathcal{U} \subseteq \mathbb{R}^n$ open, and suppose $f : \mathcal{U} \to \mathbb{R}^n$ is given. We would like to consider the ODE

$$\begin{cases} \dot{u}(t) = f(u(t)) \\ u(0) = u_0 \end{cases} \tag{2}$$

Theorem 2.1.1 (Picard-Lindelöf)**.** Suppose we have $r, K > 0$ such that $B_r(u_0) \subseteq U$ and

$$\|f(x) - f(y)\| \le K\|x - y\|$$

for all $x, y \in B_r(u_0)$. Then there exists $\varepsilon > 0$, depending on K, r and a unique C^1 solution $u : (-\varepsilon, \varepsilon) \to \mathcal{U}$ solving eq. (2).

Lecture 2

Sketch proof, see examples sheet 1. If $u \in \mathbb{C}^1$ solves eq. (2), then by the fundamental theorem of calculus, u satisfies the *weak formulation*

$$u(t) = u_0 + \int_0^t f(u(s)) \mathrm{d}s \tag{3}$$

Moreover, if $u \in C^0$ is a solution to eq. (3), then it is a C^1 solution to eq. (2). Moreover, if u exists, then it is a fixed point of

$$G(w) = u_0 + \int_0^t f(w(s)) \mathrm{d}s$$

Let $\mathcal{S} = \left\{w : (-\varepsilon, \varepsilon) \to \overline{B_{r/2}(u_0)}\right\}$ continuous. We want to show that $\mathcal{S}$ is a complete metric space, $G : \mathcal{S} \to \mathcal{S}$ is a contraction for ε sufficiently small, and by the contraction mapping theorem G has a fixed point. □

Remark 2.1.2.

1. The solution (in general) can't be global. Consider for example

$$\dot{u}(t) = u(t)^2 \quad \text{with} \quad u(0) = u_0 > 0$$

Solutions to this equation blow up in finite time.

2. This does not apply to

$$\dot{u}(t) = \sqrt{u(t)} \quad \text{with} \quad u(0) = 0$$

There are two solutions. Note we *can* apply the Peano existence theorem.

Now suppose f is smooth, and we have $\dot{u}(t) = f(u(t))$ is C^1. By the chain rule,

$$\ddot{u}(t) = Df(u(t)) \cdot \dot{u}(t) = f_2(u(t), \dot{u}(t))$$

which is continuous. Hence $\ddot{u}$ is continuous, and so $u \in C^2$. Repeating this, we get that $u \in C^k$ for all k. That is, u is smooth.

In principle, given $u_0 = u(0)$, we can determine

$$u^{(k)}(0) = F_k\left(u, u', \dots, u^{(k-1)}\right)\Big|_{t=0}$$

and so we can write

$$\sum_{k\geq 0} \frac{u^{(k)}(0)}{k!} t^k$$

We call this a *formal power series solution*. Does our solution $u(t)$ agree with this? That is, do we have

$$u(t) = \sum_{k\geq 0} \frac{u^{(k)}(0)}{k!} t^k$$

in a neighbourhood of 0?

Theorem 2.1.3 (Cauchy-Kovalevskaya for simple ODEs). If $f(u)$ is real analytic in a neighbourhood of u_0, then the series

$$\sum_{k\geq 0} \frac{u^{(k)}(0)}{k!} t^k$$

converges in a neighbourhood of 0 to the unique solution of eq. (2) given by Picard-Lindelöf.

2.2 Real analyticity and majorants

Suppose $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is a smooth function. Therefore, $f^{(n)}(0)$ exists for all $n \geq 0$. Does the partial sums

$$\sum_{n\geq 0} \left|f^{(n)}(0)\right| n! x^n$$

for some $|x| \leq \delta$? No, consider the function

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

This is a smooth function, with $f^{(n)}(0) = 0$ for all n.

Definition 2.2.1 (real analytic)
Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and $f : \mathcal{U} \to \mathbb{R}$ is *real analytic* (at x_0) if there exists $r > 0$, $f_\alpha \in \mathbb{R}$ such that

$$f(x) = \sum_\alpha f_\alpha (x - x_0)^\alpha$$

when $|x - x_0| < r$.

Remark 2.2.2. 1. That is, f can be written as a convergent power series and

$$f_\alpha = \frac{D^\alpha f(x_0)}{n!}$$

2. Real analyticity is a local property.
3. f is real analytic on an open set U if it is real analytic at each $x_0 \in U$.
4. We will denote the set of real analytic functions on $\mathcal{U}$ by $C^\omega(\mathcal{U})$.
5. If f is C^ω, then f is smooth (e.g. Weierstrass M-test).
6. If f is real analytic, and $\mathcal{U}$ is connected, then f is uniquely determined in $\mathcal{U}$ by its derivatives $D^\alpha f(x)$ at some point $x \in \mathcal{U}$.
7. In particular, f is real analytic if and only if for any compact $K \subseteq U$, there exists C, r such that

$$\sup_{x\in K} |D^\alpha f(x)| \le C||\alpha|!|r^{|\alpha|}$$

Exercise: Show $f(x) = 1/x$ and $f(x) = \sqrt{x}$ are real analytic for $x > 0$.

Example 2.2.3

Recall

$$\frac{1}{1-x} = \sum_{k\ge 0} x^k$$

for $|x| < 1$. Let $r > 0$, and consider

$$f(x) = \frac{r}{r-(x_1+\cdots+x_n)} = \frac{1}{1-\frac{x_1+\cdots+x_n}{r}} = \sum_{k\ge 0}\left(\frac{x_1+\cdots+x_n}{r}\right)^k$$

provided $|x_1+\cdots+x_n| \le \sqrt{n}\left(\sum_j |x_j|^2\right)^{1/2} = \sqrt{n}||x|| < r$. By the multinomial theorem (sheet 1),

$$f(x) = \sum_{k\ge 0}\frac{1}{r^k}\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}x^\alpha = \sum_\alpha \frac{|\alpha|!}{\alpha! r^{|\alpha|}}x^\alpha$$

where

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}$$

and so,

$$f(x) = \sum_\alpha f_\alpha x^\alpha$$

where

$$f_\alpha = \frac{|\alpha|!}{\alpha!}\frac{1}{r^{|\alpha|}}$$

This series is absolutely convergent near zero, since

$$\sum_\alpha \frac{|\alpha!|}{\alpha!}\frac{x^\alpha}{r^{|\alpha|}} = \sum_{k\ge 0}\left(\frac{|x_1|+\cdots+|x_n|}{r}\right)^k < \infty$$

Definition 2.2.4 (majorise)

Let $f = \sum_\alpha f_\alpha x^\alpha$ and $g = \sum_\alpha g_\alpha x^\alpha$, we say that g *majorises* f, or g *is a mojorant of* f, written $g \gg f$, if $g_\alpha \ge |f_\alpha|$ for all α.

For vector valued functions, we require each component to be a majorant.

Lemma 2.2.5 (properties of majorants).

1. If $g \gg f$, and g converges for $||x|| < r$, then f converges for $||x|| < r$.
2. If $f = \sum f_\alpha x^\alpha$ converges for $||x|| < r$, then for any $s \in (0, r/\sqrt{n})$, there exists a majorant of f which converges for $||x|| < s/\sqrt{n}$.

Proof. 1. Looking at the partial sums

$$\sum_{|\alpha| \leq k} |f_\alpha x^\alpha| = \sum_{|\alpha| \leq k} |f_\alpha||x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} \leq \sum_{|\alpha| \leq k} |g_\alpha||x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} \leq \sum_{\alpha} |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} = g(\tilde{x})$$

where $\tilde(x) = (|x_1|, \dots, |x_n|)$. So $\left\|\tilde{x}\right\| = ||x||$, and so if $||x|| < r$, then g converges at $\tilde{x}$. That is, $g(\tilde{x}) < \infty$. Therefore, we have a uniform bound on the partial sum.

2. Let $s \in (0, r/\sqrt{n})$, and set $y = (s, \dots, s)$. Then $\left\|y\right\| = s\sqrt{n}$, and by assumption,

$$f(y) = \sum_{\alpha} f_\alpha y^\alpha$$

converges, as $\left\|y\right\| = s\sqrt{n} < r$. So there exists a constant c such that $|f_\alpha y^\alpha| \leq c$. Hence

$$|f_\alpha| \leq \frac{C}{|y^\alpha|} = \frac{C}{|y_1|^{\alpha_1} \cdots |y_n|^{\alpha_n}} = \frac{C}{|s|^{|\alpha|}} \leq \frac{C}{|s|^{|\alpha|}} \frac{|\alpha|!}{\alpha!}$$

So we define

$$g(x) = \frac{Cs}{s - (x_1 + \cdots + x_n)}$$

From the above, g majorises f. □

Lecture 3

2.3 Proof of Cauchy-Kovalevskaya for ODEs

Proof of theorem 2.1.3. We will use the method of majorants. Without loss of generality $u_0 = 0$, and for simplicity, we can assume $n = 1$. We need to find the series coefficients. So

$$\dot{u} = f(u)$$

and so $\dot{u}(0) = f(u(0)) = f(0)$, that is, $u_1 = f(0)$. Next,

$$\ddot{u}(t) = f'(u(t))\dot{u}(t)$$

and so $\ddot{u}(0) = f'(0)f(0)$, that is, $u_2 = f'(0)f(0) = \frac{1}{2!}f'(0)f(0)$. Repeating,

$$u^{(3)}(0) = f''(0)f(0)^2 + f'(0)^2 f(0)$$

and so

$$u_3 = \frac{1}{3!}(f''(0)f(0)^2 + f'(0)^2 f(0))$$

Iterating this procedure,

$$u_k = P_k\left(f(0), \dots, f^{(k-1)}(0)\right)$$

where P_k is a polynomial in k-variables, with nonnegative coefficients. For example,

$$P_1(x) = x$$
$$P_2(x, y) = \frac{1}{2!}xy$$
$$P_3(x, y, z) = \frac{1}{3!}(x^2 z + xy^2)$$

Since f is real analytic, we have that

$$f(v) = \sum_{k \geq 0} f_k v^k$$

where $f_k = \frac{1}{k!} f^{(k)}(0)$. Hence we have that

$$f^{(k)}(0) = k! \cdot f_k$$

Substituting, we have that

$$u_k = Q_k(f_0, \dots, f_{k-1})$$

which again is a polynomial in k-variables and nonnegative coefficients. This polynomial is "universal".

Aim: We would like to show that the power series

$$\sum_k u_k t^k$$

converges in a neighbourhood of $t = 0$, and solves the ODE, eq. (2). Since f is analytic, we know that

$$f(u) = \sum_k f_k u^k$$

on $|u| < k$. Fixing some $s < r$, there exists a majorant

$$g(u) = \sum_k g_k u^k$$

of f, from lemma 2.2.5 (ii). Consider the auxilliary differential equation

$$\dot{w}(t) = g(w(t))$$

and $w(0) = 0$. Using the definition of g, we that

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{Cs}{s - w(t)}$$

We get that

$$w = s \pm \sqrt{s^2 - 2Cst}$$

Due to the initial data, we take the $-$ solution. That is,

$$w = s - \sqrt{s^2 - 2Cst}$$

This is real analytic, for $|t| < s/2C$. This tells us that

$$w(t) = \sum_k w_k t^k$$

converges for $|t| < s/2C$. Moreover,

$$w_k = Q_k(g_0, \dots, g_{k-1})$$

since Q_k is "universal".

Claim 2.3.1. w majorises u.

By construction, g majorises f, i.e. $g_k \geq |f_k|$ for all k. Moreover, since Q_k has nonnegative coefficients,

$$w_k = Q_k(g_0, \dots, g_{k-1}) \geq Q_k(|f_0|, \dots, |f_{k-1}|) \geq |Q_k(f_0, \dots, f_{k-1})| = |u_k|$$

Hence by lemma 2.2.5 (i), we know that the series

$$\sum_k u_k t^k$$

converges for $|t| < s/2C$.

To conclude, set

$$u(t) := \sum_{k \geq 0} u_k t^k$$

and we need to check that it solves eq. (2). Both sides are analytic, so suffices to check the derivatives on each side agree to all orders at $t = 0$. □

Remark 2.3.2. 1. We can extend to systems, where we replace u_k with

$$u_k^j = Q_k^j(D^{|\alpha|} \mid |\alpha| \leq k)$$

For w, we can replace $w^j = w^1$ as before.

2. For the non-autonomous case,

$$u(t) = f(u, t)$$
$$u(0) = 0$$

Consider $v(t) = (u(t), t)$, then $\dot{v}(t) = (\dot{u}(t), 1) = (f(u, 1), 1) = (f(v), 1) = F(v)$ with $v(0) = 0$, and we can apply the system version.

For the PDE version, see the handout.

2.4 Cauchy-Kovalevskaya for PDEs

Let $u : \mathbb{R}^n \to R^m$, and choose $r > 0$. Consider the following problem

$$u_t = \sum_{j=1}^{n} B_j(u, x)u_{x_j} + C(u, x)$$

on $||x||^2 + t^2 < r^2$, with

$$u = 0$$

on $||x||^2 + t^2 < r^2$ and $t = 0$. The B_j are matrices, B_j and C are real analytic.

Theorem 2.4.1 (Cauchy-Kovalevskaya for first order systems)**.** Suppose B_j, C are real analytic, for small $r > 0$. Then there exists a unique real analytic function

$$u = \sum_{\alpha} u_\alpha x^\alpha$$

solving the above PDE.

Idea. Compute

$$u_\alpha = \frac{D^\alpha u}{\alpha!}$$

in terms of B_j, C, and show that the power series converges for small r. We use the PDE to find all derivatives. □

Example 2.4.2

Consider the system

$$u_t = v_x - f$$
$$v_t = -u_x$$

with $u = v = 0$ on $t = 0$. The boundary conditions give us that

$$u(0, 0) = v(0, 0) = 0$$

We would like to determine u_α for all α. By differentiating the boudary conditions,

$$\partial_x^n u(x, 0) = \partial_x^n v(x, 0) = 0$$

for all n. That is, for the case $\alpha = (n, 0)$. From the PDE,

$$u_t(x, 0) = 0 - f = -f \qquad v_t(x, 0) = 0$$

This then means that
$$\partial_x^n \partial_t u(x,0) = -\partial_x^n f(x,0)$$
and
$$\partial_x^n \partial_t v(x,0) = 0$$
for all $n \geq 1$.

Next, if $\alpha = (n,2)$ use the PDE and we get
$$u_{tt}(x,0) = f_t(x,0)$$
and
$$v_{tt}(x,0) = f_x(x,0)$$
The same method as above gives us that
$$\partial_x^n \partial_t^2 u(x,0) = -(\partial x)^n \partial_t f(x,0)$$
$$\partial_x^n \partial_t^2 v(x,0) = (\partial_x)^{n+1} f(x,0)$$
Repeating this, we can compute all of the derivatives.

Lecture 4

2.5 Reduction to first order systems

Example 2.5.1

Consider $u : \mathbb{R}^3 \to \mathbb{R}$ satisfying
$$u_{tt} = uu_{xy} - u_{xx} + u_t$$
with conditions
$$u|_{t=0} = u_0(x,y) \quad \text{and} \quad u_t|_{t=0} u_1(x,y)$$
where u_0, u_1 are real analytic near $0 \in \mathbb{R}^3$. Note
$$f(t,x,y) = u_0 + tu_1$$
is real analytic near $0 \in \mathbb{R}^3$. Note
$$f|_{t=0} = u_0 \quad \text{and} \quad \partial_t f|_{t=0} = u_1$$
Set $w(t,x,y) = u - f$. Then we find that
$$w_t t = ww_{xy} - w_{xx} + w_t + fw_{xy} + f_{xy} w + F$$
where
$$R = ff_{xy} - f_{xx} + f_t$$
where
$$w|_{t=0} = \partial_t w|_{t=0} = 0$$
Observe F is real analytic, and independent of w and its derivatives. Let $x = (x,y,t) = (x^1, x^2, x^3)$, and set
$$v = (w, w_x, w_y, w_z) = (v^1, v^2, v^3, v^4)$$
Then
$$\begin{aligned} v_t^1 &= w_t = v^4 \\ v_t^2 &= w_{xt} = v_{x_1}^4 \\ v_t^3 &= w_{yt} = v_{x_2}^4 \\ v_t^4 &= v^1 v_{x_2}^2 - v_{x_1}^2 + v^4 + f v_{x_2}^2 + f_{xy} v^1 + F \end{aligned}$$

Define

$$B_1 = \begin{pmatrix} & & & 0 \\ & & & 1 \\ & & & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} & & & \\ & & & \\ & & & 1 \\ v^1 + f & 0 & 0 & 0 \end{pmatrix}$$

and

$$c = \begin{pmatrix} v^4 \\ 0 \\ 0 \\ v^4 + f_{xy} v^1 + F \end{pmatrix}$$

We can see then

$$\frac{\partial}{\partial x_3} v = B_1 v_{x_1} + B_2 v_{x_2} + c$$

In this case, B_1, B_2, c are real analytic functions of x, v, and so we can apply Cauchy-Kovalevskaya.

More generally, consider the scalar quasilinear problem

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, u, x) D^\alpha u + a_0(D^{k-1}u, \dots, u, x) = 0$$

where

$$u : B_r(0) \to \mathbb{R} \quad \text{and} \quad \frac{\partial u}{\partial x_n} = \cdots = \left(\frac{\partial}{\partial x_n}\right)^{k-1} u = 0$$

for $\|x'\| < r, x_n = 0$.

Define

$$v = \left(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \left(\frac{\partial}{\partial x_n}\right)^{k-1} u\right) = (v^1, \dots, v^m) \in \mathbb{R}^m$$

We would like to get a first order system in v. That is, express

$$\frac{\partial v^j}{\partial x_n}$$

in terms of v^j and $\frac{\partial v}{\partial x_j}$ for $j = 1, \dots, m-1$. If $j = 1$, then

$$\frac{\partial v^1}{\partial x_n} = \frac{\partial u}{\partial x_n} = v^\ell$$

for some ℓ. If $2 \le j \le m-1$, then

$$v^j = D^\alpha u$$

for some $|\alpha| \le k-1$, such that $\alpha_n < k-1$. So

$$\frac{\partial v^j}{\partial x_n} = D^\alpha \frac{\partial u}{\partial x_n} = \frac{\partial^{|\alpha|+1}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n+1}} u$$

If $|\alpha| \le k-2$, then $|\alpha| + 1 \le k-1$,a nd so $\frac{\partial v^j}{\partial x_n} = v^\ell$ for some ℓ.

If $|\alpha| = k-1$, and $\alpha_n < k-1$, then there exists $p \neq n$ such that $\alpha_p \ge 1$. So we have that

$$\frac{\partial v^j}{\partial x_n} = \frac{\partial}{\partial x_n}\left(\frac{\partial^{|\alpha|} u}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}}\right) = \frac{\partial}{\partial x_p}\left(\frac{\partial^{|\alpha|} u}{\partial_1^{\alpha_1} \cdots \partial_p^{\alpha_p - 1} \cdots \partial_n^{\alpha_n - 1}}\right) = \frac{\partial v^\ell}{\partial x_p}$$

for some ℓ. Finally, to compute

$$\frac{\partial v^m}{\partial x_n}$$

we will use the PDE. Recall the coefficients are $a_\alpha(v, x)$ for $v \in \mathbb{R}^m, x \in \mathbb{R}^n$. We assume $a_\alpha : B_\rho(0) \to \mathbb{R}$ is real analytic, and suppose

$$a_c = a_{(0,\dots,0,k)}(0) \neq 0$$

Since a_α are real analytic near zero, they are continuous. Therefore, $a_c(z, w) \neq 0$ for all $||z||^2 + ||w||^2 \leq \delta^2$, where $\delta < \rho$. Then

$$a_c \frac{\partial^k u}{\partial x_n^k} = -\left(\sum_{|\alpha|=k, \alpha_n<k} a_\alpha D^\alpha u + a_0 \right)$$

Dividing by a_c, we get

$$\frac{\partial^k u}{\partial x_n^k} = -\frac{1}{a_c} \left(\sum_{|\alpha|=k, \alpha_n<k} a_\alpha D^\alpha u + a_0 \right)$$

The right hand side can be written in terms of

$$\frac{\partial v^\ell}{\partial x_p}, v$$

for $p < n$. Therefore, if $a_c \neq 0$, we have turned the scalar quasilinear PDE into a first order PDE system, on which we can apply Cauchy-Kovalevskaya.

Definition 2.5.2

If $a_c = a_{(0,\dots,0,k)}(0)$ is non-zero, then we say the plane

$$\{x^n = 0\}$$

is *non-characteristic*. Otherwise, it is *characteristic*.

2.6 Exotic boundary conditions

Definition 2.6.1 (real analytic hypersurface)

We say $\Sigma \subseteq \mathbb{R}^n$ is a *real analytic hypersurface near* $x_0 \in \Sigma$ if there exists $\varepsilon > 0$, and a real analytic function

$$\Phi : B_\varepsilon(x_0) \to U \subseteq \mathbb{R}^n$$

where U is an open neighbourhood of $0 \in \mathbb{R}^n$, and defining $y = \Phi(x)$, with $\Phi(x_0) = 0$. Moreover, we require

(i) Φ is a bijection,

(ii) $\Phi^{-1} : U \to B_\varepsilon(x_0)$ is real analytic,

(iii) $\Phi(\Sigma \cap B_\varepsilon(x_0)) = \{y_n = 0\} \cap U$.

We can think of Φ as "straightening out" Σ.

Example 2.6.2

Spheres, planes, tori, etc. are real analytic (hyper)surfaces.

Let γ be unit normal to Σ, and consider

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \dots, u, x)D^\alpha u + a_0(D^{k-1}u, \dots, u, x)$$

where

$$u = \sum_i \gamma^i \partial_i u = (\gamma^i \partial_i)^{k-1} u = 0 \text{ on } \Sigma \tag{4}$$

Define

$$v(y) = u(\Phi^{-1}(y))$$

for $y \in U$. That is, $u(x) = v(\Phi(x))$ for $x \in B_\varepsilon(x_0)$. Using the chain rule,

$$\frac{\partial u}{\partial x_i} = \sum \frac{\partial u}{\partial y_j} \frac{\partial \Phi^j}{\partial x_i}$$

where $\Phi = (\Phi^1, \dots, \Phi^n) \in \mathbb{R}^n$. So the PDE becomes

$$\sum_{|\alpha|=k} b_\alpha D^\alpha v + b_0 = 0$$

where b_0, b_α dependis on u and $D^\alpha u$ for $|\alpha| \leq k-1$, and also Φ (which is given). The boundary conditions becomes

$$v = \frac{\partial}{\partial y_n} v = \dots = \left(\frac{\partial}{\partial y_n}\right)^{k-1} v = 0$$

on $\{y_n = 0\}$. Since Φ is real analytic, so are b_0, b_α.

Lecture 5

We would like to apply Cauchy-Kovalevskaya, therefore we need to check that whether the hypersurface $\{y_n = 0\}$ is non-characteristic. That is,

$$b_{(0,\dots,0,k)}(D^{k-1}v = 0, \dots, Dv = 0, y = 0) \neq 0$$

Note if $|\alpha| = 2$, we can compute

$$D^\alpha u = \frac{\partial^k v}{\partial y_n^k}(D\Phi^n)^\alpha + \text{terms not involving} \frac{\partial^k v}{\partial y_n^k}$$

For example, if $k = 2, n = 2, \alpha = (0, 2)$, then

$$D^\alpha u = u_{x_2 x_2} = v_{y_2 y_2} \underbrace{(\Phi^2_{x_2})(\Phi^2_{x_2})}_{=(D\Phi^2)^\alpha} + \text{terms not involving} v_{y_2 y_2}$$

Thus,

$$b_{(0,\dots,0,k)} = \sum_{|\alpha|=k} a_\alpha \left(D\Phi^n\right)^\alpha$$

Definition 2.6.3

We say that Σ is *non-characteristic at* $x_0 \in \Sigma$ if

$$b_{(0,\dots,0,k)}(0) = \sum_{|\alpha|=k} a_\alpha(0, \dots, 0, x_0) \left(D\Phi^n(x_0)\right)^\alpha \neq 0$$

Otherwise, Σ is characteristic at x_0.

