Differential Geometry

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Contents

Lecture 1

1 Manifolds

The concept of a manifold is intended to generalise curves and surfaces in $\mathbb{R}^3$.

> **Definition 1.1** (smooth structure)
> A *smooth structure* on a topological space M is a collection of *charts* (an *atlas*)
>
> $$\varphi_\alpha : U_\alpha \to V_\alpha$$
>
> where $U_\alpha \subseteq M$, $V_\alpha \subseteq \mathbb{R}^d$ are open, φ_α is a homeomorphism, such that
>
> 1. $M = \bigcup_\alpha U_\alpha$. That is, the charts cover M.

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2. Given any charts $\varphi_\alpha, \varphi_\beta$, the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a smooth map between open subsets of $\mathbb{R}^d$.

3. If φ is *compatible* (as in the previous point) with every chart in the atlas, then φ is a chart in the atlas.

In practice (i.e. using Zorn's lemma), given any collection of charts satisfying the first two conditions, we can find a maximal atlas containing it.

Remark 1.2.
1. The second condition implies that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism.
2. d is fixed, assuming that M is connected.

Definition 1.3 (manifold)
A *(smooth) manifold* M is a second countable Huasdorff space topological space, equipped with a smooth structure.
$d = \dim(M)$ is the *dimension* of M.

In practice, we may induce a topology on M from a smooth structure (without the mention of open or homeomorphism), then $D \subseteq M$ is open if for any chart φ_α, $\varphi_\alpha(D \cap U_\alpha)$ is an open subseteq of $\mathbb{R}^d$.

Remark 1.4.
1. We can also define a C^k manifold by replacing smooth with C^k as above. If $k = 0$, then we have a *topological manifold*.
2. On the other hand, we can replace $\mathbb{R}^d$ with $\mathbb{C}^n$, and obtain a *complex manifold*.

$\mathbb{R}^n$ is (trivially) a manifold.

Example 1.5 (unit sphere)
Define

$$S^n = \{x = (x_0, \dots, x_n) \in \mathbb{R}^n \mid ||x|| = 1\} \subseteq \mathbb{R}^{n+1}$$

For the charts, we can consider the stereographic projections

$$\varphi(x) = \frac{1}{1 - x_0}(x_1, \dots, x_n) \qquad \text{for } x_0 \neq 1$$

and

$$\psi(x) = \frac{1}{1 + x_0}(x_1, \dots, x_n) \qquad \text{for } x_0 \neq -1$$

In particular,

$$\psi \circ \varphi^{-1}(u) = \frac{u}{||u||^2}$$

Example 1.6
If M, N are manifolds of dimensions d, e respectively, then $M \times N$ is a manifold of dimension $d + e$.

As a corollary, $T^n = (S^1)^n$ is a manifold.

Example 1.7
If M is a manifold, $X \subseteq M$ is open, then X is a manifold.

Lecture 2

A convention, we will will assume for any manifold M, we will assume all components are of the same dimension. In particular, $\dim(M)$ is well defined.

Notation 1.8. We will write M^d to say M is a manifold of dimension d.

Example 1.9 (real projective space)
The real projective space is

$$\mathbb{RP}^n = \left\{\text{straight lines in } \mathbb{R}^{n+1} \text{ through } 0\right\}$$

Points in $\mathbb{RP}^n$ can be written using homogeneous coordinates $(x_0 : \cdots : x_n)$, where the x_i are not all zero, and

$$(x_0 : \cdots : x_n) = (\lambda x_0 : \cdots : \lambda x_n)$$

for all $\lambda \neq 0$. We will induce a topology from a smooth structure. The charts are given by (φ_i, U_i) where

$$U_i = \{x_i \neq 0\}$$

and we define

$$\varphi_i(x_0 : \cdots : x_n) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right) \in \mathbb{R}^n$$

For $i < j$,

$$\varphi_j \circ \varphi_i^{-1}(y_0, \ldots, y_n) = \left(\frac{y_1}{y_j}, \ldots, \frac{1}{y_j}, \ldots, \frac{y_n}{y_j}\right)$$

where the 1 is in the i-th position, and we omit the j-th position. Thus, $\mathbb{RP}^n$ is an n-dimensional manifold. Similarly, we can define the complex projective space $\mathbb{CP}^n$, which is a $2n$-manifold (and a n-dimensional complex manifold).

Example 1.10 (Grassmannian)

$$\mathrm{Gr}(k, n) = \{k\text{-dimensional subspaces of } \mathbb{R}^n\}$$

Then $\mathrm{Gr}(k, n)$ is a manifold of dimension $k(n-k)$. Note we can also define the Grassmannian over $\mathbb{C}$.

Sketch proof. An example of a coordinate neighbourhood is

$$U = \left\{k\text{-dimensional subspaces obtainable as the span of rows of } k \times n \text{ matrices of the form } \begin{pmatrix} I_k & * \end{pmatrix}\right\}$$

where I_k is the $k \times k$ identity matrix. In this case, the $k \times (n-k)$ block $*$ defines local coordinates. Choosing different columns, we have $\binom{n}{k}$ neighbourhoods, $U_{i_1,\ldots,i_k}$, where $i_1 < i_2 < \cdots < i_k$. $\square$

In particular, note that $\mathbb{RP}^n = \mathrm{Gr}(1, n+1)$.

Example 1.11 (Non-example)
Consider an equivalence relation on $\mathbb{R}^2$ given by

$$(x, y) \sim \left(\lambda x, \frac{1}{\lambda} y\right)$$

for any $\lambda \neq 0$. Let $X = \mathbb{R}^2 / \sim$ for the quotient, with the quotient topology. We can consider

$$\{(x, y) \mid xy = c\}$$

If $c \neq 0$, then this is one equivalence class. If $c = 0$, this splits into three, which are

- $\{(0, 0)\}$,

- $\{(x, 0) \mid x \neq 0\}$,
- $\{(0, y) \mid y \neq 0\}$

We can write
$$X = (-\infty, 0) \cup (0, \infty) \cup \{0', 0'', 0'''\}$$
We have three charts φ_i corresponding to each choice of zero. For example,
$$U_1 = (-\infty, 0) \cup \{0'\} \cup (0, \infty)$$
with the "natural" map φ_1. On the other hand, the induced topology is not Hausdorff. Note the induced topology is *not* the quotient topology.

Refer to Examples Sheet 1 Question 12 for an example of a non-second countable (non-)example.

Definition 1.12 (smooth map)
Let M, N be manifolds, a continuous map $f : M \to N$ is *smooth* if for any $p \in M$, we can find charts (φ, U) near p, (ψ, V) near $f(p)$, and the composition
$$\psi \circ f \circ \varphi^{-1}$$
is smooth. as a map on
$$\varphi(U \cap f^{-1}(V))$$

Definition 1.13 (diffeomorphism)
A smooth map $f : M \to N$ is a *diffeomorphism* if f is a bijection, and f^{-1} is smooth. In this case, we say that M, N are *diffeomorphic*.

Proposition 1.14.
1. If $U \subseteq \mathbb{R}^n$, then $f : U \to \mathbb{R}^m$ is smooth in the above definition if and only if it is smooth in the sense of multi-variate calculus.
2. Every chart $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^d$ is a smooth map of manifolds. Moreover, it is a diffeomorphism.
3. If $f : M \to N$, $g : N \to P$ are smooth, then $g \circ f : M \to P$ is smooth.

Proof. Obvious, and so omitted. □

1.1 Matrix Lie groups

Consider the group $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$. In fact, this is an open subset, and so it is a manifold. Moreover, matrix multiplication and inversion are smooth.

Definition 1.15 (Lie group)
A group G is called a *Lie group* if

- G is a manifold,
- the multiplication map $G \times G \to G$ is smooth,
- inversion $G \to G$ is smooth.

Remark 1.16. Suffices to show that the map
$$G \times G \ni (\sigma, \tau) \mapsto \sigma\tau^{-1} \in G$$

is smooth.

In particular, $\mathrm{GL}(n,\mathbb{R}$ is a Lie group. Similarly, $\mathrm{GL}(n,\mathbb{C})$ is also a Lie group. More generally, for $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{C})$, we define the norm $\|A\| = n \max_{i,j} |i, j|$. (Note in finite dimensional, all norms are equivalent). In particular,

$$\|AB\| \leq \|A\|\|B\|$$

We define the *exponential of A* as

$$\exp(A) = \sum_k \frac{A^k}{k!}$$

This series converges for all A, by the same proof as for $\exp : \mathbb{C} \to \mathbb{C}$ (or more generally, any Banach algebra). Moreover, it converges uniformly on balls

$$\{\|A\| \leq R\}$$

by the Weierstrass M-test. In particular, $\exp : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ is continuous.

Lecture 3

First of all, note that

$$f(A) = A^n$$

is a differentiable function, with

$$\|(\mathrm{d}f)_A\| \leq n|A|^{n-1}$$

Therefore, the series which we get by differentiating term by term, converges uniformly on any bounded set, again by the Weierstrass M-test.