Remark 2.6.4. Note that $\Sigma = \left\{x \in \mathbb{R}^n \mid \Phi^{(n)}(x) = y_n = 0\right\}$. This tells us that

$$D\Phi^n(x) = c(x)\gamma(x)$$

where γ is the unit normal of Σ. In particular,

$$D\Phi^n(x_0) = c(x_0)\gamma(x_0)$$

and so the non-characteristic condition is equivalent to

$$\sum_{|\alpha|=k} a_\alpha \gamma^\alpha(x) \neq 0$$

Theorem 2.6.5 (Cauchy-Kovalevskaya for non-characteristic hypersurfaces). Suppose $\Sigma \subseteq \mathbb{R}^n$ is a hypersurface, with normal γ, and consider the PDE eq. (4) as above. Suppose a_α, a_0 are real analytic near $x_0 \in \Sigma$, and Σ is non-characteristic near x_0. Then there exists a unique real analytic solution in a neighbourhood of x_0.

2.7 Characteristic surfaces

Consider the linear operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

where $a_{ij} \in \mathbb{R}$. Without loss of generality, we can assume $a_{ij} = a_{ji}$. Consider the PDE problem

$$\begin{cases} Lu = f \\ u = \gamma^i \partial_i u = 0 \quad \text{on } \Pi_\gamma = \{x \mid x \cdot \gamma = 0\} \end{cases} \tag{5}$$

That is, the boundary conditions are on the plane with unit normal γ. In particular, Π_γ is non-characteristic for eq. (5) if

$$\sum_{i,j=1}^{n} a_{ij} \gamma^i \gamma^j \neq 0$$

We would like to find non-characteristic Π_γ. Note that the left hand side is just $\langle A\gamma, \gamma \rangle$, where $A = (a_{ij})$ is a symmetric matrix, with the usual Euclidean inner product. In particular, A is diagonalisable, say $A = P^\top \Lambda P$, where P is orthogonal and Λ is diagonal. Then

$$\langle A\gamma, \gamma \rangle = \left\langle P^\top \Lambda P \gamma, \gamma \right\rangle = \langle \Lambda v, v \rangle$$

where $v = P\gamma$. If $\{\lambda_i\}$ are the eigenvalues for A, then the non-characteristic condition becomes

$$\sum_{i=1}^{n} \lambda_i (v_i)^2 \neq 0$$

Example 2.7.1 (Laplacian)

$$L = \Delta = \sum_{i=1}^{n} \partial_{1,1}^2$$

gives

$$A = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

The Laplacian is an elliptic operator.

Example 2.7.2 (Wave equation)

$$L = \partial_t^2 + \Delta$$

gives

$$A = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

Case 1: All eigenvalues have the same sign (and are all non-zero). Since v is a unit vector, the characteristic condition is impossible. That is, there are no characteristic hyperplanes Π_γ. In this case, we call L an *elliptic operator*.

Case 2: Say $\lambda_n < 0$ and $\lambda_j > 0$ for $j \neq n$ (or vice versa). In this case, we call L an *hyperbolic operator*. In particular, the characteristic condition becomes

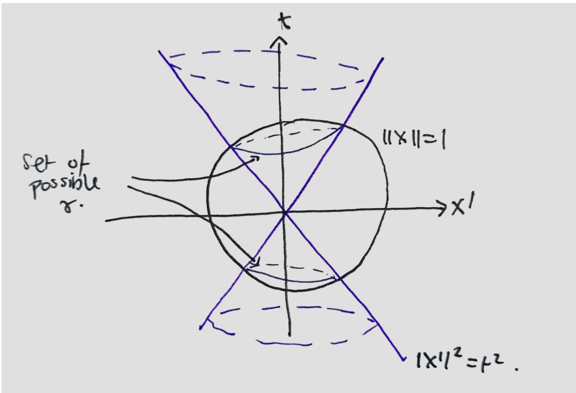
$$\sum \lambda_i v_i^2 = 0 \iff \lambda_1 v_1^2 + \sum_{j=2}^{n} \lambda_j v_j^2 = 0$$

Considering the wave equation again, we have the condition that

$$v_n^2 = \sum_{j=1}^{n-1} v_j^2$$

subject to the condition

$$||v|| = 1$$



Note that these cases are *not* exhaustive.

Now we want to different features of elliptic and hyperbolic operators. We will forget about boundary conditions, and look for solutions of the form

$$u(x) = e^{ik\cdot x}$$

for $k \in \mathbb{R}^n$. We are looking for *wave-like* solutions. Substituting,

$$L(e^{ik\cdot x}) = e^{ik\cdot x} = \sum_{j,\ell} a_{j\ell} k_j k_\ell$$

We would like to consider $Lu = 0$. Taking $k = c\gamma$, $\left\|\gamma\right\| = 1$, then the condition is equivalent to

$$c^2 \sum a_{j\ell}\gamma_j\gamma_\ell = 0$$

If L is elliptic, then the only solution is when $k = 0$. That is, there are no wave-like solutions. On the other hand, if L is hyperbolic, then we can have wave like solutions, that is,

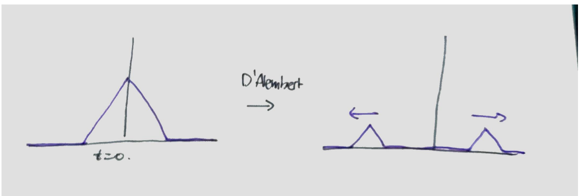
$$\sum a_{ij}\gamma_i\gamma_j = 0$$

has a solution with $\left\|\gamma\right\| = 1$. In this case, we have

$$u(x) = e^{i\lambda\gamma\cdot x}$$

gives an infinite family of solutions, indexed by $\lambda \in \mathbb{R}$.

As we take $|\lambda| \to \infty$, we see that $u'(x)$ can grow large. In particular, solutions can be rough.



By contrast, we will see that solutions to elliptic equations are smooth.

Lecture 6

Example 2.7.3
Consider the IVP for Laplace's equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = \varphi(x) \\ \partial_y u(x,0) = 0 \end{cases}$$

Is this problem well-posed? If $\varphi(x) = 0$, then 0 is a solution. On the other hand, we don't have Cauchy stability. Consider

$$u_k(x,y) = e^{-\sqrt{k}} \cos(kx)\cosh(ky)$$

See typed notes for more details.

3 Sobolev spaces

3.1 Hölder spaces $C^{k,\gamma}$

Let $U \subseteq \mathbb{R}^n$ be open, $k \in \mathbb{N}$.

Definition 3.1.1 (C^k spaces)
Define

$$C^k(U) = \{f : U \to \mathbb{R} \mid u \text{ is } k \text{ times continuously differentiable}\}$$

and define

$$C^k(\overline{U}) = \left\{u \in C^k(U) \mid u \text{ and its derivatives are bounded and uniformly continuous on } U\right\}$$

We will define the norm

$$\|u\|_{C^k(\overline{U})} = \sum_{|\alpha|\le k} \sup_{x\in U} |D^\alpha u(x)|$$

The idea is that $C^k(\overline{U})$ is the space of functions which can be extended continuously to ∂U. Note that this is is contained in, but not equal to

$$u : \overline{U} \to \mathbb{R} \text{ such that } u \text{ and its derivatives are continuous}$$

On examples sheet 2, we will show that $C^k(\overline{U})$ is a Banach space.

Definition 3.1.2 (Hölder continuous)
We say $u : U \to \mathbb{R}$ is *Hölder continuous of index* γ with $0 < \gamma \le 1$ if there exists a constant $C > 0$, such that

$$|u(x) - u(y)| \le C|x-y|^\gamma$$

for all $x, y \in U$.

If $\gamma = 1$, then we say that u is *Lipschitz continuous*.

Remark 3.1.3. If $\gamma > 1$, and u is Hölder continuous of index γ, then u is constant.

Definition 3.1.4 (0-Hölder space)
For $\gamma \in (0, 1]$, we define the 0-*Hölder space*:

$$C^{0,\gamma}(\overline{U}) = \left\{ u \in C^0(\overline{U}) \;\middle|\; u \text{ is } \gamma\text{-Hölder continuous} \right\}$$

Define the γ-Hölder seminorm by

$$[u]_{C^{0,\gamma}(\overline{U})} = \sup_{x,y \in U} \frac{|u(x) - u(y)|}{|x - y|}$$

That is, the smallest C such that u is γ-Hölder continuous. Since constant functions vanish, we add the C^0 norm, and define

$$||u||_{C^{0,\gamma}(\overline{U})} = ||u||_{C^0(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}$$

The space $C^{0,\gamma}$, with the Hölder norm $||\cdot||_{C^{0,\gamma}(\overline{U})}$ is a Banach space. We can extend this to higher order, that is,

Definition 3.1.5 (k-th Hölder space)
Define the *k-th Hölder space*

$$C^{k,\gamma}(\overline{U}) = \left\{ u \in C^k(\overline{U}) \mid D^\alpha u \in C^{0,\gamma}(\overline{U}) \text{ for all } |\alpha| = k \right\}$$

with norm

$$||u||_{C^{k,\gamma}(\overline{U})} = ||u||_{C^k(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})}$$

As above, $C^{k,\gamma}$ is a Banach space with the Hölder norm.

3.2 The Lebesgue spaces

Definition 3.2.1 (L^p space)
Let $U \subseteq \mathbb{R}^n$ be open, and suppose $1 \leq p \leq \infty$, define

$$L^p(U) = \frac{\left\{ f : U \to \mathbb{R} \mid f \text{ measurable, and with} ||u||_{L^p(U)} < \infty \right\}}{\sim}$$

where

$$||u||_{L^p} = \begin{cases} \left(\int_U |u(x)|^p \mathrm{d}x \right)^{1/p} & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x \in U} |u(x)| = \inf \left\{ C > 0 \mid |u(x)| \leq C \text{ a.e.} \right\} & p = \infty \end{cases}$$

and $u \sim v$ if $u = v$ almost everywhere.

$L^p(U)$ with the $||\cdot||_{L^p}$ norm is a Banach space. We also define local versions,

Definition 3.2.2 ($L^p_{\text{loc.}}$ space)
We say that $u \in L^p_{\text{loc.}}(U)$ if $f \in L^p(V)$ for every $V \Subset U$ ($V \subseteq K \subseteq U$, where K is compact). Equivalently,

$$L^p_{\text{loc.}}(U) = \bigcap_{V \Subset U} L^p(V)$$

Note that $L^p_{\text{loc.}}(U)$ is not Banach, on the other hand, it is a Fréchet space.

Remark 3.2.3. If $K \subseteq U$ is compact, U is open, then

$$d(K, \partial U) = \inf\{|x - y| \mid x \in K, y \in \mathbb{R}^n \setminus U\} > 0$$

We will use the space outside K as a "buffer zone".

3.3 Weak derivatives

That is, a notion of derivative for L^p.

Definition 3.3.1 (weak derivative)
Suppose $u, v \in L^1_{\text{loc.}}(U)$, α a multi-index. We say v is the *α-th weak derivative of U* if

$$\int_U u D^\alpha \phi \mathrm{d}x = (-1)^{|\alpha|} \int_U v \phi \mathrm{d}x$$

for all $\phi \in C_c^\infty(U)$. We will also call the space $C_c^\infty(U)$ the space of *test functions*.

Remark 3.3.2.
1. since $\text{supp}(D^\alpha \phi)$ and $\text{supp}(\phi)$ are compact, the integrals are finite,
2. u, v obey the correct integration by parts formula

Example 3.3.3
$u(x) = |x|$ is not differentiable at $x = 0$, but it is weakly differentiable with $v(x) = \text{sign}(x)$.

Lemma 3.3.4 (uniqueness of weak derivative)**.** Suppose $v, \tilde{v} \in L^1_{\text{loc.}}(U)$ are both the α-th weak derivative of $u \in L^1_{\text{loc.}}(U)$, then $v = \tilde{v}$ almost everywhere.

Proof. For all $\varphi \in C_c^\infty(U)$,

$$\int_U v \phi \mathrm{d}x = (-1)^{|\alpha|} \int_U u D^\alpha \phi \mathrm{d}x = \int_U \overline{v} \phi \mathrm{d}x$$

Thus,

$$\int_U (v - \overline{v}) \phi \mathrm{d}x = 0$$

for all $\phi \in C_c^\infty(U)$. Taking ϕ to be a smooth approximation to an indicator, we get the required result. ☐

Suppose u is smooth, then the weak derivative agrees with the usual derivative almost everywhere.

Notation 3.3.5. We will write $v = D^\alpha u$.

Definition 3.3.6 (Sobolev space)
Define the *Sobolev space*

$$W^{k,p}(U) = \left\{u \in L^1_{\text{loc.}}(U) \mid u \in L^p(U), \text{the weak derivatives } D^\alpha u \text{ exists for } |\alpha| \leq k,\ D^\alpha u \in L^p(U)\right\}$$

with the *Sobolev norm*

$$||u||_{W^{k,p}(U)} = \begin{cases} \left(\sum_{|\alpha|=k} \int_U |D^\alpha u|^p \mathrm{d}x\right)^{1/p} & p < \infty \\ \sum_{|\alpha| \leq k} \text{ess sup}_U |D^\alpha u| & p = \infty \end{cases}$$

When $p = 2$, we write $H^k = W^{k,2}$.

Definition 3.3.7 ($W_0^{k,p}$)

We denote by $W_0^{k,p}(U)$ the completion of $C_c^\infty(U)$ with respect to the $W^{k,p}$ norm. Analogously, we define $H_0^k = W_0^{k,2}(U)$.

The $_0$ denotes that the function vanishes on the boundary.

Lecture 7

Example 3.3.8

Let $n > 2, \lambda > 0$, and take $U = B_1(0) \subseteq \mathbb{R}^n$ the open ball. Consider

$$u(x) = \begin{cases} |x|^{-\lambda} & \text{for } x \neq 0 \\ \text{anything} & x = 0 \end{cases}$$

When is $u \in W^{p,1}(U)$?

First of all, we compute

$$\int_U \frac{1}{|x|^\lambda} \mathrm{d}x = C \int_0^1 r^{n-1-\lambda} \mathrm{d}r$$

which is finite if and only if $\lambda < n$. Moreover, $u \in L^p(U)$ if and only if $\lambda p < n$.

Let $\phi \in C_c^\infty(B_1(0) \setminus \{0\})$, if u has a weak derivative v, then

$$v_i = D_i u = -\frac{\lambda x_i}{|x|^{\lambda+2}}$$

on $B_1(0) \setminus 0$. Thus,

$$|Du| = \frac{|\lambda|}{|x|^{\lambda+1}}$$

Hence $v_i \in L^1\text{loc.}(U)$ if and only if $\lambda + 1 < n$. Suppose $\lambda + 1 < n$, then we claim that

$$v_i = \begin{cases} -\frac{\lambda x_i}{|x|^{\lambda+1}} & x \neq 0 \\ \text{anything} & x = 0 \end{cases}$$

is a weak derivative of u on U.

For $\phi \in C_c^\infty(U)$, by Stokes' theorem,

$$(-1) \int_{U \setminus B_\varepsilon(0)} u \phi_{x_i} \mathrm{d}x = \int_{U \setminus B_\varepsilon(0)} D_i u \phi \mathrm{d}x - \int_{\partial B_\varepsilon(0)} u \phi n \cdot \mathrm{d}S$$

Therefore, we can estimate

$$\left| \int_{\partial B_\varepsilon} u \phi n \cdot \mathrm{d}S \right| = |\phi|_{L^\infty} \left| \int_{\partial B_\varepsilon} \varepsilon^{-\lambda} n \cdot \mathrm{d}S \right| \leq C \varepsilon^{n-1-\lambda} \to 0$$

as $\lambda \to 0$. Thus, by the dominated convergence theorem,

$$-\int_U u \phi_{x_i} \mathrm{d}x = \int_U v_i \phi \mathrm{d}x$$

Remark 3.3.9.

1. Weak derivatives can exist even when the function is not continuous.
2. Since $D_i u \in L^p(U)$ if and only if $p(\lambda + 1) < n$, we see that

$$u \in W^{1,p}(U) \iff \lambda < \frac{n}{p} - 1$$

and if $p > n$, we see that λ must be negative, and so it is continuous.

Heuristically, larger p gives us nicer functions.

Theorem 3.3.10. $W^{k,p}(U)$ is a Banach space for $k \in \mathbb{N}, 1 \leq p \leq \infty$.

Proof. First, we need to show that it is a normed vector space. This is straightforward, and for the triangle inequality we will need to use Minkowski's inequality

$$\left(\sum_{i=1}^{\ell}(a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p}$$

For completeness, we will use the completeness of L^p. Let (u_j) be a Cauchy sequence on $W^{k,p}(U)$.

Note that $\left\|D^\alpha v\right\|_{L^p(U)} \leq \|v\|_{W^{k,p}(U)}$ for $|\alpha| \leq k$. Setting $v = u_j$, we see that $(D^\alpha u_j)_j$ is a Cauchy sequence in L^p. But we know that L^p is complete, and so there exists a function $u^\alpha \in L^p(U)$, with $D^\alpha u_j \to u^\alpha$ in L^p for all $|\alpha| \leq k$. We will set $u = u^{(0,\dots,0)}$.

Claim 3.3.11. u^α is the α-th weak derivative of u. That is, $D^\alpha u$ exists and $D^\alpha u = u^\alpha$.

Proof. Choose a test function $\phi \in C_c^\infty(U)$. Since $u_j \in W^{k,p}$, we know that $D^\alpha u_j$ exists, and

$$(-1)^{|\alpha|}\int_U u_j D^\alpha \phi \mathrm{d}x = \int_U (D^\alpha u_j)\phi \mathrm{d}x$$

for all j. Taking $j \to \infty$, using the fact that $D^\alpha u_j \to u^\alpha$ (using Hölder, or the dominated convergence theorem), we get that

$$(-1)^{|\alpha|}\int_U u D^\alpha \phi \mathrm{d}x = \int_U u^\alpha \phi \mathrm{d}x$$

and so $D^\alpha u = u^\alpha \in L^p(U)$. □

Thus, $u \in W^{k,p}(U)$. □

3.4 Approximations of Sobolev spaces

Convolution and mollifiers

Definition 3.4.1 (standard mollifier)
Let

$$\eta(x) = \begin{cases} C\exp\left(\frac{-1}{1-|x|^2}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where C is chosen such that

$$\int_{\mathbb{R}^n} \eta(x)\mathrm{d}x = 1$$

For $\varepsilon > 0$, we denote

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(x/\varepsilon)$$

We call η_ε to be the *standard mollifer*.

Remark 3.4.2.
- $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$,
- $\operatorname{supp}(\eta_\varepsilon) = \overline{B_\varepsilon(0)}$,
- $\int_{\mathbb{R}^n} \eta_\varepsilon(x)\mathrm{d}x = 1$

Definition 3.4.3

Given $U \subseteq \mathbb{R}^n$ open, define

$$U_\varepsilon = \{x \in U \mid d(x, \partial U) > \varepsilon\}$$

Definition 3.4.4 (mollification)

Given $f \in L^1_{\text{loc.}}(U)$, the *mollification of f* is

$$f_\varepsilon : U_\varepsilon \to \mathbb{R}$$
$$f_\varepsilon(x) = \eta_\varepsilon * f(x) = \int_U \eta_\varepsilon(x-y) f(y) \mathrm{d}y = \int_{B_\varepsilon(0)} \eta_\varepsilon(y) f(x-y) \mathrm{d}y$$

where $*$ denotes the convolution.

We can think of $f_\varepsilon(x)$ as the average of f in an ε-ball, weighted by η.

Theorem 3.4.5 (properties of mollification)**.** Let $f \in L^1_{\text{loc.}}(U)$, then

1. $f_\varepsilon \in C^\infty(U_\varepsilon)$,
2. $f_\varepsilon \to f$ a.e. on U as $\varepsilon \to 0$,
3. if $f \in C^0(U)$, then $f_\varepsilon \to f$ locally uniformly (i.e. uniformly on $K \subseteq U$ compact).
4. if $1 \leq p < \infty$, and $f \in L^p_{\text{loc.}}(U)$, then $f_\varepsilon \to f$ in $L^p_{\text{loc.}}(U)$. That is,

$$\left\| f_\varepsilon - f \right\|_{L^p(V)} \to 0$$

   for all $V \Subset U$.

Proof. See handout on moodle. □

In particular, there is a big improvement going from $f \in L^1_{\text{loc.}}$ to $f_\varepsilon \in C^\infty$.

Lemma 3.4.6 (local smooth approximation of Sobolev functions away from ∂U)**.** Let $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Set $u_\varepsilon = \eta_\varepsilon * u$ in U_ε. Then

1. $u_\varepsilon \in C^\infty(U_\varepsilon)$ for all $\varepsilon > 0$,
2. $u_\varepsilon \to u$ in $W^{k,p}_{\text{loc.}}(U)$. Note that for $V \Subset U$, $V \subseteq U_\varepsilon$ for ε sufficiently small.

Proof. (i) follows from the theorem. For (ii),

Claim 3.4.7.

$$D^\alpha u_\varepsilon = D^\alpha(\eta_\varepsilon * u) = \eta_\varepsilon * D^\alpha u$$

Proof. Since $u_\varepsilon \in C^\infty$, we can compute the classical derivative as follows:

$$\begin{aligned} D_x^\alpha u_\varepsilon(x) &= D_x^\alpha \int_U \eta_\varepsilon(x-y) u(y) \mathrm{d}y \\ &= \int_U (D_x^\alpha \eta_\varepsilon(x-y)) u(y) \mathrm{d}y \\ &= (-1)^{|\alpha|} \int_U (D_y^\alpha \eta_\varepsilon(x-y)) u(y) \mathrm{d}y \\ &= \int_U \eta_\varepsilon(x-y) D^\alpha u(y) \mathrm{d}y \\ &= (\eta_\varepsilon * u)(x) \end{aligned}$$

See handout for justification of swapping the integral and derivative. ☐

Note for $V \Subset U$, by (iv) of the theorem, since $D^\alpha u \in L^p(U)$, then

$$D^\alpha u_\varepsilon = \eta_\varepsilon * D^\alpha u \to D^\alpha u$$

in $L^p(V)$ as $\varepsilon \to 0$. Thus, for all $V \Subset U$, and $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\delta, V)$ such that

$$\left\|u_\varepsilon - u\right\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\le k} \left\|D^\alpha u_\varepsilon - D^\alpha u\right\|^p_{L^p(V)} \le \delta$$

for $0 < \varepsilon \le \varepsilon_0$. ☐

In conclusion, $u \in W^{k,p}(U)$ can be approximated by C^∞ functions away from ∂U.

Lecture 8

Theorem 3.4.8 (global Sobolev approximation globally away from ∂U). Suppose $U \subseteq \mathbb{R}^n$ open bounded, and suppose $u \in W^{k,p}(U)$, for $1 \le p < \infty$. Then there exists a sequence $(u_j) \in C^\infty \cap W^{k,p}(U)$, such that $u_j \to u$ in $W^{k,p}(U)$.

Exercise: Drop the assumption that U is bounded.

Remark 3.4.9. Note that we don't assume $u \in C^\infty(\overline{U})$.

Proof. **Step 1:** We have

$$U = \bigcup_{j=1}^\infty$$

where

$$U_j = \{x \in U \mid d(x, \partial U) > 1/j\}$$

and define $V_j = U_{j+3} \setminus \overline{U_{j+1}} \Subset U$. Choose $V_0 \Subset U$ such that $U = \bigcup_{j=0}^\infty V_j$. Note in particular only the consecutive V_j intersect.