Therefore, $\exp(A)$ is (at least) C^1 on $\mathrm{Mat}_n(\mathbb{C})$. We can similarly consider the higher derivatives, and we find that $\exp(A)$ is C^∞.

Remark 1.17. Some basic properties of the matrix exponential:

1. $\exp(A^\intercal) = \exp(A)^\intercal$,
2. $\exp(A^*) = \exp(A^*)$,
3. $\exp(CAC^{-1}) = C\exp(A)C^{-1}$,

In general,

$$\exp(A + B) = \exp(A)\exp(B)$$

does not hold. However, it does hold if $AB = BA$. In particular,

4. $\exp(A)\exp(-A) = \exp(-A)\exp(A) = I$

We can also define the *matrix logarithm* via the series

$$\log(I + A) = \sum_m (-1)^{m+1}\frac{A^m}{m}$$

We can show that this converges absolutely and uniformly on $\{|A| < \varepsilon\}$ for any $0 < \varepsilon < 1$. The same result holds for the term by term differentiation, and again we get that $\log(I + A)$ is C^∞ on $\{|A| < 1\}$

Suppose $\log(A)$ is defined, that is, $|A - I| < 1$, then

$$\exp(\log(A)) = A$$

since the same proof as in $\mathbb{C}$ works for any Banach algebra.

On the other hand, if we would like

$$\log(\exp(A)) = A$$

we clearly need $|\exp(A) - I| < 1$. Suppose

$$A_\theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

for some $\theta \in \mathbb{R}$. In particular,

$$A_\theta^2 = -\theta^2 I$$

and we find that

$$\exp(A_\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Setting $\theta = 2\pi$, then we find that $\exp(A_{2\pi}) = I$, but

$$\log(\exp(A_{2\pi})) = \log(I) = 0 \neq A_{2\pi}$$

Hence $|\exp(A) - I| < 1$ is a necessary, but not sufficient. On the other hand, $|A| < \log(2)$ is sufficient, since in this case,

$$\left| e^{|A|} - 1 \right| < 1$$

which is required for absolute convergence.

Example 1.18 (orthogonal group)

Consider

$$O(n) = \left\{ A \in \mathrm{GL}(n, \mathbb{R}) \mid AA^\mathsf{T} = A^\mathsf{T} A = I \right\}$$

We will show that $O(n)$ is a Lie group. Let A be an orthogonal matrix, $|A - I| < \varepsilon < 1$, then $B = \log(A)$ is defined. Moreover, $\exp(B) = A$. By continuity, we can take ε small enough such that $|B| < \log(2)$. Now

$$\exp(B) \exp\left(B^\mathsf{T}\right) = AA^\mathsf{T} = \mathrm{id}$$

and so

$$\exp(B) = A = (A^\mathsf{T})^{-1} = \exp\left(B^\mathsf{T}\right)^{-1} = \exp\left(-B^\mathsf{T}\right)$$

Taking log, since $|B| = \left|B^\mathsf{T}\right| < \log(2)$, we see that $B = -B^\mathsf{T}$.

Conversely, if B is skew-symmetric, with $|B| < \log(2)$, then

$$\exp(B) \exp\left(B^\mathsf{T}\right) = I$$

Define

$$V_0 = \left\{ B \in \mathrm{Mat}_n(\mathbb{R}) \mid |B| < \log(2), B^\mathsf{T} + B = 0 \right\}$$

and define

$$U = \exp(V_0)$$

which is an open neighbourhood of $I \in O(n)$.

Proposition 1.19. $O(n)$ has a C^∞ structure, making it into a manifold and a Lie group, with dimension $n(n-1)/2$.

Proof. We will use the subspace topology induced from $O(n) \subseteq \mathbb{R}^{n^2}$. Define

$$h : U \to V_0$$
$$h(A) = \log(A)$$

This is a homeomorphism onto its image, which is an open neighbourhood of $0 \in \mathfrak{so}(n) \cong \mathbb{R}^{n(n-1)/2}$.