Let ξ_j be a *partition of unity subordinate to* V_j. That is,

- $0 \le \xi_j \le 1$,
- $\xi_j \in C_c^\infty(V_j)$,
- $\sum_{j=0}^\infty \xi_j(x) = 1$ for all $x \in U$. Note at any point at most two ξ_j are non-zero.

Given $u \in W^{k,p}(U)$, then we see that $\xi_j u \in W^{k,p}(U)$, and $\operatorname{supp}(\xi_j u) \in V_j$.

Step 2: We would like to smooth out the split up function. Let $W_j = U_{j+4} \setminus \overline{U_j} \supseteq V_j$. Let

$$u_j = \eta_{\varepsilon_j} * (\xi_j u)$$

Fix $\delta > 0$, for each $j \ge 1$, we can choose ε_j sufficiently small such that $\operatorname{supp}(u_j) \subseteq W_j$.

By lemma 3.4.6, we have that $u_j \to \xi_j u$ in $W^{k,p}(W_j)$. With this, we can make

$$\left\|u_j - \xi_j u\right\|_{W^{k,p}(U)} = \left\|u_j - \xi_j u\right\|_{W^{k,p}(U_j)} \le \frac{\delta}{2^{j+1}}$$

Summing everything together, let $v = \sum_{j=0}^\infty u_j$. Note that each u_j is non-zero on finitely many W_j, and so at each point it is a finite sum. With this, v is smooth. Also note that $u = \sum_j \xi_j u$ on U, and so for any $V \Subset U$,

$$||v - u||_{W^{k,p}(V)} \le \sum_{j=1}^\infty \left\|u_j - \xi_j u\right\|_{W^{k,p}(V)} \le \delta \sum_{j=1}^\infty \frac{1}{2^{j+1}} = \delta$$

where we applied the triangle inequality. Since δ is independent of V, taking the sup over all $V \Subset U$, we get that

$$||v - u||_{W^{k,p}(U)} \le \delta$$

☐

Question: Can we approximate $u \in W^{k,p}(U)$ by $u \in C^\infty(\overline{U})$?
The issue here is that ∂U could be a problem. For example, we can consider ∂U to be the Cantor set.

Definition 3.4.10 ($C^{k,\delta}$-domain)

Suppose $U \subseteq \mathbb{R}^n$ is bounded and open. Then we say that ∂U is a $\mathbb{C}^{k,\delta}$*-domain* if for every $p \in \partial U$, there exists $r > 0$, and a function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, with $\gamma \in C^{k,\delta}(\mathbb{R}^{n-1})$, such that (after relabelling axes),

$$U \cap B_r(p) = \left\{(x', x_n) \in B_(p) \mid x_n > \gamma(x')\right\}$$

Theorem 3.4.11 (smooth approximation of Sobolev functions up to ∂U)**.** Let $U \subseteq \mathbb{R}^n$ be open, bounded and ∂U a $C^{0,1}$ domain. Let $u \in W^{k,p}(U)$, for some $1 \leq p < \infty$. Then there exists a sequence $(u_j) \in C^\infty(\overline{U})$, such that $u_j \to u$ in $W^{k,p}(U)$.

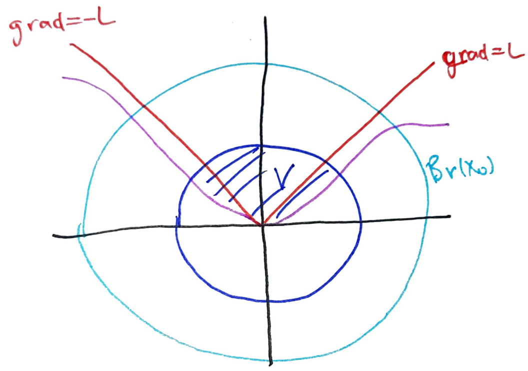
Proof. **Step 1:** Fix $x_0 \in \partial U$. Since ∂U is Lipschitz, there exists $r > 0$ and a Lipschitz function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, such that

$$U \cap B_r(x_0) = \left\{x \in B_r(x_0) \mid x_n > \gamma(x')\right\}$$

Let $V = U \cap B_{r/2}(x_0)$.

Step 2: Define the shifted point $x^\varepsilon = x + \lambda\varepsilon e_n$, for $x \in V, \varepsilon > 0$.

Claim 3.4.12. For $\lambda > 0$ large enough, $B_\varepsilon(x^\varepsilon) \subseteq U \cap B_r(x_0)$ for all $\varepsilon > 0$.



That is, we need to show that for $y \in B_\varepsilon(x^\varepsilon)$, $y_n > \gamma(y')$. As γ is Lipschitz, there exists a constant $L > 0$ such that

$$\left|\gamma(x') - \gamma(y')\right| \leq L\left|x' - y'\right|$$

So we have that

$$\left|y' - (x^\varepsilon)'\right| = \left|y' - x'\right| < \varepsilon$$

and so,

$$\gamma(y') \leq \gamma(x') + L_\varepsilon < x_n + L\varepsilon$$

by rearranging $y_n > x_n^\varepsilon - \varepsilon = x_n + \lambda\varepsilon - \varepsilon = x_n + (\lambda - 1)\varepsilon$, we see that

$$y_n > \gamma(y')$$

if $\lambda \geq L + 1$.

Define $u_\varepsilon(x) = u(x^\varepsilon)$ for $x \in V$. Set

$$v_{\delta,\varepsilon} = \eta_\delta * u_\varepsilon$$

for $0 < \delta < \varepsilon$. Then $v_{\delta,\varepsilon} \in C^\infty(\overline{V})$. Fix $\mu \geq 0$, then we note

$$\left\|v_{\delta,\varepsilon} - u\right\|_{W^{k,p}(V)} \leq \left\|v_{\delta,\varepsilon} - u_\varepsilon\right\|_{W^{k,p}(V)} + \underbrace{\left\|u_\varepsilon - u\right\|_{W^{k,p}(V)}}_{\text{translation is continuous on } W^{k,p}}$$

We can choose $\varepsilon > 0$ such that the second term is at most μ. Fix $\varepsilon > 0$, we can choose $\delta < \varepsilon$ such that the first term is at most μ, using the same proof as in lemma 3.4.6.

Step 3: Let x_0 vary over the boundary, then the Vs which we get will cover the boundary, which is compact, and so we have a finite subcover. That is, finitely many points $x_1, \dots, x_N \in \partial U$ and radii r_i, where

$$V_i = B_{r_i/2}(x_i) \cap U$$

Choose $V_0 \Subset U$ such that

$$U = V_0 \cup V_0 \cup \cdots \cup V_N$$

By step 2, we have $v_i \in C^\infty(\overline{V_i})$, such that $\left\|v_i - u\right\|_{W^{k,p}(V_i)} \leq \mu$. By lemma 3.4.6 there exists $v_0 \in C^\infty(\overline{V_0})$ such that $\left\|v_0 - u\right\|_{W^{k,p}(V_0)} \leq \mu$.

Step 4: Summing everything together, using a partition of unity $\xi_0, \dots, \xi_N$ subordinate to the open cover $V_0, \dots, V_N$. Define

$$v = \sum_{i=0}^{n} \xi_i v_i$$

This sum is finite, and so $v \in C^infty(\overline{U})$, and for $|\alpha| \leq k$,

$$\begin{aligned}
\left\|D^\alpha v - D^\alpha u\right\|_{L^p(U)} &\leq \sum_{i=0}^{N} \left\|D^\alpha(\xi_i(v_i - u))\right\|_{L^p(V_i)} \\
&\leq C_k \sum_{i=0}^{N} \left\|v_i - u\right\|_{W^{k,p}(V_i)} \\
&= C_k(1+N)\mu
\end{aligned}$$

But μ was arbitrary, and so we are done. □

Lecture 9

To conclude, we consider some examples of functions:

- $|x| \notin C^\infty(-1,1)$, but it is in $W^{1,1}(-1,1)$.
- $1/x$ is C^∞ and $L^1_{\text{loc.}}$ on $(0,1)$, but not $C^\infty(\overline{(-1,1)})$ or $W^{1,1}$.

and so, $C^\infty(U), C^\infty(\overline{U}) \nsubseteq W^{k,p}(U)$.

3.5 Extensions and traces

Suppose $U \subseteq \mathbb{R}^n$ is open and bounded, $u \in W^{k,p}(U)$. We would like to extend u to $\overline{u} : \mathbb{R}^n \to \mathbb{R}$. What happens if we set

$$\overline{u} = \begin{cases} u & \text{on } U \\ 0 & \text{on } U^c \end{cases}$$

This is okay for L^p, but not for $W^{k,p}$ as the derivatives become an issue. Moreover, we can expect at most $\overline{u} \in W^{k,p}(\mathbb{R}^n)$.

Theorem 3.5.1 (Calderon, Stein). Suppose $U \subseteq \mathbb{R}^n$ is open and bounded, and ∂U is C^1. Choose V bounded in $\mathbb{R}^n$, with $U \Subset V$. Let $1 \leq p < \infty$. Then there exists a bounded linear operator

$$\begin{aligned}
E : W^{1,p}(U) &\to W^{1,p}(\mathbb{R}^n) \\
u &\mapsto \overline{u}
\end{aligned}$$

such that for all $u \in W^{1,p}(U)$,

(i) $\overline{u}|_U = u$ a.e.

(i) $\operatorname{supp}(E(u)) \subseteq V$,

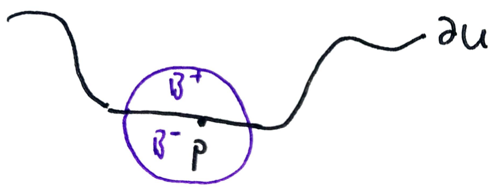
(i) There exists a constant C depending only on U, V, p, such that

$$\left\|E(u)\right\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$$

We call Eu the *extension of U to R^n*.

Proof. **Step 1:** Fix $p \in \partial U$, and suppose ∂U is flat near p. We may assume there exists $r > 0$ such that

$$\begin{aligned} B^+ &= B_r(p) \cap \{x_n \geq 0\} \subseteq \overline{U} \\ B^- &= B_r(p) \cap \{x_n < 0\} \subseteq \mathbb{R}^n \setminus \overline{U} \end{aligned}$$



Suppose also that $u \in C^1(\overline{U})$.Denote $x' = (x_1, \dots, x_{n-1})$. We define

$$\overline{u}(x) = \begin{cases} u(x) & \text{if } x \in B^+ \\ -3u(x', -x_n) + 4u(x', -x_n/2) & \text{if } x \in B^- \end{cases}$$

which is called a *higher order reflection of u from B^+ to B^-*

Claim 3.5.2. $\overline{u} \in C^1(B_r(p))$

Proof. Clearly $\overline{u}$ is continuous. Computing the derivatives:

$$\partial_{x_n}\overline{u} = \begin{cases} \partial_{x_n} u(x) & x \in B^+ \\ 3\partial_{x_n} u(x', -x_n) - 2\partial_{x_n} u(x', -x_n/2) & x \in B^- \end{cases}$$

Similarly,

$$\partial_{x_i}\overline{u} = \begin{cases} \partial_{x_i}\bar{(u)} & x \in B^+ \\ -3\partial_{x_i} u(x', -x_n) + 4\partial_{x_i} u(x', -x_n/2) & x \in B^- \end{cases}$$

and so the derivative is continuous. □

We can also check that the inequality holds in this case.

Step 2: Suppose ∂U is not flat near p. Since ∂U is C^1, there exists $r > 0$ and $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, such that

$$U \cap B_r(p) = \{x \mid x_n > \gamma(x')\}$$

Define

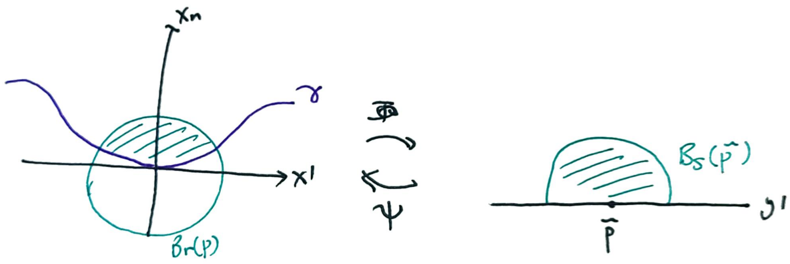
$$\begin{aligned} \Phi &: \mathbb{R}^n \to \mathbb{R}^n \\ \Phi(x) &= (x_1, \dots, x_{n-1}, x_n - \gamma(x')) \end{aligned}$$

We can see that Φ maps ∂U to $\{y_n = 0\}$, and it is invertible with C^1 inverse

$$\Psi(y) = (y_1, \dots, y_{n-1}, y_n + \gamma(y'))$$

with

- $\Phi(U \cap B_r(p)) \subseteq \{y_n > 0\}$
- $\det(D\Phi) = \det(D\Psi) = 1$



Moreover, there exists a neighbourhood W of p, with $\Phi(W) = B_s(\tilde{p})$ for some $s > 0$. In this case,

$$\Phi(U \cap W) = B_s(\tilde{p}) \cap \{y_n > 0\} = B^+$$

Define $v(y) = u(\Psi(y))$ for $y \in B_+$. Then v is C^1, and so by step 1 there exists an extension $\overline{v} \in C^1(B_s(\tilde{p}))$ with $\overline{v}|_{B^+} = v$ and

$$\|\overline{v}\|_{W^{1,p}(B_s(\tilde{p}))} \le C\|v\|_{W^{1,p}(B^+)}$$

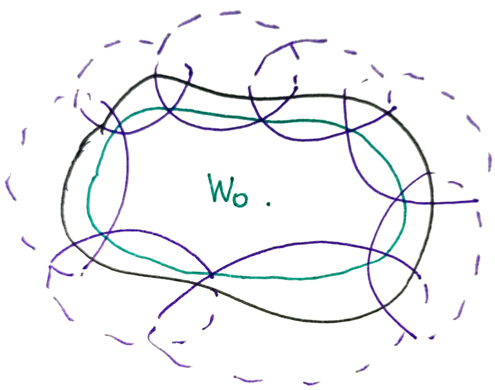
Define $\overline{u}(x) = \overline{v}(\Phi(x))$, then $\overline{u} \in C^1(W)$, and

$$\|\overline{u}\|_{W^{1,p}(W)} \le C\|u\|_{W^{1,p}(U)}$$

which we will see on examples sheet 1.

Step 3: Now we have local extensions near all $p \in \partial U$. We assume U is bounded, and so we have an open cover $\{W_0, \ldots, W_N\}$, with

$$U \subseteq \bigcup_{i=0}^{N} W_i$$



and we have extensions $\overline{u}_i \in C^1(W_i)$. Let $(\xi_i)_{i=0}^N$ be a partition of unity subordinate to $\{W_i\}$. Let

$$\overline{u} = \sum_{i=0}^{N} \xi_i \overline{u_i}$$

where $\overline{u_0} = u$. Then $\overline{u}|_U = u$ ae., and

$$\overline{u} \in C_c^1(\mathbb{R}^n)$$

with

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(U)}$$

We may assume $\text{supp}(\overline{u}) \subseteq V$, since $U \Subset V$, for example by using a cutoff function.

Step 4: Given $u \in W^{1,p}(U)$, by theorem 3.4.11, there exists a sequence $(u_j) \in C^\infty(\overline{U})$ with $u_j \to u$ in $W^{1,p}(U)$.

Claim 3.5.3. $(E(u_j))_j$ is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$.

Proof. By the previous steps, we have that $E(u_j) \in W^{1,p}(\mathbb{R}^n)$. By linearity,

$$\|E(u_j) - E(u_k)\|_{W^{1,p}(\mathbb{R}^n)} = \|E(u_j - u_k)\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_j - u_k\|_{W^{1,p}(U)}$$

But we know that (u_j) is convergent, and thus Cauchy in $W^{1,p}(U)$. □

Since $W^{1,p}(\mathbb{R}^n)$ is complete, the sequence converges and we define

$$E(u) = \lim_j E(u_j)$$

□

Remark 3.5.4. If ∂U is C^k, then we have the extension operators

$$E : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^n)$$

Given $u \in C^k(\overline{U})$, we set

$$\overline{u} = \begin{cases} u(x) & x \in B_+ \\ \sum_{j=1}^k c_j u(x', -x_n/j) & x \in B_- \end{cases}$$

To match at the boundary, we need

$$\sum_{j=1}^k c_j \left(\frac{-1}{j}\right)^m = 1$$

for all $m = 0, \ldots, k-1$.

Lecture 10

Traces

If we have $u \in C^0(\overline{U})$, then $u|_{\partial U}$ makes sense. But for $u \in W^{k,p}(U)$, then $u|_{\partial U}$ does not make sense, as ∂U has measure zero.

Theorem 3.5.5. Let U be open bounded and ∂U is C^1. Then there exists a bounded linear operator

$$T : W^{1,p}(U) \to L^p(\partial U)$$

called the *trace of u on ∂U*, such that

(i) $T(u) = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C^0(\overline{U})$,

(ii) $\|T(u)\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ for all $u \in W^{k,p}(U)$, where C depends only on U, p.

Remark 3.5.6. We have $u, Du \in L^p$, which is giving us the control of u on the boundary.

Sketch proof. See examples sheet 2. Suppose $u \in C^1(\overline{U})$, and ∂U is flat near p. Let

$$B^+ = B_r(p) \cap \{x_n \geq 0\} \subseteq \overline{U}$$
$$B^- = B_r(p) \cap \{x_n < 0\} \subseteq \mathbb{R}^n \setminus U$$

as before, and let Γ be the portion of ∂U within $B_r(p)$. Choose $\xi \in C_c^\infty(B_r(p))$ such that $0 \leq \xi \leq 1$ on $B_r(p)$, and $\xi = 1$ on $B_{r/2}(p)$. Then

$$\begin{aligned}\int_\Gamma \left|u(x',0)\right|^p dx' &\leq \int_{B_r(p)\cap\{x_n=0\}} \xi\left|u(x',0)\right|^p dx' \\ &\underbrace{=}_{\text{FTC}}(-1)\int_{B^+} \partial x_n(\xi|u|^p) dx_n dx' \\ &= (-1)\int_{B^+} |u^p|\partial_{x_n}\xi + p|u|^{p-1}\operatorname{sign}(u)\partial_{x_n} u\xi dx \\ &\underbrace{\leq}_{\text{Young's inequality}} C_p\int_{B^+} |u|^p + |Du|^p dx \\ &= C_p\|u\|^p_{W^{1,p}(U)}\end{aligned}$$

In Sheet 2, we will extend to general ∂U using a partition unity, and the fact that it is compact. Then define

$$T(u) = u|_{\partial U}$$

for $u \in C^1(\overline{U})$, and we have that

$$\left\|T(u)\right\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$$

Using density of $C^\infty(\overline{U})$ in $W^{1,p}(U)$, we are done. □

Remark 3.5.7.

- The map T above is not surjective, however in the case of $T : \mathrm{H}^s \to \mathrm{H}^{s-1/2}$, it is surjective.
- Recall $W_0^{k,p}(U)$ is the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. If $u \in W_0^{k,p}(U)$, then there exists $(u_j) \subseteq C_c^\infty(U)$, such that $u_j \to u$ in $W^{k,p}(U)$. In particular,

$$T(u) = T(\lim_j u_j) = \lim_j T(u_j) = 0$$

  In fact, the converse is true also. If $T(u) = 0$, then $u \in W_0^{1,p}(U)$.
- if $u \in W^{k,p}(U)$, then we can define trace operators for $Du, \ldots, D^{k-1}(U)$.

3.6 Sobolev inqualities

In this case, the basic idea is that we can trade differentiability (measured by k) for integrability (measured by p). Note it does not work the other way. For example, if $f' \in L^1(\mathbb{R})$ then $f \in L^\infty(\mathbb{R})$, but the converse is not true.

The idea is that we will prove estimates of the form

$$\|u\|_{L_q(\mathbb{R}^n)} \leq C\left\|Du\right\|_{L^p(\mathbb{R}^n)} \left(+\|u\|_{L^p(\mathbb{R}^n)}\right)$$

We have three cases:

1. $1 \leq p < n$,
2. $p = n$,
3. $n < p \leq \infty$.

Case 1: $1 \leq p < n$

Lemma 3.6.1. Let $n \geq 2$, and $f_1, \ldots, f_n \in L^{n-1}(\mathbb{R}^{n-1})$. Then for $1 \leq i \leq n$, define

$$\widetilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$$

and

$$f(x) = \prod_{i=1}^n f_i(\widetilde{x}_i) : \mathbb{R}^n \to \mathbb{R}$$

Then $f \in L^1(\mathbb{R}^n)$, with

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{n-1}(\mathbb{R}^{n-1})}$$

Proof. We induct on n. The case $n = 2$ gives

$$f(x_1, x_2) = f_1(x_1) f_2(x_2)$$

But

$$\begin{aligned}
\|f\|_{L^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |f_1(x_1)||f_2(x_2)| \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \int_{\mathbb{R}} |f_1(x_1)| \mathrm{d}x_1 \int |f_2(x_2)| \mathrm{d}x_2 \\
&= \|f_1\|_{L^1(\mathbb{R})} \|f_2\|_{L^1(\mathbb{R})}
\end{aligned}$$

Suppose the result is true for n. Write

$$F(x) = f_1(\widetilde{x}_1) \cdots f_n(\widetilde{x}_n)$$

and so $f(x) = F(x) f_{n+1}(\tilde{x}_{n+1})$. Fix x_{n+1} and integrate over $x_1, \ldots, x_n$:

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(\xi_1, \ldots, \xi_n, x_{n+1})| \mathrm{d}\xi_1 \cdots \mathrm{d}x_n &= \int_{\mathbb{R}^n} |F(\xi, x_{n+1})||f_{n+1}(\xi)| \mathrm{d}\xi \\
&\underbrace{=}_{\text{Hölder}} \|F(\cdot, x_{n+1})\|_{L^{n/(n-1)}(\mathbb{R}^n)} \|f_{n+1}\|_{L^n(\mathbb{R}^n)}
\end{aligned}$$

By the induction hypothesis, if $q = n/(n-1)$, then

$$\begin{aligned}
\|F(\cdot, x_{n+1})\|_{L^{n/(n-1)}(\mathbb{R}^n)} &= \|F(\cdot, x_{n+1}^q)\|_{L^1(\mathbb{R}^n)}^{1/q} \\
&\leq \prod_{i=1}^{n} \|f_i(\cdot, x_{n+1})^{n/(n-1)}\|_{L^{n-1}(\mathbb{R}^n)}^{(n-1)/n} \\
&= \prod_{i=1}^{n} \|f(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}
\end{aligned}$$

Integrating over x_{n+1},

$$\begin{aligned}
\|f\|_{L1(\mathbb{R}^{n+1})} &\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \int_{\mathbb{R}} \prod_{i=1}^{n} \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})} \mathrm{d}x_{n+1} \\
&\leq \|f_{n+1}\|_{L^n(\mathbb{R}^n)} \prod_{i=1}^{n} \left(\int_{\mathbb{R}} \|f_i(\cdot, x_{n+1})\|_{L^n(\mathbb{R}^{n-1})}^n \mathrm{d}x_{n+1} \right)^{1/n} \\
&= \prod_{i=1}^{n+1} \|f_i\|_{L^n(\mathbb{R}^n)}
\end{aligned}$$

Where we used the generalised Hölder inequality with $p = n$, that is,

$$\left\| \prod_i f_i \right\|_{L^1} \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}}$$

where $\sum \frac{1}{p_i} = 1$. □

Theorem 3.6.2 (Gagliado-Nirenberg-Sobolev inequality)**.** Suppose $1 \leq p < n$, set

$$p^* = \frac{np}{n-p}$$

for the *Sobolev conjugate of p*. Then we have a continuous embedding

$$W^{1,p}(\mathbb{R}^n) \subseteq L^{p^*}(\mathbb{R}^n)$$

That is, there exists a constant C depending only on n, p, such that

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^q(\mathbb{R}^n)} \tag{6}$$

Remark 3.6.3.
1. $p^* > p$.
2. Nothing is said about $\|Du\|_{L^{p^*}}$.