Given $C \in O(n)$, define

$$U_C = \{CA \mid A \in U\}$$

and

$$h_C : U_C \to V_0$$
$$h_C(A) = \log\left(C^{-1}A\right)$$

Again, this is a homeomorphism onto its image. Since $C \in U_C$, we have an open cover. Moreover,

$$h_{C_2} \circ h_{C_1}^{-1}(B) = h_{C_2}(C_1 \exp(B)) = \log\left(C_2^{-1} C_2 \exp(B)\right)$$

which is a smooth map between open neighbourhoods of $\mathfrak{so}(n)$. Therefore, $O(n)$ is a smooth manifold. Similarly, for consider the map

$$F(A_1, A_2) = A_1 A_2^{-1}$$

In local coordinates,

$$\begin{aligned}
F_{\text{loc}}(B_1, B_2) &= h_{C_1 C_2^{-1}}(F(h_{C_1}^{-1}(B_1), h_{C_2}^{-1}(B_2))) \\
&= \log\left((C_1 C_2^{-1})^{-1} C_1 \exp(B_1)(C_2 \exp(B_2))^{-1}\right) \\
&= \log\left(C_2 \exp(B_1) \exp(-B_2) C_2^{-1}\right)
\end{aligned}$$

and we can see that this is a smooth map as a function of B_1, B_2. □

This method works for more groups, and is sometimes called the Cayley construction (for matrix Lie groups).

2 Tangent spaces to manifolds

Consider a local curve in a ball in $\mathbb{R}^n$, defined by a smooth parametrisation $x(t) = (x_1(t), \dots, x_n(t))$. Suppose we have that $x(0) = p \in \mathbb{R}^n$. In this case, its tangent vector at p is

$$\dot{x}(0) \in \mathbb{R}^n$$

If $y(x)$ is a smooth change of coordinates near p, then by the chain rule, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} y(x(t)) = J(y)_p \dot{x}(0)$$

where $J(y)$ is the Jacobian matrix. More explicitly,

$$\dot{y}_i(0) = \sum_j \frac{\partial y_j}{\partial x_j} \dot{x}_j(0)$$

Definition 2.1 (tangent vector)
A *tangent vector* a to a manifold M^n at a point $p \in M$, is an assignment to each coordinate chart (U, φ) with $p \in U$, an n-tuple of coordinates $(a_1, \dots, a_n) \in \mathbb{R}^n$, such that if (U', φ') is another chart near p, $(x_i), (x'_i)$ the respective local coordinates,

$$a'_i = \sum_j \frac{\partial x'_i}{\partial x_j} a_j$$

This definition is called the *tensorial definition*, where we focus on the transformation law. There are several other definitions, for example, derivations, equivalence of curves to first order and so on.

Definition 2.2 (tangent space)
The *tangent space* to M at a point p, denoted $\mathrm{T}_p M$ is the space of all tangent vectors to M at p. It is naturally a vector space (of dimension n over $\mathbb{R}$).

Lecture 4

A choice of chart (with local coordinates (x_i) say), determines a linear isomorphism

$$\mathrm{T}_p M \to \mathbb{R}^n$$

where $n = \dim(M)$. We denote by $\frac{\partial}{\partial x_i}$, the basis corrsponding to the usual standard basis of $\mathbb{R}^n$, under this linear isomorphism.

If we want to emphasise the point $p \in M$, we may write

$$\left(\frac{\partial}{\partial x_i}\right)_p \quad \text{or} \quad \left.\frac{\partial}{\partial x_i}\right|_p$$

Then by the usual linear algebra, we have that

$$\left(\frac{\partial}{\partial x_i}\right)_p = \sum_j \left(\frac{\partial x'_j}{\partial x_i}\right)_p \left(\frac{\partial}{\partial x'_j}\right)_p$$

An alternative view on tangent vectors is as *derivations*. Given

$$a = \sum a_i \frac{\partial}{\partial x_i} \in \mathrm{T}_p M$$

we can define a *first-order derivation* at p,

$$a : C^\infty(M) \to \mathbb{R}$$

where

$$a(f) = \sum_i a_i \frac{\partial f}{\partial x_i}$$

in local coordinates. This definition is independent of the choice of local coordinates. We can define this in a coordinate independent way, by defining

$$a(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x(t))$$

where x is a local curve on M, with $\dot{x}(0) = a$. Moreover, every derivation satisfies the *Leibniz rule*

$$a(fg) = f \cdot a(g) + a(f) \cdot g$$

Conversely, every linear map $a : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule above arises as a derivation corresponding to a tangent vector.

Example 2.3

Let $r : \mathbb{R}^2_{u,v} \supseteq D \to S \subseteq \mathbb{R}^3$ be a regular parametrisation for a surface S. Then $\varphi = r^{-1}$ defines a chart. The partial derivatives r_u, r_v correspond to

$$\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$$

respectively.

For a Lie group G, the tangent spaces has an "infinitessimal counterpart" of the group structure.