Intuition

Consider $f : \mathbb{R}^2 \to \mathbb{R}$. L^p measures the width and the height of the function. For example, if we have

$$f_1 = A1_W$$

then

$$\|f_1\|_{L^p} = |A| \operatorname{vol}(W)^{1/p} = |A| V^{1/p}$$

Now consider $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\phi(x) \leq 1$, and let

$$f_2(x) = \phi(x)e^{i\omega \cdot x}$$

Then we know $|f_2|(x) \leq 1$, and $\operatorname{supp}(f_2) \subseteq C$ is uniformly bounded in ω. With this,

$$\partial_1 f_2 = \phi' e^{i\omega\cdot x} + i\phi\omega_1 e^{i\omega\cdot x}$$

Thus, Df_2 is *not* uniformly bounded in ω.

Next, consider

$$f_3(x) = |\omega|^{-k}\phi(x)e^{i\omega\cdot x}$$

and so, we have a uniform bound in ω of $D^\ell f_3$ for $\ell \leq k$.

Finally, let

$$f_4(x) = A\phi(x/R)e^{i\omega\cdot x}$$

Then

$$\|f_4\|^p_{W^{1,p}} \sim \int_{|x|\leq R} \left|A\phi e^{i\omega\cdot x}\right|^p + \int_{|x|\leq R} \left|\frac{A}{R}\phi' e^{i\omega\cdot x} + A\phi\omega e^{i\omega\cdot x}\right|^p \sim |A|C^{1/p}|\omega|$$

Now recall the uncertainty principle:

$$\delta_x\delta_p \geq \frac{\bar{h}}{2} > 0$$

Thus,

$$\text{volume} \times \text{frequency} \geq c > 0$$

Thus, a function with frequency ω must be spread out on a ball of radius $\geq 1/\omega$. Thus, the support must have measure $\geq \omega^{-n}$. With this, $\omega \geq V^{-n}$, and so

$$\|f\|_{W^{1,p} \geq |A|V^{1/p-1/n}} = |A|V^{p^*} = \|f\|_{L^{p^*}}$$

See "Tery Tao uncertainty principle" for more details.

Lecture 11

Remark 3.6.4.
- If $u \equiv 1$, then it wouldn't satisfy eq. (6), and so it is essential that we are in $W^{1,p}$.
- Proof follows from density of $C_c^\infty(\mathbb{R}^n)$ in $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

Proof. **Step 1:** We can assume $u \in C_c^\infty(\mathbb{R}^n)$, and first consider the case $p = 1$. By FTC and compact support,

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \mathrm{d}y_i$$

which means that

$$|u(x)| \leq \underbrace{\int_{-\infty}^{\infty} |Du(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)|}_{=f_i(\tilde{x}_i)\mathrm{d}y_i}$$

Thus,

$$|u(x)|^n \leq f_1(\tilde{x}_1) \cdots f_n(\tilde{x}_n)$$

Integrating over $x \in \mathbb{R}^n$,

$$\left\| |u| \right\|_{L^1(\mathbb{R}^n)}^{n/(n-1)} \leq \left\| \prod_{i=1}^n (f_i(\tilde{x}_i))^{1/(n-1)} \right\|_{L^1\mathbb{R}^n} \leq \prod_{i=1}^n \left\| f_i^{1/(n-1)} \right\|_{L^{n-1}(\mathbb{R}^{n-1})} = \left\| Du \right\|_{L^1(\mathbb{R}^n)}^{n/(n-1)}$$

With this,

$$\| u \|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq \left\| Du \right\|_{L^1(\mathbb{R}^n)}$$

and in this case, $p^* = n/(n-1)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,1}(\mathbb{R}^n)$, result follows by density.

Step 2: Now suppose $p > 1$, consider $v(x) = |u(x)|^\gamma$. Then

$$Dv = \gamma \operatorname{sign}(u) |u|^{\gamma-1} Du$$

and

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{\gamma n/(n-1)} \mathrm{d}x \right)^{(n-1)/n} &= \left\| |u|^\gamma \right\|_{L^{n/(n-1)}(\mathbb{R}^n)} \\
&\leq \left\| D(|u|^\gamma) \right\|_{L^1(\mathbb{R}^n)} \\
&\leq \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |Du| \mathrm{d}x \\
&\underbrace{\leq}_{\text{Hölder}} \gamma \left(|u|^{(\gamma-1)p/(p-1)} \mathrm{d}x \right)^{1-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |Du|^p \right)^{1/p}
\end{aligned}$$

Choose γ to match the exponents of u in the integrals, i.e.

$$\gamma = \frac{p(n-1)}{n-p}$$

In particular,

$$\frac{\gamma n}{n-1} = p^*$$

and so we have that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \right)^{(n-1)/n} \leq \frac{p(n-1)}{n-p} \left(\int_{\mathbb{R}^n} |u|^{p^*} \right)^{(p-1)/p} \left\| Du \right\|_{L^p(\mathbb{R}^n)}$$

Which then implies that

$$\| u \|_{L^{p^*}(\mathbb{R}^n)} \leq \underbrace{\frac{p(n-1)}{n-p}}_{=:C} \left\| Du \right\|_{L^p(\mathbb{R}^n)} \leq C \| u \|_{W^{1,p}(\mathbb{R}^n)}$$

We can then conclude by density. □

Note in particular $C \to \infty$ as $p \to n$.

Corollary 3.6.5 (GNS for $U \subseteq \mathbb{R}^n$). Suppose $U \subseteq \mathbb{R}^n$ is open and bounded, with C^1 boundary. Let $1 \leq p < n$. If $p^* = \frac{np}{n-p}$, then

$$W^{1,p}(U) \subseteq L^{p^*}(U)$$

and the embedding is continuous. That is, there exists $C = C(U, p, n)$ such that

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}$$

for all $u \in W^{1,p}(U)$.

Proof. Exercise. Use the extension theorem and the GNS inequality for $\mathbb{R}^n$. □

Corollary 3.6.6 (Poincaré inequality). Let $U \subseteq \mathbb{R}^n$ be open and bounded. Suppose $u \in W_0^{1,p}(U)$, for some $1 \leq p < n$. Then there exists a constant $C = C(p, q, n, U)$ such that

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for each $1 \leq q \leq p^*$. In particular, as $1 \leq p \leq p^*$, we get

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

Remark 3.6.7.
(i) On $W^{1,p}(U)$ with U bounded, the $W^{1,p}$ norm is equivalent to $\|Du\|_{L^p(U)}$.
(ii) We do need that $u \in W_0^{1,p}$, to kill off constant functions, which have derivative zero.

Proof. We will use that $W_0^{1,p}(U)$ is the closure of $C_c^\infty(U)$ under the $W^{1,p}(U)$ norm. That is, given $u \in W_0^{1,p}(U)$, there exists $u_n \in C_c^\infty(U)$ such that $\|u_n - u\|_{W^{1,p}(U)} \to 0$. Since u_n is smooth, and vanishes near ∂U, we can extend $\overline{u_n} = 0$ on $\mathbb{R}^n \setminus U$, with $\overline{u_n} \in C_c^\infty(\mathbb{R}^n)$.

Applying theorem 3.6.2,

$$\|\overline{u_n}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\overline{u_n}\|_{L^p(\mathbb{R}^n)}$$

Sending $n \to \infty$ and noting that $\overline{u}$ vanishes on $\mathbb{R}^n \setminus U$, we get the result for $q = p^*$. In general, we use Hölder: since $|U| < \infty$,

$$\|u\|_{L^q(U)} \leq C\|u\|_{L^{p^*}(U)} \leq C'\|Du\|_{L^p(U)}$$

□

Case 2: $p = n$

In this case, $p^* \to \infty$, and so we may expect

$$\|u\|_{L^\infty(U)} \leq C\|u\|_{W^{1,n}}$$

But this is false for $n > 1$. One dimensional PDEs are boring, so we won't continue with this case.

Case 3: $n < p < \infty$

We might expect in this case that it is "better than L^∞", i.e. continuous.

Theorem 3.6.8 (Morrey's inequality). Let $n < p < \infty$, then there exists $C = C(p, n)$ such that for $u \in C_c^\infty(\mathbb{R}^n)$,

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \text{for} \quad \gamma = 1 - \frac{n}{p} < 1$$

That is, we have an embedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\gamma}(\mathbb{R}^n)$$

Proof. Let Q be an open cube of side length r, containing 0, and set

$$\overline{u} = \frac{1}{|Q|}\int_Q u(x)\mathrm{d}x$$

for the average of u over Q. Then

$$|\overline{u} - u(0)| \leq \frac{1}{|Q|}\int_Q |u(x) - u(0)|\mathrm{d}x$$

Since $u \in C_c^\infty(\mathbb{R}^n)$, by the fundamental theorem of calculus and the chain rule,

$$\begin{aligned} u(x) - u(0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(u(tx))\mathrm{d}t \\ &= \sum_{i=1}^n \int_0^1 x^i \frac{\partial u}{\partial x^i}\mathrm{d}t \end{aligned}$$

Thus,

$$|u(x) - u(0)| \leq r\sum_{i=1}^n \int_0^1 |\partial_{x_i} u(tx)|\mathrm{d}t$$

This then gives us that

$$\begin{aligned} |\overline{u} - u(0)| &\leq \frac{r}{|Q|}\int_Q\int_0^1 \sum_{i=1}^n |\partial_{x_i} u(tx)|\mathrm{d}t\mathrm{d}x \\ &= \frac{r}{|Q|}\int_0^1 t^{-n}\left(\sum_{i=1}^n \int_{tQ} \partial_{x_n} u(y)\mathrm{d}y\right)\mathrm{d}t \\ &\leq \frac{r}{|Q|}\int_0^1 t^{-n}\left(\sum_i \left\|\partial_{x_i} u\right\|_{L^p(tQ)} |tQ|^{1/q}\right)\mathrm{d}t \\ &\leq \frac{r}{r^n}\int_0^1 t^{-n}\left\|Du\right\|_{L^p(\mathbb{R}^n)} t^{n/q} r^{n/q}\mathrm{d}t \\ &= \frac{r^{1-n/p}}{1-n/p}\left\|Du\right\|_{L^p(\mathbb{R}^n)} \end{aligned}$$

That is,

$$|\overline{u} - u(0)| \leq \frac{r^\gamma}{\gamma}\left\|Du\right\|_{L^p(\mathbb{R}^n)}$$

By translation invariance,

$$|\overline{u} - u(x)| \leq \frac{r^\gamma}{\gamma}\left\|Du\right\|_{L^p(\mathbb{R}^n)}$$

for all $x \in Q$. Thus, by the triangle inequality,

$$|u(x) - u(y)| \leq |u(x) - \overline{u}| + |\overline{u} - u(y)| \leq 2\frac{r^\gamma}{\gamma}\left\|Du\right\|_{L^p(\mathbb{R}^n)}$$

for all $x, y \in Q$. But for $x, y \in Q$, there exists a cube Q of side length $r = 2|x - y|$ such that $x, y \in Q$, which means that

$$\frac{|u(x) - u(y)|}{|x-y|^\gamma} \leq C\left\|Du\right\|_{L^p(\mathbb{R}^n)}$$

Note the left hand side is independent of r, and so the inequality is true for all $x, y \in \mathbb{R}^n$. Thus,

$$[u]_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\left\|Du\right\|_{L^p(\mathbb{R}^n)}$$

Finally, we would like to control $\sup_{x\in\mathbb{R}^n}|u(x)|$, note that any $x \in \mathbb{R}^n$ belongs to a cube with side length 1. In particular,

$$\begin{aligned} |u(x)| &\leq |\overline{u}| + |u(x) - \overline{u}| \\ &\leq \int_Q |u(x)|\mathrm{d}x + C\left\|Du\right\|_{L^p} \\ &\leq \|u\|_{L^p(\mathbb{R}^n)}\left\|1\right\|_{L^q(Q)} + C\left\|Du\right\|_{L^p(\mathbb{R}^n)} \\ &\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \end{aligned}$$

Note the constant is independent of the choice of x. That is,

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

□

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Corollary 3.6.9. For $n < p < \infty$, and $u \in W^{1,p}(\mathbb{R}^n)$, then there exists a unique u^* with $u^* = u$ a.e., and u^* is continuous with

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

Corollary 3.6.10. Suppose $n < p < \infty$, $u \in W^{1,p}(U)$ for $U \subseteq \mathbb{R}^n$ open bounded, with ∂U being C^1. Then there exists a unique $u^* \in C^{0,\gamma}(\mathbb{R}^n)$, $\gamma = 1 - \frac{n}{p}$, $u = u^*$ a.e. on U, and

$$\|u^*\|_{C^{0,\gamma}(U)} \leq C\|u\|_{W^{1,p}(U)}$$

where C depends only on U, p, n.

Proof. By the extension theorem, there exists $\overline{u} \in W^{1,p}(\mathbb{R}^n)$, with $\operatorname{supp}(u)$ compact, $\overline{u} = u$ a.e. on U. Thus, there exists a sequence $(u_j) \in C_c^\infty(\mathbb{R}^n)$ with $u_j \to \overline{u}$ in $W^{1,p}(\mathbb{R}^n)$. Note by Morrey's inequality,

$$\|u_m - u_j\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u_m - u_j\|_{W^{1,p}(\mathbb{R}^n)}$$

and so (u_j) is Cauchy in the Banach space $C^{0,\gamma}(\mathbb{R}^n)$, and so there exists a limit $\overline{u}^* \in C^{0,\gamma}(\mathbb{R}^n)$. Then $u^* = \overline{u}^*|_U$ satisfies the requirements. □

In summary, if $U \subseteq \mathbb{R}^n$ is open, bounded and has C^1 boundary, then:

- if $1 \leq p < n$, then we have a continuous embedding

  $$W^{1,p}(U) \hookrightarrow L^{p^*}(U)$$

  where

  $$\frac{1}{p_*} = \frac{1}{p} - \frac{1}{n}$$

  and so $p^* > p$.

- If $n < p < \infty$, then

  $$W^{1,p}(U) \hookrightarrow C^{0,\gamma}(U)$$

  where

  $$\gamma = 1 - \frac{n}{p}$$

Example 3.6.11

Let $n = 3$, $u \in W^{2,2}$. Then $u, Du \in W^{1,2}$. $p = 2 < 3$, and we have $p^* = 6$, hence $u, Du \in L^6$. Thus, $u \in W^{1,6}$, and $6 > 3 = n$ so $u \in C^{0,1/2}$.

4 Second order elliptic boundary value problems

In this section, let U be an open bounded subset of $\mathbb{R}^n$, with C^1 boundary.

For $u \in C^2(\overline{U})$, define

$$Lu = -\sum_{i,j=1}^{n}(a^{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{7}$$

where a^{ij}, b^i, c are functions on U. We assume a^{ij}, b^i, c are L^∞, and $a^{ij} = a^{ji}$. This form is called *divergence form*, since it looks like

$$\text{grad} \cdot (A \text{grad} u)$$

If $a^{ij} \in C^1(\overline{U})$, then we can rewrite L in *non-divergence form*

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i(x) u_{x_i} + cu$$

We will study the divergence form, since it is adapted to Hilbert space methods. The non-divergence form is better for maximum principles, and Dirichlet energies. The second form is the topic of the Part III Elliptic PDEs course.

Definition 4.0.1
We say L is *elliptic* if

$$\sum_{i,j} a_{ij} \xi_i \xi_j > 0$$

for all $x \in U, \xi \in \mathbb{R}^n \setminus 0$. We say that L is *uniformly elliptic* if there exists a constant $\theta > 0$, such that

$$\sum_{i,j} a^{ij}(x) \xi_i \xi_j \geq \theta \|\xi\|^2$$

for all $x \in U, \xi \in \mathbb{R}^n$.

Note some references call uniformly elliptic: strongly or strictly elliptic.

4.1 Weak formulation and Lax-Milgram

We will consider the boundary value problem

$$\begin{cases} Lu = f & \text{on } U \\ u|_{\partial U} = 0 \end{cases} \tag{8}$$

with $f \in L^2(U), a^{ij}, b^i, c \in L^\infty(U)$.

Suppose $u \in C^2(\overline{U})$ solves eq. (8) pointwise a.e.. Take any $v \in C^2(\overline{U})$ with $v|_{\partial U} = 0$, we get (using summation convention)

$$\begin{aligned} \int_U f v \mathrm{d}x &= \int_U -v(a^{ij} u_{x_i})_{x_j} + v b^j u_{x_j} + cuv \mathrm{d}x \\ &= \underbrace{-\int_{\partial U} v a^{ij} u_{x_i} n_j \mathrm{d}S}_{=0} + \int_U a^{ij} u_{x_i} v_{x_j} + b u_{x_i} v + cuv \mathrm{d}x \end{aligned}$$

and so,

$$\int_U v f \mathrm{d}x = B[u, v] \tag{9}$$

for all $v \in C^2(\overline{U})$ with $v|_{\partial U} = 0$, where

$$B[u, v] = \int_U a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + cuv \mathrm{d}x$$

With this, if $u \in C^2(\overline{U})$ solves eq. (8), then eq. (9) holds. Conversely, if $u \in C^2(\overline{U})$ with $u|_{\partial U} = 0$ and satisfying eq. (9), then by integration by parts, we get that

$$\int_U (f - Lu) v \mathrm{d}x = 0$$

for all $v \in C_c^\infty(U)$. Thus, $Lu = f$ pointwise a.e. on U.

In conclusion, if $u \in C^2(\overline{U})$, with $u|_{\partial U} = 0$, then u satisfies eq. (8) if and only if it satisfies eq. (9).

But we note that eq. (9) makes sesne for $v \in H_0^1(U)$ and $u \in H^1$. To encode the boundary conditions, we can assume $u \in H_0^1(U)$.

Definition 4.1.1 (weak solution)
We say that $u \in H_0^1(U)$ is a *weak solution* of eq. (8) for given $f \in L^2(U)$ if

$$B[u, v] = \langle f, v \rangle_{L^2(U)}$$

for all $v \in H_0^1(U)$.

Theorem 4.1.2 (Lax-Milgram). Let H be a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$. Suppose $B : H \times H \to \mathbb{R}$ is bilinear, such that there exists constants $\alpha, \beta > 0$ such that

(i) (Boundedness) $|B[u, v]| \leq \alpha ||u|| \, ||v||$ for all $u, v \in H$,

(ii) (Coercivity) $\beta ||u||^2 \leq B[u, u]$ for all $u \in H$.

Then if $f \in H^*$, there exists a unique $u \in H$ such that

$$B(u, v) = \langle f, v \rangle$$

for all $v \in H$.

We will defer the proof to the next lecture.

Example 4.1.3
Recall that $H^k = W^{k,2}$ is a Hilbert space. Consider the boundary value problem

$$\begin{cases} Lu = -\Delta u + cu = f & \text{on } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where $c \geq 0$, $f \in L^2(U)$. In this case,

$$B[u, v] = \int_U \text{grad}u \cdot \text{grad}v + cuv dx$$

For boundedness, by Hölder (or Cauchy-Schwarz),

$$|B[u, v]| \leq (1 + c) ||u||_{H^1} ||v||_{H^1}$$

For coercivity,

$$B[u, u] = \left\|\text{grad}u\right\|_{L^2(U)}^2 + c||u||_{L^2(U)}^2 \geq \left\|\text{grad}u\right\|_{L^2(U)}^2 \geq \tilde{C} ||u||_{H^1(U)}^2$$

where for the last inequality, we used the Poincare inequality. Thus, we can apply Lax-Milgram with $H = H_0^1$.

Corollary 4.1.4 (of Lax-Milgram, stability). With the assumptions of Lax-Milgram, let u_i be the unique solution to

$$B[u_i, v] = \langle f_i, v \rangle$$

for all $v \in H$. Then

$$\left\|u_1 - u_2\right\|_H \leq \frac{1}{\beta} \left\|f_1 - f_2\right\|_{H^*}$$

Proof. Since $B[u_i, v] = \langle f_i, v \rangle$, by bilinearity,

$$B[u_1 - u_2, v] = \langle f_1 - f_2, v \rangle$$

Choosing $v = u_1 - u_2$, then

$$\beta ||v||^2 \leq B[u_1 - u_2, v] = \langle f_1 - f_2, v \rangle \leq \left\|f_1 - f_2\right\| ||v||$$

where we use Cauchy-Schwarz for the last inequality. □

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Proof of theorem 4.1.2. **Step 1:** For each fixed $u \in H$, define $\varphi_u(v) = B[u,v]$. This is a bounded linear functional on H, i.e. $\varphi_u \in H^*$. Applying the Riesz representation theorem, there exists a unique $w_u \in H$ such that

$$\varphi_u(v) = (w_u, v) = B[u,v]$$

for all $v \in H$. In particular, we have a map

$$\begin{aligned} A : H &\to H \\ u &\mapsto w_u \end{aligned}$$

and we have that $B[u,v] = (Au, v)$ for all $v \in H$.

Step 2: A is bounded. If $\lambda_1, \lambda_2 \in \mathbb{R}$, $u_1, u_2 \in H$, then for each $v \in H$, we have the following:

$$\begin{aligned} (A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v) \\ &= (\lambda_1 Au_1 + \lambda_2 Au_2, v) \end{aligned}$$

and so A is linear. Moreover,

$$\|Au\|^2 = (Au, Au) = B[u, Au] \le \alpha \|u\| \|Au\|$$

Hence $\|Au\| \le \alpha \|u\|$. Thus A is bounded, with $\|A\| \le \alpha$.

Step 3: We will show that A is injective and $A(H)$ is closed.

$$\beta \|u\|^2 \le B[u,u] = (Au, u) \le \|Au\| \|u\|$$

With this, $\beta\|u\| \le \|Au\|$. That is, A is bounded below, hence A is injective and the image is closed[1].

Step 4: We will show that A is surjective. Since $A(H)$ is a closed subspace of H, which is a Hilbert space, and so we can write

$$H = A(H) \oplus A(H)^\perp$$

With this, it suffices to show $A(H)^\perp = 0$. For $w \in A(H)^\perp$,

$$\beta \|w\|^2 \le B[w,w] = (Aw, w) = 0$$

$\|w\| = 0$, and so $A(H)^\perp = 0$. With this, A is bijective with bounded inverse.

Step 5: We would like to solve the following problem: Given $f \in H^*$, we would like to find u such that $B[u,v] = \langle f, v\rangle$ for all $v \in H$. By the Riesz representation theorem, there exists a unique $w_f \in H$ such that $\langle f, v\rangle = (w_f, v)$ for all $v \in H$. Now let $u = A^{-1} w_f$. Then

$$B[u,v] = (Au, v) = (w_f, v) = \langle f, v\rangle$$

i.e. $B[u, \cdot] = f$.

Step 6: For uniqueness, if u_1, u_2 satisfy $B[u_j, \cdot] = f$, then

$$B[u_1 = u_2, v] = 0$$

for all $v \in H$. Setting $v = u_1 - u_2$, and using coercivity we are done. □

Theorem 4.1.5 (energy estimates for B)**.** Suppose

$$Lu = -(a^{ij} u_{x_i})_{x_j} + b^i u_{x_i} + cu$$

where $a^{ij} = a^{ji}, b^i, c \in L^\infty(U)$, and suppose L is uniformly elliptic. If

$$B[u,v] = \int a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + cuv dx$$

[1] If (Au_j) is Cauchy, then so is (u_j)

is the associated bilinear form. Then there exists $\alpha, \beta > 0, \gamma \geq 0$, such that for all $u, v \in H_0^1(U)$.

(i) $|B[u,v]| \leq \alpha ||u||_{H_0^1(U)} ||v||_{H_0^1(U)}$,

(ii) (Garding's inequality) $\beta ||u||^2_{H_0^1(U)} \leq B[u,u] + \gamma ||u||^2_{L^2(U)}$

Remark 4.1.6. In PDE theory, "energy" refers to L^2.

Proof. For (i),

$$
\begin{aligned}
|B[u,v]| &\leq \sum_{i,j} \left\|a^{ij}\right\|_{L^\infty(U)} \int_U |Du||Dv| \mathrm{d}x + \sum_i \left\|b^i\right\|_{L^\infty(U)} \int_U |Du||v| \mathrm{d}x + ||c||_{L^\infty(U)} \int_U |u||v| \mathrm{d}x \\
&\leq \alpha ||u||_{H_0^1} ||v||_{H_0^1}
\end{aligned}
$$

where in the last step we used Cauchy-Schwarz, and collecting terms.

For (ii), we will use uniform ellipticity.

$$
\begin{aligned}
\theta \int_U |Du|^2 \mathrm{d}x &\leq \int_U a^{ij} u_{x_i} u_{x_j} \mathrm{d}x \\
&= B[u,u] - \int_U (b^i u_{x_i} u + cu^2) \mathrm{d}x \\
&=\leq B[u,u] + \sum_i \left\|b^i\right\|_{L^\infty} \int_U |Du||u| \mathrm{d}x + ||c||_{L^\infty(U)} \int_U u^2 \mathrm{d}x
\end{aligned}
$$

By Young's inequality with

$$
|ab| = \sqrt{2\varepsilon}|a| \frac{|b|}{\sqrt{2\varepsilon}}
$$

we get that

$$
\int_U |Du||u| \mathrm{d}x \leq \varepsilon \int_U |Du|^2 \mathrm{d}x + \frac{1}{2\varepsilon} \int_U |u|^2 \mathrm{d}x
$$

Choose ε such that

$$
\varepsilon \sum_i \left\|\beta_i\right\|_{L^\infty(U)} \leq \frac{\theta}{2}
$$

This then gives us that

$$
\frac{\theta}{2} \int_U |Du|^2 \mathrm{d}x \leq B[u,u] + c \int_U |u|^2 \mathrm{d}x
$$

Adding to this the Poincaré inequality, we get that

$$
\beta ||u||^2_{H_0^1(U)} \leq B[u,u] + \gamma ||u||^2_{L^2(U)}
$$

□

Remark 4.1.7. If B is a bilinear form for the operator L with $b^i = c = 0$, then $\gamma = 0$. In this case, we get

$$
\theta \int_U |Du|^2 \mathrm{d}x \leq B[u,u]
$$

and if we add the Poincaré inequality, we get

$$
||u||^2_{H_0^1(U)} \leq cB[u,u]
$$

which is Garding's inequality with $\gamma = 0$. In this case, we can apply Lax-Milgram directly.

On the other hand, if $\gamma > 0$, we can't apply Lax-Milgram.

Theorem 4.1.8. Let L be as above, then there exists a $\gamma \geq 0$ such that for any $\mu \geq \gamma$, and any $f \in L^2(U)$, there exists a unique solution $u \in H_0^1(U)$ to the boundary value problem

$$\begin{cases} L_\mu u = Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{10}$$

Moreover, there exists $C > 0$ such that

$$\|u\|_{H^1(U)} \leq C\|f\|_{L^2(U)}$$

Proof. Let γ be from Garding's inequality, i.e.

$$\beta\|u\|_{H_0^1}^2 \leq B[u,u] + \gamma\|u\|_{L^2(U)}^2$$

Let $\mu \geq \gamma$, and set

$$B_\mu[u,v] = B[u,v] + \mu\,(u,v)_{L^2}$$

which is the bilinear form for the operator L_μ in eq. (10). In this case, B_μ satisfies the conditions of Lax-Milgram.

Lecture 14

Given $f \in L^2(U)$, and set $\langle f, v\rangle = (f,v)_{L^2(U)}$. This is a bounded linear functional on $L^2(U)$, i.e. $f \mapsto (f,\cdot) \in (L^2(U))^*$. In particular, this is a bounded linear functional on H_0^1. We can apply Lax-Milgtam to find a unique $u \in H_0^1(U)$ with

$$B_\mu[u,v] = \langle f, v\rangle = (f,v)_{L^2(U)}$$

for all $v \in H_0^1$. Finally,

$$\beta\|u\|_{H_0^1(U)}^2 \leq B_\mu[u,u] = (f,u)_{L^2(U)} \leq \|f\|_{L^2(U)}\|u\|_{L^2(U)} \leq \|f\|_{L^2(U)}\|u\|_{H_0^1(U)}$$

□

So far, the solutions only live in H_0^1, and we need to pay a price for the μ.

4.2 Compactness results in PDEs

Recall the following results:

- Bolzano-Weierstraß- closed unit ball in $\mathbb{R}^n$ is sequentially compact.
- Recall for a metric space, the following are equivalent:
  (i) compactness,
  (ii) sequential compactness,
  (iii) completeness and totally boundedness
- If H is an infinite dimensional Hilbert space, then $\{x \in H \mid \|x\| \leq 1\}$ is not compact.

We will consider a weaker topology to recover compactness, since the topology induced by the norm is too strong.

Definition 4.2.1 (weak convergence)
Suppose H is a Hilbert space, $(u_j) \subseteq H$ a sequence, then we say that u_j *converges weakly to* $u \in H$ if for all $w \in H$,

$$(u_j, w) \to (u, w)$$

and we write $u_j \rightharpoonup u$.

Remark 4.2.2. If the weak limit exists, then it is unique.

Proposition 4.2.3 (Banach-Alaoglu for a separable Hilbert space)**.** Let H be a separable Hilbert space, and suppose we have a bounded sequence $(u_n) \subseteq H$. Then (u_n) has a weakly convergent subsequence.

That is, the closed unit ball in H is weakly sequentially compact.

Proof. Diagonal argument, see AoF. Or deduce from the below, since any Hilbert space is reflexive, and so the weak and weak-$*$ topologies agree. □

Theorem 4.2.4 (Banach Alaoglu)**.** Let X be a Banach space, then the closed unit ball in X^* is compact in the weak-$*$ topology on X^*.

Lemma 4.2.5 (Poincaré again)**.** Suppose $u \in H^1(\mathbb{R}^n)$, and $Q = (\xi_1, \xi_1 + L) \times \cdots \times (\xi_n, \xi_n + L)$ be a cube with side lengths L. Then

(i)

$$\|u\|^2_{L^2(Q)} \leq \frac{1}{|Q|}\left(\int_Q u \mathrm{d}x\right)^2 + \frac{nL^2}{2}\|Du\|^2_{L^2(Q)}$$

(ii)

$$\|u - \overline{u}\|_{L^2(Q)} \leq \frac{nL^2}{2}\|Du\|^2_{L^2(Q)}$$

where

$$\overline{u} = \frac{1}{|Q|}\int_Q u(x)\mathrm{d}x$$

In particular, if $\overline{u} = 0$ we recover the previous Poincaré inequality.

Proof. For (i), since ∂Q is Lipschitz, we apply the approximation theorem, to get $C^\infty(\overline{Q})$ are dense in $H^1(Q)$. Consider $u \in C^\infty(\overline{Q})$. For $x, y \in Q$, we use the fundamental theorem of calculus to get

$$u(x) - u(y) = \int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t}u(t, x_2, \cdots, x_n)\mathrm{d}t + \int_{y_2}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t}u(y_1, t, x_3, \cdots, x_n)\mathrm{d}t + \cdots + \int_{y_n}^{x_n} \frac{\mathrm{d}}{\mathrm{d}t}u(y_1, \ldots, y_{n-1}, t)\mathrm{d}t$$

Squaring, to get

$$u(x)^2 + u(y)^2 - 2u(x)u(y) \leq n\left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t}u(t, x_2, \ldots, x_n)\mathrm{d}t\right)^2 + \cdots + n\left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t}u(y_1, \ldots, y_{n-1}, t)\mathrm{d}t\right)^2$$

where we use Cauchy-Schwarz to get that

$$(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$$

Integrating over $x, y \in Q$,

$$\int_Q \int_Q (\text{LHS})\mathrm{d}x\mathrm{d}y = 2|Q|\|u\|^2_{L^2(Q)} = 2\left(\int_Q u(x)\mathrm{d}x\right)^2$$

For the right hand side,

$$\begin{aligned} I_1 &= \left(\int_{y_1}^{x_1} \frac{\mathrm{d}}{\mathrm{d}t}u(t, x_2, \cdots, x_n)\mathrm{d}t\right)^2 \leq \left(\int_{y_1}^{x_1} \mathrm{d}t\right)\left(\int_{y_1}^{x_1}\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t, x_2, \ldots, x_n)\right)^2 \mathrm{d}t\right) \\ &\leq L\int_{\xi_1}^{\xi_1 + L}\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t, x_2, \ldots, x_n)\right)^2 \mathrm{d}t \end{aligned}$$

Hence combining the terms,

$$2|Q| \|u\|_{L^2(Q)}^2 - 2\left(\int u(x)\mathrm{d}x\right)^2 \leq L^2 n|Q| \|Du\|_{L^2(Q)}^2$$

Rearranging gives the result.

For (ii), consider $\eta \in C_c^\infty$, with $\eta = 1$ on Q. Then

$$\int_Q (U - \overline{u}\eta)\mathrm{d}x = 0$$

and we can then use (i). □

Recall if $1 \leq p < n$, we have an embedding $W^{1,p} \hookrightarrow L^{p^*}$.

Theorem 4.2.6 (Rellich-Kondrachov). Suppose $U \subseteq \mathbb{R}^n$ be open with C^1 boundary. Let (u_n) be a bounded sequence in $H^1(U)$. Then there exists $u \in H^1(U)$, and a subsequence (u_{n_j}) such that $u_{n_j} \rightharpoonup u$ in $H^1(U)$, and $u_{n_j} \to u$ in $L^2(U)$.

Proof. By the extension theorem, we have an extension $\overline{u_n} \in H^1(\mathbb{R}^n)$, with $\mathrm{supp}(\overline{u_n}) \subseteq Q$ for some cube Q. Moreover, the extension operator $E : H^1(U) \to H_0^1(Q)$ is bounded. In particular,

$$\|\overline{u_n}\|_{H^1(Q)} \leq C\|u_n\|_{H^1(U)} \leq CK$$

for some K. Now $H_0^1(Q)$ is a separable Hilbert space, so by Banach-Alaoglu there exists $u \in H_0^1(Q)$, with $\overline{u_{n_j}} \rightharpoonup u$ in $H_0^1(Q)$, and

$$\|u\|_{H^1(Q)} \leq c$$

We claim that $w_j = \overline{u_{n_j}} \to u$ in $L^2(Q)$.

To see this, fix $\delta > 0$ and divide Q into k subcubes $\{Q_a\}_{a=1}^k$, of side lengths $0 < \ell < \delta$, intersecting only on their faces. Then

$$\|w_j - u\|_{L^2(Q)}^2 \leq \sum_{a=1}^k \|w_j - u\|_{L^2(Q)}^2 \leq \sum_{a=1}^k \left(\frac{1}{|Q_a|}\left(\int_{Q_a}(w_j - u)\mathrm{d}x\right)^2\right) + \frac{n^2\delta^2}{2}\|Dw_j - Du\|_{L^2(Q)}^2$$

Fix $\varepsilon > 0$, since $w_j, u \in H_0^1(Q)$, then $\|Dw_j - Du\|_{L^2(Q)}^2 \leq C$ for some C. Fix $\delta > 0$ such that

$$\frac{n^2\delta^2}{2}\|Dw_j - Du\|_{L^2(Q)}^2 < \frac{\varepsilon}{2}$$

This then fixes k. Note that the map

$$f \mapsto \int_Q f(x)\mathrm{d}x$$

is a bounded linear functional on $H_0^1(Q)$, and so by weak convergence,

$$\int_{Q_a}(w_j - u)\mathrm{d}x \to 0$$

This is true for all a. Since k is fixed and finite, choose j large enough so that

$$\sum_{a=1}^k \left(\frac{1}{|Q_a|}\left(\int_{Q_a}(w_j - u)\mathrm{d}x\right)^2\right) < \frac{\varepsilon}{2}$$

Using this, $\|w_j - u\|_{L^2(Q)}^2 < \varepsilon$. □

Lecture 15

4.3 Fredholm alternative and spectra of elliptic PDEs

Definition 4.3.1 (adjoint, compact)
Let H be a Hilbert space, and consider $K \in \mathcal{B}(H)$.

1. the *adjoint of* K, denoted $K^\dagger$ is the unique operator, such that

$$\left(x, K^\dagger y\right) = (Kx, y)$$

for all $x, y \in H$.
We say that K is *self-adjoint* if $K^\dagger = K$.

2. K is *compact* if for each bounded sequence $(u_j) \subseteq H$, there exists a subsequence (u_{j_k}) such that $K(u_{j_k})$ converges strongly in H.

Example 4.3.2 (Key example)
Let $K : L^2(U) \to H^1(U)$ be a bounded linear operator. SInce $H^1 \hookrightarrow L^2$, we can think of $K \in \mathcal{B}(L^2(U))$.

Claim 4.3.3. $K \in \mathcal{B}(L^2(U))$ is compact.

Proof. Let $(u_j) \subseteq L^2(U)$ is a bounded sequence, then

$$\left\|K(u_j)\right\|_{H^1} \leq \left\|K\right\|\left\|u_j\right\|_{L^2(U)}$$

and so $(K(u_j))$ is a bounded sequence in H^1. By Rellich-Kondrachov, there exists a subsequence (u_{j_k}), such that $K(u_{j_k})$ converges strongly in $L^2(U)$. □

The idea is that if we are looking at the equation

$$\Delta u = f$$

we can view this as a map

$$\begin{aligned} \mathrm{H}^1(U) &\to L^2(U) \\ u &\mapsto f \end{aligned}$$

Finding a solution is the inverse map $K : L^2(U) \to H^1(U)$, with $K(f) = u$. This map will be compact.

Theorem 4.3.4 (Fredholm alternative for compact operators)**.** Let H be a Hilbert space, $K \in \mathcal{B}(H)$ compact. Then

(i) $\ker(I - K)$ is finite dimensional,

(ii) $\mathrm{im}(I - K)$ is closed,

(iii) $\mathrm{im}(I - K) = \ker(I - K^\dagger)^\perp$,

(iv) $\ker(I - K) = 0$ if and only if $\mathrm{im}(I - K) = H$,

(v) $\dim(\ker(I - K)) = \dim(\ker(I - K^\dagger))$

Proof. Appendix D.5 of Evans. □

Note (iii) and (iv) are referred to as the *Fredholm alternative*. Applied to linear algebra, we would like to consider the equation

$$Ax = b$$

We have the alternative:

(a) A is invertible, A^{-1} exists, and so the inhomogeneous problem has a unique solution,

(b) $\ker(A)$ is non-trivial. The homogeneous equation $Ax = 0$ admits non-trivial solutions. Moreover, from (iii), $\text{im}(A) = \ker(A^\top)^\perp$, and so the inhomogeneous equation has a solution if and only if $b \in \ker(A^\top)^\perp$, and so

$$y^\top b = 0$$

for all $y \in \ker(A^\top)$, i.e. $A^\top y = 0$.

Restating (iii) and (iv), we have

(I) for each $f \in H$, $(I - K)u = f$ has a unique solution,

(II) or the homogeneous equation $(I - K)u = 0$ has a non-trivial solution. In this case, the space of solutions $(I - K)u$ is finite dimensional, and $(I - K)u = f$ has a solution if and only if $f \in \ker(I - K^\dagger)^\perp$.

Definition 4.3.5 (resolvent and spectrum)
Let H be a real Hilbert space, $A \in \mathcal{B}(H)$. The *resolvent (set)* of A is

$$\rho(A) = \{\lambda \in \mathbb{R} \mid A - \lambda I \text{ is invertible}\}$$

The *real spectrum* of A is

$$\sigma(A) = \mathbb{R} \setminus \rho(A)$$

We also define the *point spectrum*

$$\sigma_p(A) = \{\eta \in \sigma(A) \mid \ker(A - \eta I) \neq 0\}$$

If $Aw = \eta w$, we call w an *eigenvector*.

Remark 4.3.6. $\rho(A)$ is open, and $\sigma(A)$ is closed.

Theorem 4.3.7 (spectrum of compact operator)**.** Suppose H is a separable infinite dimensional Hilbert space, with $K \in \mathcal{B}(H)$ compact. Then

(i) $0 \in \sigma(K)$,

(ii) $\sigma(K) \setminus \{0\} = \sigma_p(K) \setminus \{0\}$,

(iii) $\sigma(K) \setminus \{0\}$ is countable. Say $\sigma(K) \setminus \{0\} = \{\lambda_i\}_{i \in \mathbb{N}}$, then (up to reordering) $\lambda_i \to 0$,

(iv) if K is in addition self-adjoint, then there exists a countable orthonormal basis for H consisting of eigenvectors for K.

Proof. II Linear Analysis. □

4.3.1 Application to elliptic PDEs

Consider eq. (7) as before, with L uniformly elliptic on $U \subseteq \mathbb{R}^n$. The bilinear form associated to L is

$$B[u, v] = \int_U a^{ij} u_{x_i} v_j + b^i u_{x_i} v + cuv \mathrm{d}x$$

Definition 4.3.8 (formal adjoint, adjoint bilinear form)
We define the formal adjoint to L as

$$L^\dagger v = -\sum_{i,j} (a^{ij} v_{x_i})_{x_j} - \sum_i b^i u_{x_i} + \left(c - \sum_i b^i_{x_i} \right) v$$

and the adjoint bilinear form is given by

$$B^\dagger[v, u] = B[u, v]$$

We say $v \in H_0^1(U)$ is a *weak solution* of the *adjoint problem*

$$\begin{cases} L^\dagger v = f & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases}$$

if

$$B^\dagger[w, v] = (f, w)_{L^2}$$

for all $w \in H_0^1(U)$.

Note if $b^i \in C^1(U)$, then $B^\dagger$ is the same bilinear form as B.

Theorem 4.3.9 (Fredholm alternative for elliptic boundary value problem). Consider for bounded U with C^1 boundary,

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{11}$$

Then

(I) for each $f \in L^2(U)$, eq. (11) admits a unique weak solution, or

(II) there exists a non-trivial weak solution to the homogeneous problem (i.e. $f = 0$), and $\dim(N) = \dim(N^\dagger)$, where

$$N = \{\text{weak solutions to homogeneous equation}\} \subseteq H_0^1(U)$$

and

$$N^\dagger = \{\text{weak solutions to the homogenous adjoint equation}\}$$

With this, eq. (11) has a unique solution if and only if

$$(f, v)_{L^2} = 0$$

for all $v \in N^\dagger$.

Proof. By theorem 4.1.8, there exists $\gamma > 0$ such that for every $f \in L^2(U)$, there exists a unique weal solution $u \in H_0^1(U)$ to

$$\begin{cases} L_\gamma u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

where $L_\gamma u = Lu + \gamma u$. We also have an associated bilinear form

$$B_\gamma[u, v] = B[u, v] + \gamma\,(u, v)_{L^2}$$

and we have that

$$B_\gamma[u, v] = (f, v)_{L^2}$$

for all $v \in H_0^1$, and $\|u\|_{H^1} \leq C\|f\|_{L^2}$.

Write $L_\gamma^{-1}(f) = u$ for the solution operator. This is well defined as the solution exists and is unique. We can check that this is linear. The inequality above shows that

$$\|L_\gamma^{-1}(f)\|_{\mathrm{H}^1} \leq C\|f\|_{L^2}$$

and so

$$L_\gamma^{-1} : L^2 \to H_0^1$$

is bounded, hence $L_\gamma^{-1} : L^2 \to L^2$ is compact.

Observe that for $g \in L^2$, then $L_\gamma^{-1}(g) = w$ if and only if $B_\gamma[w, v] = (g, v)$ for all $v \in H_0^1(U)$. Now suppose $u \in H_0^1$ is a weak solution to eq. (11), that is,

$$B[u, v] = (f, v)$$

for all $v \in H_0^1$, and so

$$B_\gamma[u, v] = \langle f + \gamma u, v \rangle$$

for all $v \in H_0^1$. Thus, u solves eq. (11) weakly if and only if

$$u = L_\gamma^{-1}(f + \gamma u) = L_\gamma^{-1}(f) + \gamma L_\gamma^{-1}(u)$$

which is true if and only if $(I - K)u = h$, where

$$K = \gamma L_\gamma^{-1} \quad \text{and} \quad h = L_\gamma^{-1}(f)$$

Lecture 16

Observe the map $K : L^2 \to L^2$ is also compact, and so we can apply the Fredholm theorem for compact operators, and either:

(I) for all $h \in L^2$, $u - Ku = h$ has a unique solution $u \in L^2$,

(II) there exists $0 \neq u \in L^2$ with $u - Ku = 0$.

Suppose (I) holds. Set $h = L_\gamma^{-1}(f)$, then there exists a unique $u \in L^2$ with

$$u = \gamma L_\gamma^{-1}(u) + L_\gamma^{-1}(u)$$

Since $L_\gamma^{-1} : L_2 \to H_0^1$, $u \in H_0^1$ and by the above, u is a weak solution of eq. (11).

Now suppose (II) holds. Then there exists $u \in L^2$ non-zero, with $u = Ku = \gamma L_\gamma^{-1}(u)$. As above, $u \in H_0^1$. Use the definition of L_γ^{-1} to see that

$$B[u, v] + \gamma (u, v)_{L^2} = (\gamma u, v)_{L^2}$$

for all $v \in L^2$. Hence $B[u, v] = 0$ for all $v \in H_0^1$. That is, $u \in N$. Moreover, $\dim(N) = \dim(\ker(I - K)) = \dim(\ker(I - K^\dagger))$.

Claim 4.3.10. Let $v \in L^2$, then $(I - K^\dagger)v = 0$ if and only if $B^\dagger[v, w] = 0$ for all $w \in H_0^1$.

Proof.

$$\begin{aligned} (I - K^\dagger)v = 0 &\iff (v, w)_{L^2} = (v, Kw)_{L^2} \text{ for all } w \in L^2 \\ &\iff (v, w)_{L^2} = \left(v, \gamma L_\gamma^{-1}(w)\right)_{L^2} \text{ for all } w \in L^2 \end{aligned}$$

But note that any weak solution to

$$\begin{cases} L_\gamma \overline{w} = \overline{f} & \text{on } U \\ \overline{w} = 0 & \text{on } \partial U \end{cases}$$

obeys

$$B[\overline{w}, \varphi] + \gamma (\overline{w}, \varphi)_{L^2} = \left(\overline{f}, \varphi\right)_{L^2}$$

So if we take $\overline{f} = w$, then we have $\overline{w} = L_\gamma^{-1}(w)$. Hence

$$B[L_\gamma^{-1}(w), v] + \gamma \left(L_\gamma^{-1}(w), v\right) = (w, v)_{L^2}$$

Hence the above is true if and only if

$$\begin{aligned} &\iff B[L_\gamma^{-1}(w), v] + \gamma \left(L_\gamma^{-1}(w), v\right)_{L^2} = \left(v, \gamma L_\gamma^{-1}(w)\right)_{L^2} \text{ for all } w \in L^2 \\ &\iff B[L_\gamma^{-1}(w), v] = 0 \text{ for all } w \in L^2 \\ &\iff B^\dagger[v, L_\gamma^{-1}(w)] = 0 \text{ for all } w \in L^2 \end{aligned}$$

On examples sheet 3, $\operatorname{im}(L_\gamma^{-1})$ is dense, and so we have that

$$v = K^\dagger v \iff B^\dagger[v, w] = 0$$

□

It remains to prove that eq. (11) has a weak solution if and only if $(f, v)_{L^2} = 0$ for all $v \in N^\dagger$. Now note

$$\text{eq. (11) has a solution} \iff (I - K)u = L_\gamma^{-1} f \iff L_\gamma^{-1}(f) \in \operatorname{Im}(I - K) = \ker(I - K^\dagger)^\perp$$

That is, we need $\left(v, L_\gamma^{-1}(f)\right)_{L^2} = 0$ for all $v \in \ker(I - K^\dagger)$. But for all $v \in \ker(I - K^\dagger)$,

$$0 = \left(v, L_\gamma^{-1}(f)\right)_{L^2} = \left(v, \frac{1}{\gamma} K f\right)_{L^2} - \frac{1}{\gamma}(K^\dagger v, f) = \frac{1}{\gamma}(v, f)_{L^2}$$

and so $(v, f)_{L^2} = 0$ for all $v \in \ker(I - K^\dagger)$. □

> **Remark 4.3.11.** In this proof, given L, we see that for γ large, L_γ is a bounded inverible linear map, the map $L_\gamma^{-1} = (L + \gamma I)^{-1}$ is called the *resolvent of L*. The fact that $L_\gamma^{-1} : L^2 \to L^2$ is compact is expressed as *L has compact resolvent*.