Definition 2.4 (Lie algebra)

A *Lie algebra* is a vector space $\mathfrak{g}$, wiht a bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the *Lie bracket*, with

- anticommutativity $[x, y] = [-y, x]$,
- the Jacobi idenity
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Theorem 2.5. Let $G \subseteq \mathrm{GL}(n, F)$ be a matrix Lie group (where $F = \mathbb{R}$ or $\mathbb{C}$), such that log defines a chart near $I \in G$. Let

$$\mathfrak{g} = \mathrm{T}_I G$$

is identified with a (real) subspace of $\mathrm{Mat}_n(F)$. Then $\mathfrak{g}$ is a Lie algebra, with

$$[x, y] = xy - yx$$

We call $\mathfrak{g}$ the *Lie algebra of G*. We may also write

$$\mathfrak{g} = \mathrm{Lie}(G)$$

Proof. Clearly $\mathfrak{g}$ is a vector space, and $[\cdot,\cdot]$ is anticommutative and satisfies the Jacobi identity. Therefore, all we need to show is that if $x, y \in \mathfrak{g}$, then so is $[x, y]$.

For $B_1, B_2 \in \mathfrak{g}$, consider

$$A(t) = \exp(B_1 t)\exp(B_2 t)\exp(-B_1 t)\exp(-B_2 t)$$

This defines a smooth path in G, with $A(0) = I$. Recall

$$\exp(Bt) = I + Bt + \frac{1}{2}B^2t^2 + o(t^2)$$

as $t \to 0$. We obtain that

$$A(t) = I + [B_1, B_2]t^2 + o(t^2)$$

Letting $B(t) = \log(A(t))$, for small t, we have that $B(t) = [B_1, B_2]t^2 + o(t^2)$. But we also have that

$$\exp(A(t)) = B(t)$$

Moreover, by definition, $B(t) \in \mathfrak{g}$ for t small, as $B(t)$ is in the image of the logarithm chart. So

$$\frac{B(t)}{t^2} \in \mathfrak{g}$$

as $\mathfrak{g}$ is a vector space. So

$$\lim_{t\to 0}\frac{B(t)}{t^2} = [B_1, B_2] \in \mathfrak{g}$$

since $\mathfrak{g} \leq \mathrm{Mat}_n(F)$ is closed. □

If we take $G = O(n)$, then we have that

$$\mathfrak{g} = \mathfrak{o}(n) = \left\{A \in \mathrm{Mat}_n(\mathbb{R}) \mid A^{\mathsf{T}} + A = 0\right\}$$

is the space of skew-symmetric matrices.

Definition 2.6

Let M be a manifold. Then the *tangent bundle of M* is

$$\mathrm{T}M = \bigsqcup_{p\in M}\mathrm{T}_pM$$

Theorem 2.7. $\mathrm{T}M$ has a smooth structure, making it into a manifold of dimension $2\dim(M)$.

Proof. The topology on $\mathrm{T}M$ will be induced from the smooth structure. Let (φ, U) be a chart on M. Say the local coordinates corresponding to φ are $x_1, \ldots, x_n$. So we can write $a \in \mathrm{T}_pM$ as

$$\sum a_i \frac{\partial}{\partial x_i}$$

Define

$$U_T = \bigsqcup_{p\in U}\mathrm{T}_pM$$

$$\varphi_T(p, a) = (\varphi(p), (a_i))$$

It is clear that the sets of the form U_T cover, so all we need to check is that the change of coordinate maps are smooth. Let (ψ, V) be another chart on M, with local coordinates y_i. Then

$$\psi \circ \varphi^{-1}(x, a) = (y, b)$$

and it is easy to see that $y = \psi \circ \varphi^{-1}(x)$, and

$$b_i = \sum_j \frac{\partial y_i}{\partial x_j} a_j$$

Hausdorff and second countable follow from the fact that M and $\mathbb{R}^n$ are. □

Lecture 5

We will use the notation $\pi : TM \to M$ for the canonical projection map

$$\pi(p, a) = p$$

Proposition 2.8. The map $\pi : TM \to M$ is smooth.

Proof. In local coordinates given by φ and φ_T, we have

$$\pi(x, a) = x$$

□

Remark 2.9. In general, the tangent bundle TM need not be diffeomorphic to $M \times \mathbb{R}^n$.

Definition 2.10 ((smooth) vector field)

A *(smooth) vector field* X on a manifold M is a smooth map $X : M \to TM$, such that $\pi(X(p)) = p$. That is, $X(p) \in T_pM$.

Example 2.11

$X = 0$ is a vector field. Similarly, X supported on a coordinate neighbourhood is a well defined vector field.

X is C^∞ if and only if for each coordinate neighbourhood U, with local coordinates x_i, we have

$$X = \sum_i a_i(p) \frac{\partial}{\partial x_i}$$

Smoothness of X becomes the requirement that each $a_i : U \to \mathbb{R}$ is smooth.