Theorem 4.3.12. Under the same assumptions as in theorem 4.3.9,

(i) there exists a countable set $\Sigma \subseteq \mathbb{R}$ such that the boundary value problem

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{12}$$

has a weak solution for all $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.

(ii) if Σ is infinite, then $\Sigma = \{\lambda_k\}_{k \in \mathbb{N}}$. After reordering, then

$$\lambda_1 < \lambda_2 < \cdots$$

with $\lambda_k \to \infty$ as $k \to \infty$.

(iii) for each $\lambda \in \Sigma$, there exists a finite dimensional space

$$\mathcal{E}(\lambda) = \left\{u \in H^1 \mid u \text{ is a weak solution to the homog. problem } Lu = \lambda u\right\}$$

We call $\lambda \in \Sigma$ an *eigenvalue* of L, and elements of $\mathcal{E}(\lambda)$ are the corresponding *eigenfunctions*.

Proof. Choose $\gamma > 0$ as in eq. (11). Choose $\mu \geq \gamma$, then $L_\mu = L + \mu I$ is invertible, and

$$L_\mu^{-1} : L^2 \to L^2$$

is compact. If $\lambda \leq -\gamma$, then the problem

$$\begin{cases} Lu - \lambda u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Thus, $\Sigma \subseteq (-\gamma, \infty)$. If $\lambda > -\gamma$, then solving eq. (12) is equivalent to solving

$$\begin{cases} (L - \lambda I)u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{13}$$

Applying theorem 4.3.9 to $L - \lambda I$, eq. (13) has a unique weak solution for all $f \in L^2$ if and only if $u = 0$ is the unique solution to

$$\begin{cases} (L - \lambda I)u = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

That is, case (II) in theorem 4.3.9 does not occur. This is true if and only if $u = 0$ is the only solution to

$$\begin{cases} Lu + \gamma u = (\lambda + \gamma)u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

which is true if and only if $u = 0$ is the only solution to

$$u = L_\gamma^{-1}((\lambda + \gamma)u) = \frac{\gamma + \lambda}{\gamma} K(u)$$

which is saying $u = 0$ is the only solution to

$$K(u) = \frac{\gamma}{\gamma + \lambda} u$$

That is, $\gamma/(\gamma + \lambda)$ is *not* an eigenvalue of K. So

$$\lambda \in \Sigma \iff \mu = \frac{\gamma}{\gamma + \lambda} \text{ is an eigenvalue of } K$$

But recall theorem 4.3.7, the set of eigenvalues of K is either finite, or countably infinite and converging to zero. In the second case, if

$$\mu_k \to 0 \text{ then } \lambda_k \to \infty$$

The fact that $\mathcal{E}(\lambda)$ is finite dimensional follows from the Fredholm alternative. □

Remark 4.3.13. If $\lambda \notin \Sigma$, then there exists $C(\lambda) > 0$ such that

$$||u||_{L^2} \leq C(\lambda)\|f\|_{L^2}$$

As λ approaches an eigenvalue, $C(\lambda) \to \infty$.

4.4 Self-adjoint positive operators

Definition 4.4.1 (formally self adjoint)
The operator L is *formally self-adjoint* if $L = L^\dagger$. Equivalently, $b^i = 0$ for all i.

If L is self adjoint, then $B[u, v] = B[v, u]$.

Definition 4.4.2 (positive)
We say L is *positive* if there exists $\beta > 0$ such that

$$\beta||u||_{H^1}^2 \leq B[u, u]$$

for all $u \in H_0^1$.

That is, B is coercive.

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Theorem 4.4.3 (eigenvalues of symmetric elliptic operators)**.** Let L be uniformly elliptic, formally self-adjoint, positive operator on U. Then we can represent the eigenvalues of L as a sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

with multiplicity, i.e λ appears $\dim(\mathcal{E}(\lambda))$ times. Moreover, there exists an orthonormal basis of L^2, consisting of eigenfunctions $\{w_k\}$, with

$$\begin{cases} Lw_k = \lambda w_k & \text{on } U \\ w_k = 0 & \text{on } \partial U \end{cases}$$

and each $w_k \in H_0^1(U)$.

Proof. By positivity, Lax-Milgram implies that L is invertible. Moreover, $L^{-1} : L^2(U) \to H_0^1(U)$ is bounded. Denote $S = L^{-1} : L^2(U) \to L^2(U)$. Then S is compact.

Claim 4.4.4. S is self-adjoint.

Proof. Choose $f, g \in L^2(U)$, then $Sf = u$ means $u \in H_0^1(U)$ is the unique weak solution to

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

The same statement applies to $Sg = v$. That is,

$$B[u, w] = (f, w)_{L^2}$$

for all $w \in H_0^1$, and

$$B[v, \varphi] = (g, \varphi)_{L^2}$$

for all $\varphi \in H_0^1$. With this,

$$\begin{aligned} (Sf, g)_{L^2} &= (u, g)_{L^2} \\ &= B[v, u] \\ &= B[u, v] \\ &= (f, v)_{L^2} \\ &= (f, Sg)_{L^2} \end{aligned}$$

□

Now by theorem 4.3.7 (spectrum of compact self-adjoint operators), there exists a sequence of eigenvalues $(\mu_k)_k \subseteq \mathbb{R}$, such that $\mu_k \to 0$, and there exists $w_k \in L^2(U)$, such that $\{w_k\}$ is an orthonormal basis of L^2, with $Sw_k = \mu_k w_k$. Equivalently, $L^{-1} w_k = \mu_k w_k \in H_0^1$, and so $Lw_k = \lambda_k w_k$, where $\lambda_k = 1/\mu_k$. Positivity of λ_k follows from positivity of L, and so the positivity of S. □

4.5 Elliptic regularity

In this section, we will assume $U \subseteq \mathbb{R}^n$ is an open bounded domain, $V \Subset U$. Our goal is to improve the regulaity of the weak solutions $u \in H_0^1(U)$, to say $u \in C^2(\overline{U})$.

Example 4.5.1 (motivating examples)
Let $u \in C_c^\infty(\mathbb{R}^n)$ be a solution to

$$-\Delta u = f$$

Then

$$\begin{aligned} \int_{\mathbb{R}^n} f^2 \mathrm{d}x &= \int_{\mathbb{R}^n} (\Delta u)^2 \mathrm{d}x \\ &= \sum_{i,j} \int_{\mathbb{R}^n} (D_i D_i u)(D_j D_j u) \mathrm{d}x \end{aligned}$$

Integrating by parts twice,

$$\begin{aligned} &= \sum_{i,j} \int_{\mathbb{R}^n} (D_i D_j u)(D_i D_j u) \mathrm{d}x \\ &= \left\| D^2 u \right\|_{L^2(\mathbb{R}^n)}^2 \end{aligned}$$

That is, we have that

$$\left\| D^2 u \right\|_{L^2(\mathbb{R}^n)} \leq \left\| \Delta u \right\|_{L^2(\mathbb{R}^n)}$$

and so all second derivatives are controlled in L^2 by Δu.

However, if $u \in H^1$, then $D^2 u$ may not exist (even weakly). Thus, we will approximate the derivatives.

Definition 4.5.2 (difference quotient)
For $0 < |h| < d(V, \partial U)$ (required so we stay awy from the boundary), define the *difference quotient*

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}$$

for $x \in V$, $i = 1, \ldots, n$. Write

$$\Delta^h u = (\Delta_1^h u, \ldots, \Delta_n^h u)$$

Remark 4.5.3. Suppose $u \in L^2$, then $\Delta^h u \in L^2(V)$, and

$$D(\Delta^h u) = \Delta^h (Du)$$

Hence if $u \in H^1(U)$, then $\Delta^h u \in H^1(V)$.

Lemma 4.5.4. Suppose $u \in L^2(U)$, then $u \in H^1(V)$ if and only if there exists $C > 0$, such that for all h with $0 < |h| < \frac{1}{2} d(V, \partial U)$, with

$$\left\| \Delta^h u \right\|_{L^2(V)} \leq C$$

Moreover, there exists $\widetilde{C}$ such that

$$\frac{1}{\tilde{C}} \left\| Du \right\|_{L^2(V)} \leq \left\| \Delta^h u \right\|_{L^2(V)} \leq \tilde{C} \left\| Du \right\|_{L^2(V)}$$

That is, the difference quotient is equivalent to the derivative, sometimes written

$$\left\| Du \right\|_{L^2(V)} \simeq \left\| \Delta^h u \right\|_{L^2(V)}$$

Proof. Examples sheet 3. □

Theorem 4.5.5 (interior regularity). Suppose L is uniformly elliptic on U, and assume $a^{ij} \in C^1(U)$ and $b^i, c \in L^\infty(U)$, $f \in L^2(U)$. Suppose $u \in H^1(U)$ satisfies

$$B[u, v] = (f, v)_{L^2} \tag{14}$$

for all $v \in H_0^1(U)$, then $u \in H^2_{\text{loc.}}(U)$, and for each $V \Subset U$,

$$||u||_{H^2(V)} \leq C \left(\left\| f \right\|_{L^2(U)} + ||u||_{L^2(U)} \right)$$

with $C = C(a, b, c, V, U, n)$, but not on f or u.

Remark 4.5.6. The result says that we gain two weak derivatives by solving the equation. We can also write the inequality as

$$||u||_{H^2(V)} \leq C \left(\left\| Lu \right\|_{L^2(U)} + ||u||_{L^2(U)} \right)$$

Proof. **Step 1:** Fix $V \Subset U$, and choose W such that $V \Subset W \Subset U$. Let $\xi \in C_c^\infty(W)$, such that $0 \leq \xi \leq 1$, $\xi|_V = 1$, $\xi|_{\partial W} = 0$. We can rewrite the weak equation as

$$\int_U a^{ij} D_i u D_j v \mathrm{d}x = \int_U \tilde{f} v \mathrm{d}x$$

where

$$\tilde{f} = f - b^i D_i u - cu \in L^2(U)$$

Let $v = -\Delta_k^{-h}(\xi^2 \Delta_k^h u)$ for k fixed, $0 < |h| < d(W, \partial U)$. Note by previous comments, $v \in H_0^1(W)$, and approximates $D^2 u$. Set

$$A = \int_U a^{ij} u_{x_i} v_{x_j} \mathrm{d}x$$
$$B = \int_U \tilde{f} v \mathrm{d}x$$

Observe for $\psi, \phi \in L^2(U)$ supported in W, then

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$$\int_U \psi(x) \left(\Delta_k^{-h} \phi(x)\right) \mathrm{d}x = -\int_U (\Delta_k^h \psi(x)) \phi(x) \mathrm{d}x$$

which is integration by parts for the difference quotient. Moreover,

$$\Delta_j^h (\psi\phi)(x) = \frac{\psi(x + he_k)\phi(x + he_k) - \psi(x)\phi(x)}{h} = (\tau_k^h \psi)(x) \Delta_k^h \phi(x) + (\Delta_k^h \psi)(x)\phi(x)$$

where

$$\tau_k^h \psi(x) = \psi(x + he_k)$$

is the *translation operator*.

Step 2 (Bounding A): Using the above,

$$\begin{aligned} A &= -\int_U a^{ij} u_{x_i} \Delta_k^{-h}((\xi^2 \Delta_k^h u)_{x_j}) \mathrm{d}x \\ &= \int_U \Delta_k^h (a^{ij} u_{x_i})(\xi^2 \Delta_k^h u)_{x_j} \mathrm{d}x \\ &= \int_U \left((\tau_k^h a^{ij}) \Delta_k^h u_{x_i} + (\Delta_k^h a^{ij} u_{x_i})\right) \left(\xi^2 \Delta_k^h u_{x_j} + 2\xi \xi_{x_j} \Delta_k^h u\right) \mathrm{d}x \\ &= A_1 + A_2 \end{aligned}$$

where

$$A_1 = \int_U \xi^2 (\tau_k^h a^{ij})(\Delta_k^h u_{x_i})(\Delta_k^h u_{x_j}) \mathrm{d}x$$

By uniform ellipticity,

$$\tau_k^h a^{ij} \eta_i \eta_j \geq \theta |\eta|^2$$

Applying with $\eta_i = \Delta_k^h u_{x_i}$, we get that

$$A_1 \geq \theta \int_U \xi^2 \left|\Delta_k^h (Du)\right|^2 \mathrm{d}x$$

Next,

$$A_2 = \int_U (\Delta_k^h a^{ij}) u_{x_i} \xi^2 \Delta_k^h u_{x_j} + 2\xi (\Delta_k^h a_{ij}) u_{x_i} \xi_j \Delta_k^h u + 2\xi (\tau_k^h a^{ij})(\Delta_k^h u_{x_i}) \xi_{x_i} \Delta_k^h u \mathrm{d}x$$

Since $a^{ij} \in C^2(U)$, and ξ is bounded,

$$|A_2| \leq C \int_W \xi |Du| \left|\Delta_k^h (Du)\right| + \xi |Du| \left|\Delta_k^h u\right| + \xi \left|\Delta_k^h (Du)\right| \left|\Delta_k^h u\right| \mathrm{d}x$$

We are interested in $\Delta_k^h (Du)$. We can use Young's inequality

$$\leq \varepsilon \int_W \xi^2 \left|\Delta_k^h (Du)\right|^2 \mathrm{d}x + \frac{C}{\varepsilon} \int_W |Du|^2 + \left|\Delta_k^h u\right|^2 \mathrm{d}x$$

By lemma 4.5.4,

$$\leq \varepsilon \int_W \xi^2 \left|\Delta_k^h (Du)\right|^2 \mathrm{d}x + \frac{C}{\varepsilon} \int_W |Du|^2 \mathrm{d}x$$

Set $\varepsilon = \theta/2$, and use $A_2 \geq -|A_2|$, we find that

$$A = A_1 + A_2 \geq \frac{\theta}{2}\int_W \xi^2 \left|\Delta_k^h(Du)\right|^2 \mathrm{d}x - C\int_W |Du|^2 \mathrm{d}x$$

Step 3 (Bounding B):

$$|B| \leq C\int_W (|f| + |Du| + |u|)\left|\Delta_k^{-h}(\xi^2\Delta_k^h u)\right|\mathrm{d}x$$

Applying lemma 4.5.4 again, and $(a+b)^2 \leq 2a^2 + 2b^2$,

$$\begin{aligned}\int_W \left|\Delta_k^h(\xi^2\Delta_k^h)\right|^2 \mathrm{d}x &\leq C\int_W \left|D(\xi^2\Delta_k^h u)\right|^2 \mathrm{d}x \\ &\leq C\int_W |\xi|^2|D\xi|^2\left|\Delta_k^h u\right|^2 \mathrm{d}x + C\int_W \xi^2\left|\Delta_k^h(Du)\right|^2 \\ &\leq C\int_W |Du|^2 + C\int_W \xi^2\left|\Delta_k^h(Du)\right|^2 \mathrm{d}x\end{aligned}$$

By Young's inequality on $|B|$,

$$|B| \leq \varepsilon\int_U \xi^2\left|\Delta_k^h(Du)\right|^2 \mathrm{d}x + \frac{C}{\varepsilon}\int_W (|f|^2 + u^2 + |Du|^2)\mathrm{d}x$$

Set $\varepsilon = \theta/4$.

Step 4: Since $A = B$, we have that $|A| = |B|$. Using the bounds that we have:

$$\frac{\theta}{2}\int_U \xi^2\left|\Delta_k^h(Du)\right|^2 \mathrm{d}x - C\int_W |Du|^2 \mathrm{d}x \leq |A| = |B| \leq \frac{\theta}{4}\int_U \xi^2\left|\Delta_k^h(Du)\right|^2 \mathrm{d}x + C\int_W (f^2 + u^2 + |Du|^2)\mathrm{d}x$$

Rearranging,

$$\int_U \xi^2\left|\Delta_k^h(Du)\right|^2 \mathrm{d}x \leq C\int_W f^2 + u^2 + |Du|^2 \mathrm{d}x$$

Since $\xi|_V = 1$, we get that if $u \in H^1(V)$ solves eq. (14), then

$$\int_V \left|\Delta_k^h(Du)\right|^2 \mathrm{d}x \leq C\int_W f^2 + u^2 + |Du|^2 \mathrm{d}x$$

Since C is independent of h (track every step), we can apply lemma 4.5.4, $Du \in H^2(V)$ and so $u \in H^2_{\text{loc.}}(U)$, with

$$\|u\|_{H^2(V)} \leq C\left(\left\|f\right\|_{L^2(W)} + \|u\|_{H^1(W)}\right)$$

Step 5: Removing the dependency on $\left\|Du\right\|_{L^2(W)}$. Let $\xi \in C_c^\infty(U)$ (be a different test function) with $\xi|_W = 1$. Set $v = \xi^2 u$ in eq. (14), to get

$$\int_U a^{ij}u_{x_i}(\xi^2 u)_{x_j} + b^i u_{x_i} + cu^2\xi^2 \mathrm{d}x = \int_U \xi^2 fu\mathrm{d}x$$

By the same proof as in Garding's inequality, we can rearrange to get

$$\left\|Du\right\|^2_{L^2(W)} \leq C\left(B[u,u] + \gamma\|u\|^2_{L^2(W)}\right) \leq C\left(\left\|f\right\|^2_{L^2(W)} + \|u\|^2_{L^2(W)}\right)$$

Hence we have that

$$\|u\|_{H^1(W)} \leq C\left(\left\|f\right\|_{L^2(W)} + \|u\|_{L^2(W)}\right)$$

and so we have the expression

$$\|u\|_{H^2(V)} \leq C\left(\left\|f\right\|_{L^2(W)} + \|u\|_{L^2(W)}\right)$$

□

Remark 4.5.7. 1. This is a local result. To have $u \in H^2(V)$, for $V \Subset U$, it is enough to have $f \in L^2(W)$, where $V \Subset W \Subset U$. That is, if $f \notin L^2$ near the boundary, we don't see this in our estimates.

2. We can now show that the equation $Lu = f$ holds pointwise a.e. To see this, $u \in H^2_{\text{loc.}}(U)$, and so $Lu \in L^2_{\text{loc.}}(U)$. Take $V \Subset U$. For $v \in C_c^\infty(U)$, then from eq. (14),

$$(Lu - f, v)_{L^2(U)} = 0$$

Since $Lu - f \in L^2(V)$, it holds pointwise a.e. on V.

Theorem 4.5.8 (higher order interior regularity)**.** If $a^{ij}, b^i, c \in C^m(U)$, and $f \in H^m(U)$, then $u \in H^{m+2}(U)$ and for all $V \Subset W \Subset U$,

$$||u||_{H^{m+2}(V)} \leq C \left(\|f\|_{H^m(U)} + ||u||_{L^2(U)} \right)$$

Remark 4.5.9. We also have a Hölder theory of elliptic regularity, i.e. if $f \in C^{k,\gamma}(U)$ then $u \in C^{k+2,\gamma}(U)$.

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Remark 4.5.10. Recall if m is large enough, i.e. $m > n/2$, then

$$H^{m+2}_{\text{loc.}}(U) \hookrightarrow C^2_{\text{loc.}}(U)$$

and so if $a^{ij}, b^i, c, f \in C^\infty(U)$, then u is also smooth.

Theorem 4.5.11 (boundary H^2 regularity)**.** Suppose $a^{ij} \in C^1(\overline{U}), b^i, c \in L^\infty(U)$, $f \in L^2(U)$, and ∂U is C^2. Suppose $u \in H^1_0(U)$ is a weak solution to

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{15}$$

Then $u \in H^2(U)$, and

$$||u||_{H^2(U)} \leq C \left(\|f\|_{L^2(U)} + ||u||_{L^2(U)} \right)$$

Sketch proof. We focus on the case

$$U = B_1(0) \cap \{x_n > 0\}$$

Let $V = B_{1/2}(0) \cap \{x_n > 0\}$ and $\xi \in C_x^\infty(B_1(0))$ with $\xi = 1$ on V, $0 \leq \xi \leq 1$. Since $u \in H^1_0$ is a weak solution,

$$\int_U a^{ij} u_{x_i} v_{x_j} = \int_U \tilde{f} v$$

for all $v \in H^1_0(U)$. Choose $0 < |h| < \frac{1}{4} d(\text{supp}(\xi), \partial B_1(0))$. Consider

$$v = -\Delta_k^{-h(\xi^2 \Delta_k^h u)}$$

for $k = 1, \ldots, n-1$ fixed.

Claim. $v \in H^1_0(U)$.

Proof.

$$\begin{aligned} v(x) &= \frac{-1}{h} \Delta_k^{-h} (\xi^2(x)(u(x + he_k) - u(x))) \\ &= \frac{1}{h^2} \left(\xi^2(x - he_k)(u(x) - u(x - he_k)) - \xi^2(x) u(x + he_k - u(x)) \right) \end{aligned}$$

The translation is horizontal, and $Tu|_{x_n=0} = 0$, and so

$$T(u(x \pm he_k))|_{x_n=0}$$

for all $|x| < 1 - h$. When $x_n = 0, |x| \geq 1 - h$, we have that $\xi(x) = 0$ and $\xi(x - he_k) = 0$. □

Repeating the proof of theorem 4.5.5 to conclude

$$\int_V \left|\Delta_k^h(Du)\right|^2 dx \leq C \int_U f^2 + u^2 + |Du|^2 dx$$

where C does not depend on h. Hence

$$D_k u \in H^1(V)$$

for $k = 1, \ldots, n-1$, with

$$\left\|D_k D_i u\right\|_{L^2(V)} \leq C(\left\|f\right\|_{L_2(U)} + \|u\|_{H^1(U)})$$

where $i = 1, \ldots, n$.

Hence it suffices to consider $u_{x_n x_n}$. We will use the equation for this. Write the PDE as

$$a^{nn} u_{x_n x_n} = F = -\sum_{i+j<2n} a^{ij} u_{x_i x_j} - \sum_i b^i u_{x_i} - cu + f$$

By uniform ellipticity,

$$a^{nn} = \sum a^{ij} \xi_i \xi_j \geq \theta \left\|\xi\right\|^2 = \theta > 0$$

where $\xi = (0, \ldots, 0, 1)$.

By the bound above, we can bound all of the terms of F, and so $F \in L^2(V)$, and

$$u_{x_n x_n} = \frac{1}{a^{nn}} F \in L^2(V)$$

and

$$\|u\|_{H^2(V)} \leq C(\left\|f\right\|_{L^2(U)} + \|u\|_{H^1(U)})$$

From the proof of Garding's inequality, we can replace $\|u\|_{H^1(U)}$ with $\|u\|_{L^2(U)}$ in the above.