Theorem 2.12. Suppose M^n is a manifold, and $X^{(1)}, \dots, X^{(n)}$ are smooth vector fields on M, such that for all $p \in M$,

$$X^{(1)}(p), \dots, X^{(n)}(p)$$

is a basis for T_pM. Then TM is isomorphic to the product $M \times \mathbb{R}^d$. That is, there exists $\Phi : TM \to M \times \mathbb{R}^n$ is a diffeomorphism, such that

$$\Phi(p, m) = (p, \phi_p(m))$$

and for each fixed p, $\phi_p : T_pM \to \mathbb{R}^n$ is a linear isomorphism.

Manifolds which satisfies the requirements of the theorem are called *parallelisable*.

Proof. For $(p, a) \in \mathrm{T}M$, we can write it as

$$a = \sum a_i X^{(i)}(p)$$

for some unique $a_1, \dots, a_n \in \mathbb{R}$. Using this, we can define a map

$$\Phi(p, a) = (p, (a_i)) \in M \times \mathbb{R}^n$$

It is easy to see that this is a bijection. Moreover, ϕ_p defines a linear isomorphism. Therefore, all we need to check is the smoothness. Let $\psi : U \to \mathbb{R}^n$ be a chart on M, and $\psi_T : U_T \to \mathbb{R}^n \times \mathbb{R}^n$ be the corresponding chart on $\mathrm{T}M$. In this case,

$$(\psi \times \mathrm{id}) \circ \Phi \circ \psi_T^{-1}(x, b) = (x, a)$$

such that

$$a = \sum_i a_i X^{(i)}(p) = \sum_j b_j \left(\frac{\partial}{\partial x_j}\right)_p$$

We can write $X^{(i)}$ in the $\frac{\partial}{\partial x_j}$ basis, to get

$$X^{(i)} = \sum_j X_j^{(i)} \frac{\partial}{\partial x_j}$$

Thus,

$$b_j = \sum_i a_i X_j^{(i)}(x)$$

Since the $X_j^{(i)}$ are smooth, we are done. Moreover, the inverse transformation is also smooth, and thus Φ defines a local diffeomorphism, which must then be a diffeomorphism. □

Remark 2.13.
- The converse of the theorem is also true, and it is easy, since all we need to choose is (the image of) $(x, e_1), \dots, (x, e_n)$.
- The condition that M is parallelisable is a restriction. We know that all Lie groups are parallelisable, but general manifolds are not. For example, there are no non-vanishing vector fields on S^2. More generally, S^n is parallelisable if and only if $n = 1, 3$ or 7.

Definition 2.14 (differential)
Let $F : M \to N$ be a smooth map, Define the *differential of F at $p \in M$* as

$$\begin{aligned} \mathrm{d}F_p : \mathrm{T}_pM &\to \mathrm{T}_{F(p)}N \\ \left.\frac{\partial}{\partial x_i}\right|_p &\mapsto \sum_j \frac{\partial y_i}{\partial x_j}(x(p)) \left.\frac{\partial}{\partial y_j}\right|_{F(p)} \end{aligned}$$

where x_i are local coordinates around p, given by the chart φ, and y_j are local coordinates aroung $F(p)$, given by ψ. In local coordinates, $y = y(x) = \psi \circ F \circ \psi^{-1}(x)$.

Moreover, it is easy to see this is independent of choice of local coordinate, which follows from the chain rule in multivariave calculus.

Remark 2.15. For smooth maps $F : M \to N$, $G : N \to P$,

$$\mathrm{d}(G \circ F) = \mathrm{d}G \circ \mathrm{d}F$$

Now suppose $F : M \to N$ is a diffeomorphism, X is a smooth vector field on M. Then the *pushforward of X by F* is

$$((\mathrm{d}F)X)_{F(p)} = (\mathrm{d}F)_p X(p)$$

which defines a vector field on N. We want F^{-1} to be smooth, since otherwise we don't know whether the image will be smooth.

Every vector field X on M defines a linear map

$$X : \mathbb{C}^\infty(M) \to \mathbb{C}^\infty(M)$$

where if

$$X = \sum_i X_i \frac{\partial}{\partial x_i}$$

we define locally

$$Xh = \sum_i X_i \frac{\partial h}{\partial x_i}$$

again by the chain rule, this is independent of choice of coordinates.