To finish, we cover the boundary with a finite union of V_is, and sum using a partition of unity. See Evans for details. □

Corollary 4.5.12. Under the assumptions of theorem 4.5.11, if u is the unique weak solution to the boundary value problem eq. (15), then we have that

$$\|u\|_{H^2(U)} \leq C\left\|f\right\|_{L^2(U)} = C\left\|Lu\right\|_{L^2(U)}$$

i.e. we can drop the $\|u\|_{L^2(U)}$ terms.

That is, the $\|u\|_{L^2(U)}$ measures the kernel of L, and so if the solution is unique, the kernel is zero.

Remark 4.5.13. We can get higher regularity. If $a^{ij}, b^i, c \in C^{m+1}(\overline{U}), f \in H^m(\overline{U})$, ∂U is C^{m+2}, and $u \in H_0^1$ a weak solution, then $u \in H^{m+2}(U)$ and

$$\|u\|_{H^{m+2}(U)} \leq C(\left\|f\right\|_{H^m(U)} + \|u\|_{L^2(U)})$$

Remark 4.5.14. If everything is smooth, then u is smooth. For example, if $Lu = \lambda u$, then $L - \lambda I$ is uniformly elliptic, and

$$(L - \lambda I)u = 0 \in C^\infty$$

and so the eigenfunctions are smooth.

5 Hyperbolic PDEs

We will consider *second order linear PDE* of the form

$$\sum_{i=1}^{n+1}(a^{ij}u_{y_i})_{y_j} + \sum_{i=1}^{n+1} a^i(u)u_{y_i} + a(y)u = f \tag{16}$$

with $y \in \mathbb{R}^{n+1}$, $a^{ij} = a^{ji}$, $a^{ij}, a^i, a \in C^\infty(\mathbb{R}^{n+1})$. This equation is *hyperbolic* if the quadratic form

$$q(\xi) = \sum_{i,j=1}^{n+1} a^{ij}(y)\xi_i\xi_j$$

has signature $(+, -, \dots, -)$ for all $y \in \mathbb{R}^{n+1}$. That is, at each $y \in \mathbb{R}^{n+1}$, after a change of basis, we can write q as

$$\lambda_{n+1}^2\xi_{n+1}^2 - \sum_{i=1}^{n} \lambda_i^2\xi_i^2$$

where each $\lambda_i > 0$.

We call q the *principal symbol* of the PDE. By a coordinate transformation, locally we can put eq. (16) in the form

$$u_{tt} - \sum_{i,j=1}^{n}(a^{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^{n+1} b^i(x,t)u_{x_i} + c(x,t)u \tag{17}$$

where $(y_1, \dots, y_{n+1}) = (x_1, \dots, x_n, t)$.

Note if we assume

$$\sum_{i,j} a^{ij}\xi_i\xi_j \geq \theta\|\xi^2\|$$

then since the coefficient of u_{tt} is 1, which is non-zero, then we see that

$$\{(x,t) \mid t = 0\}$$

is a non-characteristic surface of the PDE. In principle, we can solve the PDE with analytic data u, u_t at $t = 0$.

5.1 Hyperbolic initial boundary value problems

Suppose $U \subseteq \mathbb{R}^n$ is open bounded with C^1 boundary. Define

$$U_T = (0,T) \times U \qquad \Sigma_t = \{t\} \times U \qquad \partial^* U_T = [0,T] \times \partial U$$

Using this,

$$\partial U_T = \Sigma_0 \cup \Sigma_1 \cup \partial^* U_T$$

Let $u \in C^2(U_T)$, which satisfies the initial boundary value problem

$$\begin{cases} u_{tt} = \Delta u & \text{in } U_T \\ u = \psi_0 & \text{on } \Sigma_0 \\ u_t = \psi_1 & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases}$$

We will perform an energy estimate. Multiply the PDE by u_t, integrating by parts over $U_t = (0,t) \times U$ for $t \in (0,T)$, we get

$$0 = \int_{U_t} u_{tt}u_t - u_t\Delta u \mathrm{d}x\mathrm{d}t$$

In what follows, D will denote the derivative with respect to the space variables only. Recall $\mathrm{grad} \cdot (g\,\mathrm{grad} h) = \mathrm{grad} g \cdot \mathrm{grad} h + g\Delta h$, and so we get

$$\begin{aligned} &= \int_{U_t} \left(\frac{1}{2}\partial_t(u_t^2) - \mathrm{div}_x(u_tDu) + Du_tDu\right)\mathrm{d}x\mathrm{d}t \\ &= \int_{U_t} \frac{1}{2}\partial_t\left((u_t)^2 + |Du|^2\right) - \mathrm{div}_x(u_tDu)\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{2}\int_{\Sigma_t}(u_t^2 + |Du|^2)\mathrm{d}x - \frac{1}{2}\int_{\Sigma_0}(u_t)^2 + |Du|^2\mathrm{d}x \end{aligned}$$

where we use the divergence theorem, and the fact that u vanishes on $\partial^* U_t$. Hence we have that

$$\int_{\Sigma_t} u_t^2 + |Du|^2 \mathrm{d}x = \int_{\Sigma_0} \psi_1^2 + |D\psi_0|^2 \mathrm{d}x$$

Ww call this an energy estimate as the energy is conserved, where u_t is kinetic energy and $|Du|^2$ is potential energy.

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We call ths estimate above an *a priori estimate*. These are very useful.

Let $v, \overline{v} \in C^2(U_T)$ be two solutions with initial data $\phi_i, \overline{\phi_i}$. Let $u = v - \overline{v}$, $\psi_0 = \phi_0 - \overline{\phi_0}$ and $\psi_1 = \phi_1 - \overline{\phi_1}$. Then there exists $C > 0$ such that

$$\sup_{t \in [0,T]} \left(\|u(\cdot, t)\|^2_{H^1(\Sigma_t)} + \|u_t(\cdot, t)\|^2_{L^2(\Sigma_t)} \right) \leq C \left(\|\psi_0\|^2_{H^1(\Sigma_t)} + \|\psi_1\|^2_{L^2(\Sigma_0)} \right)$$

Thus, in this case, we have uniqueness and continuous dependence on initial conditions.

Define

$$Lu = -\sum_{i,j=1}^n (a^{ij}(x,t)u_{x_i})x_j + \sum_{i=1}^n b^i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u$$

with $a^{ij} = a^{ji}, b^i, b, c \in C^1(\overline{U_T})$. Suppose there exists $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $(x,t) \in U_T$, $\xi \in \mathbb{R}^n$.

We will consider the initial boundary value problem

$$\begin{cases} u_t t + Lu = f & \text{in } U_T \\ u = \psi_0, u_t = \psi_1 & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases} \tag{18}$$

We would like to find the weak formulation.

Suppose $u \in C^2(\overline{U_T})$ is a solution to eq. (18). Multiply by $v \in C^2(\overline{U_T})$, such that $v = 0$ on $\partial^* U_T \cup \Sigma_T$.

Integrating over U_T,

$$\begin{aligned} \int_{U_T} fv \mathrm{d}x\mathrm{d}t &= \int_{U_T} (u_{tt}v + Luv)\mathrm{d}x\mathrm{d}t \\ &= \int_{U_T} (-u_t v_t + a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i} v + bu_t v + cuv)\mathrm{d}x\mathrm{d}t + \left[\int_{\Sigma_t} u_t v \mathrm{d}x\right]_{t=0}^T - \int_0^T \int_{\partial\Sigma_t} a^{ij}u_{x_i} v \mathrm{d}\mathbf{S}\mathrm{d}t \\ &= \int_{U_T} (-u_t v_t + a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i} v + bu_t v + cuv)\mathrm{d}x\mathrm{d}t - \int_{\Sigma_0} \psi_1(x)v(x,0)\mathrm{d}x \end{aligned}$$

Thus, we have the equation

$$\int_{U_T} fv \mathrm{d}x\mathrm{d}t = \int_{U_T} (-u_t v_t + a^{ij}u_{x_i}v_{x_j} + b^i u_{x_i} v + bu_t v + cuv)\mathrm{d}x\mathrm{d}t - \int_{\Sigma_0} \psi_1(x)v(x,0)\mathrm{d}x \tag{19}$$

Now suppose eq. (19) holds for all $v \in C^2(\overline{U_T})$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$. If $v \in C_c^\infty(\overline{U_T})$, then we can undo the integration by parts, and we get that

$$\int_{U_T} v(u_{tt} + Lu - f)\mathrm{d}x\mathrm{d}t = 0$$

Since v is arbitrary, $u_{tt} + Lu = f$ on U_T.

Now if $v \in C^\infty(\overline{U_T})$, then we get that

$$\int_{U_T} (u_{tt} + Lu - f)\mathrm{d}x\mathrm{d}t = \int_{\Sigma_0} (\psi_1 - u_t)v\mathrm{d}x$$

Hence

$$\int_{\Sigma_0} (\psi_1 - u_t) v \mathrm{d}x = 0$$

for all $v \in C^\infty(\overline{U_T})$ with $v = 0$ on $\partial^* U_T \cup \Sigma_T$. Now let $v(x,t) = \chi(t)\varphi(x)$, with $\chi \in C^\infty([0,T])$ and $\varphi \in C_c^\infty(\Sigma_0)$. We require that $\chi = 1$ near $t = 0$ and $\chi = 0$ near $t = T$, hence

$$v|_{\Sigma_0} = \varphi$$

Hence

$$\int_{\Sigma_0} (\psi_1(x) - u_t(x,0))\varphi(x) \mathrm{d}x = 0$$

and so $\psi_1 = u_t$ on Σ_0.

Definition 5.1.1 (weak solution)

Suppose $f \in L^2(U_T)$, $\psi_0 \in H_0^1(\Sigma_0)$, $\psi_1 \in L^2(\Sigma_0)$, $a^{ij} = a^{ji}, b^i, b, c \in C^1(\overline{U_T})$. We say that $u \in H^1(U_T)$ is a *weak solution to the hyperbolic initial boundary value problem* eq. (18) if $u|_{\Sigma_0} = \psi_0$, $u|_{\partial^* U_T} = 0$ in the trace sense, and eq. (19) for all $v \in H^1(U_T)$, with $v = 0$ on $\partial^* U_T \cup \Sigma_T$ in the trace sense.

Theorem 5.1.2 (uniqueness). A weak solution, if it exists, is unique.

Proof. If $v, \overline{v}$ are two weak solutions with the same initial data, then we can use the linearity of the PDE problem, $u = v - \overline{v}$ is a weak solution with $f = 0$, $\psi_0 = 0$ and $\psi_1 = 0$.

The idea is to use an energy to show that $||u|| = 0$ to show that $u = 0$. We would like to pick $v = u_t$, as we did for the wave equation, but

1. v may not be in $H^1(U_T)$, since we only know that $u \in H^1$.
2. v may not vanish on Σ_T.

We set

$$v(x,t) = \int_t^T e^{-\lambda s} u(x,s) \mathrm{d}t$$

where we will choose λ later. We can see that v is in $H^1(U_T)$, and $v = 0$ on $\partial^* U_T \cup \Sigma_T$. Moreover,

$$v_t = -e^{-\lambda t} u(x,t)$$

We will take this v as the test function. This gives us

$$\int_{U_T} u_t u e^{-\lambda t} - e^{\lambda t} a^{ij} v_{t x_i} v_{x_j} + b^i u_{x_i} v + b u_t v + (c-1) u v - e^{\lambda t} v v_t \mathrm{d}x\mathrm{d}t = 0$$

Integrating by parts,

$$\int_{U_T} u_t u e^{-\lambda t} - e^{\lambda t} a^{ij} v_{t x_j} v_{x_i} + \underbrace{(b^i u v)_{x_i}}_{a} + \underbrace{(b u v)_t}_{b} - (b^i_{x_i} u v + b^i u v_{x_i} + b_t u v + b u v_t) + (c-1) u v - \frac{1}{2}\partial_t (v^2 e^{\lambda t}) + \frac{1}{2}\lambda v^2 e^{\lambda t} \mathrm{d}x\mathrm{d}t$$

The terms a and b vanish by boundary conditions. Hence we have that

$$\begin{aligned}\int_{U_T} \frac{1}{2}\frac{\partial}{\partial t}\left(u^2 e^{-\lambda t} = a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - v^2 e^{\lambda t}\right) \mathrm{d}x\mathrm{d}t + \frac{\lambda}{2}\int_{U_T} (u^2 e^{-\lambda t} + a^{ij} e^{\lambda t} v_{x_i} v_{x_j} + v^2 e^{\lambda t}) \mathrm{d}x\mathrm{d}t \\ = \int_{U_T} \frac{1}{2} a^{ij}_j v_{x_i} v_{x_j} e^{\lambda t} + (b^i_{x_i} + b_t + 1 - c) u v + b^i v_{x_i} u + b u v_t \mathrm{d}x\mathrm{d}t\end{aligned}$$

Call the first line A and the second B. For A,

$$A = e^{\lambda T} \int_{\Sigma_T} \frac{1}{2} u^2 \mathrm{d}x + \frac{1}{2}\int_{\Sigma_0} a^{ij} v_{x_i} v_{x_j} + v^2 \mathrm{d}x + \frac{\lambda}{2}\int_{U_T} u^2 e^{-\lambda t} + e^{\lambda t} a^{ij} v_{x_i} v_{x_j} + v^2 e^{\lambda t} \mathrm{d}x\mathrm{d}t$$

Hence we have that

$$A \geq \frac{\lambda}{2}\int_{U_T} u^2e^{-\lambda t} + \theta|Du|^2e^{\lambda t} + v^2e^{\lambda}t\mathrm{d}x\mathrm{d}t$$

Also,

$$\begin{aligned}
B &\leq C(a_t^{ij})\int_{U_T} e^{\lambda t}|Dv|^2\mathrm{d}x + C(b,b^i,c)\int_{U_T}|u||v|\mathrm{d}x + C(b^i)\int_{U_T}|u||Dv|\mathrm{d}x + C(b)\int_{U_T}u^2e^{-\lambda t}\mathrm{d}x \\
&\leq \frac{C}{\theta}\int_{U_T} e^{\lambda t}\theta|Dv|^2 + C\int_{U_T} e^{-\lambda t}|u^2| + e^{\lambda t}(|v|^2|Dv|^2)\mathrm{d}x\mathrm{d}t \\
&\leq C\int_{U_T}\theta|Dv|^2e^{\lambda t} + u^2e^{-\lambda t} + v^2e^{\lambda t}\mathrm{d}x\mathrm{d}t
\end{aligned}$$

Now using that $|A| = |B|$,

$$\left(\frac{\lambda}{2} - C\right)\int_{U_T}\underbrace{(u^2e^{-\lambda t} + \theta|Dv|^2 + v^2e^{-\lambda t})}_{\leq 0}\mathrm{d}x\mathrm{d}t \leq 0$$

Taking $\lambda > 2C$, the integral must be zero, and so

$$\int_{U_T} u^2e^{-\lambda t}\mathrm{d}x\mathrm{d}t = 0$$

Hence $u = 0$ a.e. □

Lecture 21

Theorem 5.1.3 (existence of solutions). Given $\psi_0 \in H_0^1(U), \psi_1 \in L^2(U), f \in L^2(U_T)$, then there exists a unique weak solution of eq. (19) $u \in H^1(U_T)$, with

$$||u||_{H^1(U_T)} \leq C\left(\|\psi_0\|_{H^1(U)} + \|\psi_1\|_{H^1(U)} + \|f\|_{L^2(U_T)}\right)$$

Proof (Galerkin's method). We will project everything onto the finite dimensional subspace of L^2, given by the first N eigenfunctions of the Dirichlet Laplacian. Taking $N \to \infty$ gives the result.

Step 1: Recall the eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ of $L = -\Delta$ with Dirichlet boundary conditions form an orthonormal basis of $L^2(U)$. We have that $\varphi_k \in H_0^1(U)$, and by elliptic regularity $\varphi_k \in C^\infty(U)$ privided ∂U is C^∞. With this,

$$(\varphi_k, \varphi_\ell)_{L^2(U)} = \delta_{k\ell}$$

and if $u \in L^2(U)$ then

$$u = \sum_{k=1}^{\infty}(u,\varphi_k)_{L^2(U)}\varphi_k$$

with convergence in $L^2(U)$.

Step 2: First consider $\psi_0, \psi_1 \in C_c^\infty(U)$, $f \in C_c^\infty(U_T)$. These spaces are dense in $\mathrm{H}_0^1(U)$, $L^2(U)$ and $L^2(U_T)$ respectively. Define

$$u^N(x,t) = \sum_{k=1}^{N} u_k(t)\varphi_k(x)$$

Assume $u_k(t) \in C^2(\overline{(0,T)})$, and that u^N is a weak solution to eq. (19). Take $v(x,t) = \rho(t)\varphi_\ell(x)$ for the test function, with $\rho \in C_c^\infty((0,T))$ arbitrary. Substituting into eq. (19), we get

$$\int_{U_T} -(u_t^N\dot{\rho}\varphi_\ell + a^{ij}u_{x_j}^N(\varphi_\ell)_{x_i}\rho + b^iu_{x_i}^N\rho\varphi_\ell + bu_t^N\rho\varphi_\ell + cu\rho\varphi_\ell - f\rho\varphi_\ell)\mathrm{d}x\mathrm{d}t = 0$$

Note

$$\int_{U_T} -u_t^N\dot{\rho}\varphi_\ell\mathrm{d}x\mathrm{d}t = \int_{U_T} u_{tt}^N\rho\varphi_\ell\mathrm{d}x\mathrm{d}t$$

and so our identity looks like

$$\int_0^t \int_{\Sigma_t} G(x,t)\rho(t)\mathrm{d}x\mathrm{d}t = 0$$

But ρ is arbitrary, and so

$$\int_{\Sigma_t} G(x,t)\mathrm{d}x = 0$$

for all t. With this,

$$\left(u_{tt}^N, \varphi_\ell\right)_{L^2(\Sigma_t)} + \int_{\Sigma_t} a^{ij} u_{x_j}^N (\varphi_\ell)_{x_i} + b^i (u^N)_{x_i} \varphi_\ell + b u_t^N \varphi_\ell + c u^N \varphi_\ell \mathrm{d}x = (f, \varphi_\ell)_{L^2(\Sigma_t)} \tag{20}$$

But eq. (20) holds for all t, and $\ell = 1, \dots, N$. By orthogonality,

$$\left(u_{tt}^N, \varphi_\ell\right)_{L^2(\Sigma_t)} = \sum_{k=1}^M (\ddot{u}_k(t)\varphi_k, \varphi_\ell)_{L^2(\Sigma_t)} = \ddot{u}_\ell(t)$$

With this, we get that for $\ell = 1, \dots, N$, then

$$\ddot{u}_\ell(t) + \sum_{k=1}^N (\alpha_{\ell,k}(t) u_k(t) + \beta_{\ell,k}(t)\dot{u}_k(t)) = f_\ell(t)$$

where

$$\begin{aligned}
\alpha_{\ell,k}(t) &= \int_{\Sigma_t} a^{ij}(\varphi_\ell)_{x_j}(\varphi_k)(x_i) + b^i(\varphi_\ell)_{x_i}\varphi_k + c\varphi_\ell\varphi_k \mathrm{d}x \\
\beta_{\ell,k}(t) &= \int_{\Sigma_t} b(x,t)\varphi_k\varphi_\ell \mathrm{d}x \\
f_\ell(t) &= \int_{\Sigma_t} f(x,t)\varphi_\ell(x)\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
u_\ell(0) &= (\psi_0, \varphi_\ell)_{L^2(\Sigma_0)} \\
\dot{u}_\ell(0) &= (\psi_1, \varphi_\ell)_{L^2(\Sigma_0)}
\end{aligned}$$

This is a system of N second order ODEs, linear in u_k, with coefficients which are bounded uniformly in C^1 for $t \in [0, T]$. By Picard-Lindelöf, there exists a unique solution $u_k \in C^2([0, T])$. Moreover,

$$u^N, u_t^N \in H^1(U_T)$$

Step 3: We would like a uniform estimates

$$\left\|u^N\right\|_{\mathrm{H}^1(U_T)} \le C$$

which are independent of N. Multiply eq. (20) by $e^{-\lambda t}\dot{u}_\ell(t)$, sum over $1, \dots, n$, and integrate over $[0, \tau] \subseteq [0, T]$.