Let $f \in C^\infty(N)$, $F : M \to N$ be a diffeomorphism. Then $f \circ F \in C^\infty(M)$. In coordinates x_i on M, y_j on N given by F,

$$\frac{\partial}{\partial x_i}(f \circ F) = \sum_j \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$

Therefore,

$$X(f \circ F) = \sum_{i,j} X_i \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i} = (((\mathrm{d}F)X)\, f) \circ F$$

That is, the diagram

$$\begin{array}{ccc} C^\infty(N) & \xrightarrow{\cdot \circ F} & C^\infty(M) \\ \downarrow{\scriptstyle (\mathrm{d}F)X} & & \downarrow{\scriptstyle X} \\ C^\infty(N) & \xrightarrow{\cdot \circ F} & C^\infty(M) \end{array}$$

commutes.

Let X, Y be vector fields on M. X and Y can be consider as a first order linear differential operator. Then the composition XY is not, but the *Lie bracket*

$$[X, Y] = XY - YX$$

is a well defined first order linear differential operator. That is, it defines an $\mathbb{R}$-linear map $C^\infty(M) \to C^\infty(M)$, satisfying

$$[X, Y](fg) = f \cdot [X, Y]g + g \cdot [X, Y]f$$

One way to see this is by the symmetry of mixed partial derivatives. Thus, the space of all vector fields on M forms a Lie algebra. Note that it is infinite dimensional, and so it can't come from a Lie group.

Lecture 6

Suppose in local coordinates $X = \sum_i X_i \frac{\partial}{\partial x_i}$ and $Y = \sum_i Y_i \frac{\partial}{\partial x_i}$, then

$$[X, Y] = \sum_{i,j} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_k}$$

Notation 2.16. We write $V(M)$ for the vector space of all C^∞ vector fields on M.

2.1 Left invariant vector fields

Let G be a Lie group, $e \in G$ is the identity element of G, and $\mathfrak{g} = \mathrm{T}_e G$ for the tangent space of G at e, i.e. its Lie algebra.

Given $g \in G$, we can define the *left translation* $L_g : G \to G$,

$$L_g(h) = gh$$

L_g is a diffeomorphism, as it is smooth with smooth inverse $L_g^{-1} = L_{g^{-1}}$. Consider $\xi \in \mathfrak{g}$, and consider

$$X_\xi(g) = (\mathrm{d}L_g)_e \xi \in \mathrm{T}_g G$$

Since L_g is a diffeomorphism, $(\mathrm{d}L_g)_e$ is a linear isomorphism $\mathfrak{g} \to \mathrm{T}_g G$, with inverse

$$(\mathrm{d}L_g)_e^{-1} = (\mathrm{d}L_{g^{-1}})_g$$

Lemma 2.17. $X_\xi(g)$ as defined above, is smooth. Thus, X_ξ is a smooth vector field.

Proof. Consider the map $L : G \times G \to G$, with $L(g, h) = gh$. Then L is smooth. Fix $g_0 \in G$, the local coordinate expression of L in a neighbourhood of (g_0, e) is

$$\widehat{L} : U_{g_0} \times U_e \to V_{g_0}$$

Explicitly, $\widehat{L} = \varphi_{g_0} \circ L \circ (\varphi_{g_0}^{-1} \times \varphi_e^{-1})$. With this,

$$\widehat{L_g} = \widehat{L}(\varphi_{g_0}(g), \cdot) : U_e \to V_{g_0}$$

where g is in a neighbourhood of g_0. With this, consider the derivative $D_2\widehat{L}$ of $\widehat{L}$ in the V_e coordinates. Since $D_2\widehat{L}$ depends smoothly on the V_{g_0} coordinates on L, X_ξ has smooth coefficients in a neighbourhood of g_0. □

Proposition 2.18. If $\xi_1, \dots, \xi_n \in \mathfrak{g}$ are linearly independent. That for all $g \in G$,

$$X_{\xi_1}(g), \dots, X_{\xi_n}(g)$$

are linearly independent.

Proof. Clear from definition as $(\mathrm{d}L_g)_e$ is invertible. □

Theorem 2.19. Every Lie group is parallelisable.

Proof. From the proposition and theorem 2.12. □

Also, consider $g, h \in G$, and $\xi \in \mathfrak{g}$. In this case,

$$(\mathrm{d}L_g)_h X_\xi(h) = (\mathrm{d}L_g)_h(\mathrm{d}L_h)_e(\xi) = (\mathrm{d}L_g \circ \mathrm{d}L_h)_e(\xi) = (\mathrm{d}L_{gh})_e(\xi) = X_\xi(gh)$$

That is,

$$(\mathrm{d}L_g)X_\xi = X_\xi \circ L_g$$

Definition 2.20 (left-invariant vector field)
A vector field X on a Lie group G, with

$$(\mathrm{d}L_g)X = X \circ L_g \tag{1}$$

is called a *left-invariant vector field*. We write

$$\ell(G) \subseteq V(G)$$

for the subspace of all left-invariant vector fields on G.