For example,

$$\sum_{\ell=1}^N \int_{-\lambda t} \dot{u}_\ell(t) \int_{\Sigma_t} u_{tt}^N \varphi_\ell \mathrm{d}x\mathrm{d}t = \int_{U_\tau} e^{-\lambda t} u_{tt}^N u_t^N \mathrm{d}x\ ddt$$

We find that

$$\int_{U_\tau} \left(u_{tt}^N u_t^N + a^{ij} u_{x_i}^N u_{tx_j}^N + b^i u_{x_i}^N u_t^N + b(u_t^N)^2 + c u^N u_t^N\right) e^{-\lambda t}\mathrm{d}x\mathrm{d}t = \int_{U_\tau} f u_t^N e^{-\lambda t}\mathrm{d}x\mathrm{d}t$$

Similar to the proof of uniqueness, we can rearrange this as

$$\begin{aligned}
\tilde{A} &= \int_{U_\tau} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(Q_a e^{-\lambda t}\right)\mathrm{d}x\mathrm{d}t + \frac{\lambda}{2}\int_{U_\tau} Q_a e^{-\lambda t}\mathrm{d}x\mathrm{d}t \\
\tilde{B} &= \int_{U_\tau} \left(\frac{1}{2} a_t^{ij} u_{x_i}^N u_{x_j}^N - b^i u_{x_i}^N u_t^N = b(u_t^N)^2 + (1-c) u^N u_t^N + f u_t^N\right) e^{-\lambda t}\mathrm{d}x\mathrm{d}t \\
\tilde{A} &= \tilde{B}
\end{aligned}$$

where

$$Q_a = (u_t^N)^2 + a^{ij}u^N_{x_i}u^N_{x_j} + (u^N)^2$$

Let

$$Q_\theta = (u_t^N)^2 + \theta\left|Du^N\right|^2 + (u^N)^2$$

Using uniform ellipticity, Young's inequality, $e^{-\lambda t} \leq 1$, and so on, we get that

$$\begin{aligned}\tilde{B} &\leq \int_{U_\tau} Q_\theta e^{-\lambda t}\mathrm{d}x\mathrm{d}t + \left\|f\right\|^2_{L^2(U_\tau)} \\ \tilde{A} &\geq e^{\lambda\tau}\int_{\Sigma_\tau} Q_\theta \mathrm{d}x - \frac{1}{2}\int_{\Sigma_\tau} Q_\theta\mathrm{d}x + \frac{\lambda}{2}\int_{U_\tau} Q_\theta e^{-\lambda t}\mathrm{d}x\mathrm{d}t\end{aligned}$$

Note $\left|\tilde{A}\right| = \left|\tilde{B}\right|$, for $\lambda/2 - C \geq 1/2$, we get

$$\begin{aligned}e^{-\lambda\tau}\int_{\Sigma_\tau} Q_\theta\mathrm{d}x + \int_0^\tau\int_{\Sigma_t} Q_\theta e^{-\lambda t}\mathrm{d}x\mathrm{d}t &\leq \int_{\Sigma_0} Q_a\mathrm{d}x + C\left\|f\right\|^2_{L^2(U_\tau)} \\ &\leq C\left(\left\|u^N(\cdot,0)\right\|^2_{H^1(\Sigma_0)} + \left\|\dot{u}^N(\cdot,0)\right\|^2_{L^2(\Sigma_0)} + \left\|f\right\|^2_{L^2(U_T)}\right)\end{aligned}$$

for all $\tau \in [0, T]$. Taking sup,

$$\begin{aligned}\sup_\tau\left(\left\|u^N(\cdot,\tau)\right\|^2_{H^1(\Sigma_t)} + \left\|\cdot u^N(\cdot,\tau)\right\|^2_{L^2(\Sigma_\tau)}\right) + \int_0^T\left(\left\|u^N(\cdot,t)\right\|^2_{H^1(\Sigma_t)} + \left\|\cdot u^N(\cdot,t)\right\|^2 - L^2(\Sigma_t)\right)\mathrm{d}t \\ \leq Ce^{\lambda T}\left(\left\|u^N(\cdot,0)\right\|^2_{H^1(\Sigma_0)} + \left\|\dot{u}^N(\cdot,0)\right\|^2_{L^2(\Sigma_0)} + \left\|f\right\|^2_{L^2(U_T)}\right)\end{aligned}$$

Since

$$u^N(0) = \sum_{k=1}^N(\psi_0, \varphi_k)\varphi_k \to \psi_0$$

as $N \to \infty$, if $\psi_0 \neq 0$, then for large N, then

$$\left\|u^N(0)\right\|_{H^1(\Sigma_0)} \leq 2\left\|\psi_0\right\|\left\|\psi_0\right\|_{H^1(\Sigma_0)}$$

Similarly,

$$\left\|\dot{u}^N\right\|_{L^2(\Sigma_0)} \leq 2\left\|\psi_1\right\|_{L^2(\Sigma_0)}$$

The right hand sides are independent of N, and so

$$\left\|u^N\right\|_{H^1(U_T)} \leq C_1 = C\left(\left\|\psi_0\right\|_{H^1(\Sigma_0)} + \left\|\psi_1\right\|_{H^1(\Sigma_0)} + \left\|f\right\|_{L^2(U_T)}\right)$$

The right hand side is the uniform estimate which we want. Now Lecture 22

$$u^N \in H^1_\partial(U_T) = \left\{\phi \in H^1(U_T) \mid \phi|_{\partial^* U_T} = 0\right\}$$

which is a closed subspace of $H^1(U_T)$, and so it is weakly sequentially compact. Hence there exists a subsequence (u^{N_i}) such that

$$u^{N_i} \rightharpoonup u \in H^1_\partial(U_T)$$

Moreover,

$$\|u\|_{H^1(U_T)} = \liminf_{i\to\infty}\left\|u^{N_i}\right\|_{H^1(U_T)} \leq C_1$$

Step 4: We want to show that u is the desired weak solution. We can relabel the u^{N_i} as u^N. Fix $m \leq N$, and consider

$$v = \sum_{k=1}^m v_k(t)\varphi_k(x)$$

where $v_k \in H^1((0,T))$ with $v_k(T) = 0$. Then v is a test function for the weak formulation. From eq. (20) (replace ℓ with k), multiply the equation by $v_k(t)$, and sum from $k = 1, \ldots, m$, we get that

$$\left(u^N_{tt}, v\right)_{L^2(\Sigma_t)} + \int_{\Sigma_t} a^{ij}u^N_{x_i}v_{x_j} + b^i u^M_{x_i}v + bu^N_t v + cu^N v\mathrm{d}x = (f, v)_{L^2(\Sigma_t)}$$

Now integrating over $[0, T]$, integrating by parts, and using the fact that $v(T) = 0$, we get that

$$-\int_{\Sigma_0} u_t^N v\mathrm{d}x + \int_{U_T} -u_N^t v_t + a^{ij} u_{x_i}^N v_{x_j} + b^i u_{x_i}^N v + b u_t v + c u^N v\mathrm{d}x\mathrm{d}t = \int_{U_T} f v\mathrm{d}x\mathrm{d}t$$

But for the first term, since $N > m$,

$$\int_{\Sigma_0} u_t^N v\mathrm{d}x = \int_{\Sigma_0} \psi_1 v\mathrm{d}x$$

Passing to the weak limit,

$$-\int_{\Sigma_0} \psi_1 v\mathrm{d}x + \int_{U_T} (-u_t v_t + a^{ij} u_{x_i} v_{x_j} + b^i u_{x_i} v + b u_t v + c u v)\mathrm{d}x\mathrm{d}t = \int_{U_T} f v\mathrm{d}x\mathrm{d}t \tag{21}$$

But this is precisely the weak formulation. We leave as an exercise that the space of such v is dense in $H^1_\partial(U_T)$, and so eq. (21) holds for all $v \in H^1_\partial(U_T)$.

Step 5: It remains to show that $u|_{\Sigma_0} = \psi_0$. For each fixed k, define

$$\begin{aligned}\Phi_k : H^1(U_T) &\to \mathbb{R} \\ w &\mapsto \int_{\Sigma_0} w\varphi_k \mathrm{d}x\end{aligned}$$

This is a bounded linear map. To see this,

$$|\Phi_k(w)| \le \int_{\Sigma_0} |w\varphi_k| \le \|w\|_{L^2(\Sigma_0)} \left\|\varphi_k\right\|_{L^2(\Sigma_0)} \le \|w\|_{L^2(\partial U_T)} \le C\|w\|_{H^1(U_T)}$$

where in the last step we use the trace theorem. By weak convergence,

$$\Phi_k(u^N) \to \Phi_k(u)$$

Thus,

$$\int_{\Sigma_0} \psi_0 \varphi_k \mathrm{d}x = \int_{\Sigma_0} u^N(x, 0)\varphi_k(x)\mathrm{d}x \to \int_{\Sigma_0} u(x, 0)\varphi_k(x)\mathrm{d}x$$

Hence

$$\int_{\Sigma_0} (\psi_0 - u(x, 0))\, \varphi_k \mathrm{d}x = 0$$

for all k. Hence $u = \psi_0$ on Σ_0. □

Remark 5.1.4. This proof fails when $T = \infty$, or U is unbounded. See Hille-Yosida, in Brezis' book.

Definition 5.1.5 (Bochner space)

If X is a Banach space, the *Bochner space* $L^p((0, T); X)$ is

$$L^p((0, T); X) = \left\{u : (0, T) \to X \mid \|u\|_{L^p((0,T);X)} < \infty\right\}$$

where

$$\|u\|_{L^p((0,T);X)} = \left(\int_0^T \|u\|_X^p \mathrm{d}t\right)^{1/p}$$

for $1 \le p < \infty$, and

$$\|u\|_{L^\infty((0,T);X)} = \operatorname*{ess\,sup}_{t\in(0,T)} \left\|u(t)\right\|_X$$

Remark 5.1.6. In step 3, we showed that

$$\|u\|_{H^1(U_T)} \le C_1$$

In fact, the weak solution satisfies

$$\left\|u_t\right\|_{L^\infty((0,T);L^2(U))} + \|u\|_{L^\infty((0,T);H^1(U))} \le C_1$$

> Thus, instead of $H^1(U_T)$, we can consider
> $$u \in L^\infty((0,T); H^1(U))$$

5.2 Finite speed of propagation

A crucial feature of hyperbolic equation is that there is a finite speed of propagation.

> **Definition 5.2.1** (spacelike, timelike)
> Let $\Sigma \subseteq \mathbb{R}^{n+1}$ be a hypersurface, given by
> $$\Sigma = \left\{(x,t) \in \mathbb{R}^{n+1} \mid F(x,t) = 0\right\}$$
> Define
> $$w(F_{x_i}, F_t) = (F_t)^2 - a^{ij} F_{x_i} F_{x_j}$$
> We say that Σ
>
> - spacelike if $w > 0$,
> - timelike if $w < 0$,
> - characteristic (in PDE theory) or null (in GR) if $w = 0$.

> **Example 5.2.2**
> The plane $t = 0$ is spacelike.

> **Example 5.2.3**
> The cylinder
> $$F = |x - x_0|^2 - R^2$$
> is timelike

Let $S_0 \subseteq U$ be an open set with smooth boundary. Let $\tau : S_0 \to (0,T)$ be a smooth function such that $\tau|_{\partial S_0} = 0$. Let
$$S' = \text{Graph}(\tau) = \{(x, \tau(x)) \mid x \in S_0\}$$
If $F(x_1, \dots, x_n, t) = t = \tau(x)$, then we see that S' is spacelike if
$$1 - a^{ij}\tau_{x_i}\tau_{x_j} > 0$$
Equivalently,
$$a^{ij}(x)\tau_{x_i}\tau_{x_j} < 1$$
Let
$$D = \{(x,t) \in U_T \mid x \in S_0, 0 \leq t \leq \tau(x)\}$$

Exercise: if $a^{ij}\xi_i\xi_j \leq \mu|\xi|^2$, for some $\mu > 0$, then we can show that such S_0, S' exists.

> **Theorem 5.2.4** (domain of dependence). If S' is spacelike, u a weak solution to eq. (19), then $u|_D$ depends only on $\psi_0|_{S_0}, \psi_1|_{S_0}, f|_D$.

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Proof. The proof is similar to the proof of uniquness. By linearity, it suffices to show that $u|_D = 0$ if $\psi_0|_{S_0} = 0, \psi_1|_{S_0} = 0$ and $f|_D = 0$. Take a test function
$$v(x,t) = \begin{cases} \int_t^\tau e^{-\lambda s} u(x,s) ds & (x,t) \in D \\ 0 & \text{otherwise} \end{cases}$$

We leave the proof that $v \in H^1(U_T)$, with $v = 0$ on $\partial^* U_T \cup \Sigma_T$, and

$$\begin{aligned} v_{x_i} &= \tau_{x_i} e^{-\lambda \tau(x)} u(x, \tau(x)) + \int_t^{\tau(x)} e^{-\lambda s} u_{x_i}(x, s) \mathrm{d}s \\ v_t &= -e^{-\lambda t} u(x, t) \end{aligned}$$

on D. These vanish outside of D. Inserting this into the definition of the weak solution, we find that

$$\begin{aligned} \overline{A} &= \int_D \frac{1}{2} \partial_t \left(u^2 e^{-\lambda t} - a^{ij} v_{x_i} v_{x_j} e^{\lambda t} - v^2 e^{\lambda t} \right) \mathrm{d}x \mathrm{d}t \\ A &= \frac{\lambda}{2} \int_D (u^2 e^{-\lambda t} + a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + v^2 e^{\lambda t}) \mathrm{d}x \mathrm{d}t \\ B &= \int_D \frac{1}{2} a^{ij} v_{x_i} v_{x_j} e^{\lambda t} + (b^i_{x_i} + b_t + c) uv + b^i v_{x_i} u + b u u_t \mathrm{d}x \mathrm{d}t \\ \overline{A} + A &= B \end{aligned}$$

By Fubini,

$$\int_D \mathrm{d}x \mathrm{d}t = \int_{S_0} \mathrm{d}x \int_0^{\tau(x)} \mathrm{d}t$$

Using $v|_{S'} = 0$, and $v_{x_i}|_{S'} = \tau_{x_i} u(x_i, \tau(x)) e^{-\lambda \tau(x)}$, we get that

$$\overline{A} = \frac{1}{2} \int_{S_0} u^2(x, \tau(x)) e^{\lambda \tau(x)} (1 - a^{ij} \tau_{x_i} \tau_{x_j}) \mathrm{d}x + \frac{1}{2} \int_{S_0} (a^{ij} v_{x_i} v_{x_j} + v^2)|_{t=0} \mathrm{d}x$$

Continuing as in the proof of uniqueness,

$$\left(\frac{\lambda}{2} - c \right) \int_D u^2 e^{-\lambda t} + \theta |Du|^2 e^{\lambda t} + v^2 e^{\lambda t} \mathrm{d}x \mathrm{d}t \le 0$$

If λ is large, this forces $u|_D = 0$. □

> **Remark 5.2.5.** No signal can travel faster than a fixed speed. Let $x_0 \in U$ and S_0 some ball about x_0. If $(x_0, t) \in D$, then any data outside S_0 does not influence $u(x_0, t)$. Only after $t > \tau(x_0)$ will the function be determined by data outside s_0.
>
> Therefore, everything is local in hyperbolic PDEs.

5.3 Hyperbolic regularity

So far, we have shown existence to and uniqueness of weak solutions to

$$u_{tt} + Lu = f$$

with given initial and boundary conditions. Given $\psi_0 \in H^1_0(U), \psi_1 \in L^2(U), f \in L^2(U_T)$, we have shown

$$||u||_{L^\infty_t H^1_x} + \left\| u_t \right\|_{L^\infty_t L^2_x} + ||u||_{H^1(U_T)} \le C \left(\left\| \psi_0 \right\|_{H^1(U)} + \left\| \psi_1 \right\|_{L^2(U)} + \left\| f \right\|_{L^2(U)} \right)$$

where $L^\infty_t H^1_x = L^\infty((0, T), H^1(U))$. In this case, we did not manage to improve the regulaity when compared to the initial conditions.

> **Example 5.3.1**
>
> Suppose $u \in C^\infty(U_T)$ which solves
>
> $$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } U_T \\ u = \psi_0, u_t = \psi_1 & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases}$$

Let $w = u_t$. Then

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } U_T \\ w = \psi_1, w_t = \Delta\psi_0 & \text{on } \Sigma_0 \\ w = 0 & \text{on } \partial^* U_T \end{cases}$$

Using the above estimate, we have that

$$\|w\|_{L_t^\infty H_0^1} + \|w_t\|_{L_t^\infty L_x^2} + \|w\|_{H^1(U_T)} \le C\left(\|\psi_1\|_{H^1(U)} + \|\Delta\psi_0\|_{L^2(U)}\right)$$

Hence we have control over $u_{tt}, u_{x_i t}$ in $L^2(U)$ in terms of initial data. To control $u_{x_i x_j}$, we use elliptic regularity. In particular,

$$\|u\|_{H^2(U)} \le C\|\Delta u\|_{L^2(U)} = C\|u_{tt}\|_{L^2(U)}$$

All together,

$$\|u\|_{L_t^\infty H_x^2} + \|u_t\|_{L_t^\infty H_x^1} + \|u_{tt}\|_{L_t^\infty L_x^2} \le C\left(\|\psi_0\|_{H^2} + \|\psi_1\|_{H^1}\right)$$

Theorem 5.3.2 (hyperbolic regularity). Suppose $a^{ij}, b^i, b, c \in C^2(\overline{U_T})$, with ∂U being C^2. Then for $\psi_0 \in H^2(U) \cap H_0^1(U)$, $\psi^1 \in H_0^1(U)$, $f, f_t \in L^2(U_T)$, then the unique weak solution $u \in H^1(U_T)$ satisfies

$$\begin{aligned} u &\in H^2(U_T) \cap L_t^\infty H_x^2 \\ u_t &\in L_t^\infty H_0^1 \\ u_{tt} &\in L_t^\infty L_x^2 \end{aligned}$$

Proof. By approximation, we can assume f, ψ_0, ψ_1 are smooth. As in the Galerkin method, use

$$u^N(x,t) = \sum_{k=1}^N u_k(t)\varphi_k(x)$$

Consider the ODE for $u_k(t)$. The coefficients are C^2, and so u_k is C^3.

Since u^N is C^3, we can differentiate eq. (20) with respect to t, to get

$$\begin{aligned} &\left(u_{ttt}^N, \varphi_k\right)_{L^2(\Sigma_0)} + \int_{\Sigma_t} a^{ij} u_{tx_i}^N (\varphi_k)_{x_j} + b^i u_{tx_i}^N \varphi_k + b u_{tt}^N \varphi_k + c u_t^N \varphi_k \mathrm{d}x \\ &= (f_t, \varphi_u)_{L^2(\Sigma_t)} - \int_{\Sigma_t} a_t^{ij} u_{x_i}^N (\varphi^k)_{x_j} + b_t^i u_{x_i}^N \varphi_k + b_t u_t^N \varphi_k + c_t u^N \varphi_k \mathrm{d}x \end{aligned}$$

Multiply the above by $\ddot{u}_k e^{-\lambda t}$, sum from $k = 1$ to N, integrating $\int_0^\tau \mathrm{d}t$, we get that

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|u_t^N\|_{H^1(\Sigma_t)}^2 + \|u_{tt}^N\|_{L^2(\Sigma_t)}^2\right) + \|u_t\|_{H^1(U_t)}^2 \\ &\quad \le e^{\lambda t} C\left(\|\psi_0\|_{H^1(\Sigma_t)}^2 + \|\psi_1\|_{L^2(\Sigma_0)}^2 + \|f\|_{L^2(U_T)}^2 + \|u_t^N\|_{H^1(\Sigma_0)}^2 + \|u_{tt}^N\|_{L^2(\Sigma_0)}^2 + \|f_t\|_{L^2(U_T)}^2\right) \end{aligned}$$

First, note that

$$\|u_{tt}^N\|_{H^1(\Sigma_0)} \le C\|\psi_1\|_{H^1(\Sigma_0)}$$

and using eq. (20) again,

$$\begin{aligned} \|u_{tt}^N\|_{L^2(\Sigma_0)}^2 &= -\int_{\Sigma_0} a^{ij} u_{x_i}^N u_{ttx_j}^N + b^i u_{x_i}^N u_{tt}^N + b u_t^N u_{tt}^N + c u^N u_{tt}^N \mathrm{d}x + \left(f, u_{tt}^N\right)_{L^2(\Sigma_0)} \\ &= \int_{\Sigma_0} (a^{iju_{x_j}^N})_{x_i} u_{tt}^N + \text{stuff} \end{aligned}$$

By Cauchy-Schwarz,

$$\|u_{tt}^N\|_{L^2(\Sigma_0)} \le C\left(\|u^N\|_{H^2(\Sigma_0)} + \|u_t^N\|_{L^2(\Sigma_0)} + \|f\|_{L^2(\Sigma_0)}\right)$$

We would like to control $\|u^N\|_{H^2(\Sigma_0)}$ uniformly in N.

$$\left(\Delta u^N, \Delta u^N\right)_{L^2(\Sigma_0)} = \left(u^N, \Delta^2 u^N\right)_{L^2(\Sigma_0)} = \left(\psi_0, \Delta^2 u^N\right)_{L^2(\Sigma_0)} = \left(\Delta \Psi_0, \Delta u^N\right)_{L^2(\Sigma_0)}$$

In particular,

$$\|\Delta u^N\|_{L^2(\Sigma_0)} \leq \|\Delta \psi_0\|_{L^2(\Sigma_0)} \leq \|\psi_0\|_{H^2(\Sigma_0)}$$

Using elliptic regularity,

$$\|u^N\|_{H^2(\Sigma_0)} \leq \|\psi_0\|_{H^2(\Sigma_0)}$$

In summary, we have

$$\|u_t^N\|_{L_t^\infty H_x^1} + \|u_{tt}^N\|_{L_t^\infty L^2} + \|u_{tt}^N\|_{H^1(U_T)} \leq C_2$$

where C_2 is independent of n. By Banach Alaoglu, we have that

$$\begin{aligned} u_t &\in H^1(U_T) \\ u_t &\in L_t^\infty H_0^1 \\ u_{tt} &\in L_t^\infty L_x^2 \end{aligned}$$

For the spacial derivatives, use the fact that

$$Lu = f - u_{tt}$$

by elliptic regularity on Σ_t, then

$$\|u\|_{H_x^2} \leq \|Lu\|_{L_x^2} \leq \|f\|_{L_x^2} + \|u_{tt}\|_{L_x^2} \leq CC_2$$

and so $u \in L_t^\infty H_x^2$ as required. □

Lecture 24

6 Heat equation

Consider $u : \mathbb{R} \to \mathbb{R}$, $h > 0$, and consider the average value $\overline{u}$ of u on $(-h, h)$, i.e.

$$\overline{u} = \frac{1}{2h} \int_{-h}^{h} u(x) \mathrm{d}x$$

Taylor expanding,

$$u(x) = \sum_k \frac{\partial^k u(0) h^k}{k!}$$

Substituting,

$$\begin{aligned} \overline{u} &= \frac{1}{2h} \int u(0) + u'(0)h + \frac{u''(0)h^2}{2} + \mathcal{O}\left(h^3\right) \mathrm{d}x \\ &= u(0) + \frac{u''(0)h^2}{12} + \mathcal{O}\left(h^4\right) \\ &= u(0) + \frac{\Delta u(0)h^2}{12} + \mathcal{O}\left(h^4\right) \end{aligned}$$

That is, the Laplacian measures how much the function varies from its average in a neighbourhood of 0. More generally, we have the mean value property for the Laplacian. That is,

$$\Delta u(p) = \lim_{r \to 0} C(n, r) \int_{S_r(p)} u(x) - u(p) \mathrm{d}x$$

where $S_r(p)$ is the sphere of radius r about p.

Consider the heat equation

$$u_t = \Delta u$$

If the average in a neighbourhood of p is hotter than at p, then the temperature will increase at p.

Consider the initial boundary value parabolic equation

$$\begin{cases} u_t - \Delta u = f & \text{on } U_T \\ u = \psi & \text{on } \Sigma_0 \\ u = 0 & \text{on } \partial^* U_T \end{cases}$$

Multiply the PDE by u, we get

$$\frac{1}{2}\partial_t(u^2) - \text{div}_x(uDu) + |Du|^2 = fu$$

Integrating over $[0, t] \times U$, we get

$$\frac{1}{2}\int_{[}\Sigma_t]u^2 \text{d}x + \int_{U_t} |Du|^2 \text{d}x\text{d}t = \int_{U_t} uf \text{d}x\text{d}t + \frac{1}{2}\int_{\Sigma_0} \psi^2$$

By Young's inequality,

$$\int_{U_t} uf \leq \varepsilon \int_{U_t} U^2 \text{d}x\text{d}t + \frac{4}{\varepsilon}\int_{U_t} f^2 \text{d}x\text{d}t$$

All together,

$$\int_{\Sigma_t} u^2 \text{d}x + \int_{U_t} (u^2 + |Du|^2)\text{d}x\text{d}t \leq C\left(\int_{U_t} f^2 \text{d}x\text{d}t + \int_{\Sigma_0} \psi^2 \text{d}x\right)$$

Here, we see that energy is not conserved, but it is decreasing in time. Taking the sup over $t \in [0, T]$, we have that

$$||u||^2_{L^\infty_t L^2(U)} + ||u||^2_{L^2_t H^1(U)} \leq C\left(\|f\|_{L^2(U_T)} + \|\psi\|_{L^2(\Sigma_0)}\right)$$

For regularity, assume that we have a smooth solution to the heat equation. Multiply the equation by u_t, to get

$$u_t^2 + \text{div}_x(u_t Du) + \frac{1}{2}\partial_t|Du|^2 = u_t f$$

Apply Young's inequality to get

$$\frac{1}{2}u_t^2 + \frac{1}{2}\partial_t|Du|^2 \leq \frac{1}{2}f^2 + \text{div}_x(u_t Du)$$

Again, integrate on $U \times [0, t]$ to get

$$\frac{1}{2}\int_{U_t} u_t^2 \text{d}x\text{d}t + \frac{1}{2}\int_{\Sigma_t} |Du|^2 \leq \frac{1}{2}\int_{U_t} f^2 \text{d}x\text{d}t + \frac{1}{2}\int_{\Sigma_0} |Du|^2 \text{d}x$$

Taking sup over $t \in [0, T]$, we find that

$$\|u_t\|_{L^2(U_T)} + \|Du\|_{L^\infty_t L^2(U)} \leq C\left(\|f\|_{L^2(U_T)} + \|\psi\|_{H^1(\Sigma_0)}\right)$$

Using the PDE, at each time t,

$$-\Delta = f - u_t$$

and $u = 0$ on ∂U. Hence by elliptic estimates,

$$||u||_{H^2(U)} \leq \|\Delta u\|_{L^2(U)} \leq \|f\|_{L^2(U)} + \|u_t\|_{L^2(U)}$$

Integrating over time,

$$||u||_{L^2_t H^2(U)} \leq C\left(\|f\|_{L^2(U_T)} + \|u_t\|_{L^2(U_T)} \leq C\left(\|f\|_{L^2(U_T) + \|\psi\|_{H^1(\Sigma_0)}}\right)\right)$$

Again, we have a gain in regularity.

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