It is easy to see that $\ell(G)$ is finite dimensional, since any $X \in \ell(G)$ is X_ξ for $\xi = X(e) \in \mathfrak{g}$. This also shows that the map $\mathfrak{g} \to \ell(G)$ given by $\xi \mapsto X_\xi$ is a linear isomorphism.

Thus,

$$\dim(\ell(G)) = \dim(\mathfrak{g}) = \dim(G)$$

Theorem 2.21. $\ell(G)$ is a Lie subalgebra of $V(G)$. More precisely, $[X_\xi, X_\eta]$ is left invariant for any $\xi, \eta \in \mathfrak{g}$.

Proof. Recall that

$$X(f \circ F) = (((\mathrm{d}F)X)\, f) \circ F$$

We will take $F = L_g$, $f \in C^\infty(G)$, $X = [X_\xi, X_\eta]$. We will show that X satisfies eq. (1).

$$\begin{aligned}
\big((\mathrm{d}L_g)[X_\xi, X_\eta](f)\big) \circ L_g &= [X_\xi, X_\eta](f \circ L_g) \\
&= X_\xi X_\eta(f \circ L_g) - X_\eta X_\xi(f \circ L_g) \\
&= X_\xi \cdot (\mathrm{d}L_g)X_\eta(f) \circ L_g - X_\eta \cdot (\mathrm{d}L_g)X_\xi(f) \circ L_g \\
&= ((\mathrm{d}L_g)X_\xi)((\mathrm{d}L_g)X_\eta)(f) \circ L_g - ((\mathrm{d}L_g)X_\eta)((\mathrm{d}L_g)X_\xi)(f) \circ L_g \\
&= ([(\mathrm{d}L_g)X_\xi, (\mathrm{d}L_g)X_\eta]f) \circ L_g \\
&= [X_\xi \circ L_g, X_\eta \circ L_g](f) \circ L_g \\
&= ([X_\xi, X_\eta] \circ L_g)(f) \circ L_g
\end{aligned}$$

But f, g are arbitrary, and as such, we have

$$(\mathrm{d}L_g)[X_\xi, X_\eta] = [X_\xi, X_\eta] \circ L_g$$

which means that $[X_\xi, X_\eta] \in \ell(G)$. □

With this, for all $\xi, \eta \in \mathfrak{g}$, there exists $\zeta \in \mathfrak{g}$ such that

$$[X_\xi, X_\eta] = X_\eta$$

This defines a Lie algebra structure on $\mathfrak{g}$.

Theorem 2.22. Let G be a matrix Lie group and log defines charts on G. Then the map

$$\mathfrak{g} \ni \xi \mapsto X_\xi \in \ell(G)$$

is an isomorphism of Lie algebras, where on $\mathfrak{g}$ we use the matrix commutator.

We will prove this later.

3 Submanifolds

Let M be a manifold, $N \subseteq M$ and N is a manifold. Consider the inclusion map $\iota : N \hookrightarrow M$.

Definition 3.1 (embedded submanifold)
Suppose

(i) ι is smooth,

(ii) $(\mathrm{d}\iota)_p : \mathrm{T}_pN \to \mathrm{T}_pM$ is injective for all p,

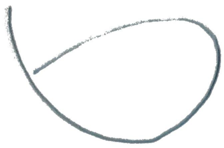
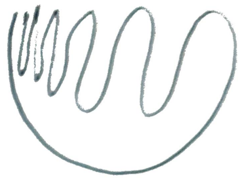
(iii) ι is a homeomorphism onto its image.

Then we say that N is an *embedded submanifold* of M.

Remark 3.2. For (iii), $D \subseteq N$ is open if and only if there exists $U \subseteq M$ open, with $D = D \cap N$.

Lecture 7

The condition (iii) is required, since we don't want an example which looks like a figure-8, or other 'weird' topological spaces, such as

Remark 3.3. If ι only satisfies (i) and (ii), we call N an *immersed submanifold*.

Remark 3.4. A slight generalisation of this is that: if N, M are manifolds, a map $\psi : N \to M$ is called an *embedding*, denoted $\psi : N \hookrightarrow M$, if ψ is injective, and satisfies conditions (i), (ii) and (iii). That is, $\psi(N) \subseteq M$ is an immersed submanifold.

If $d\Psi$ is injective for all p, then we call ψ an *immersion*.

Convention: Submanifold will mean embedded submanifold unless otherwise stated.

Example 3.5 (parametrised urves and surfaces in $\mathbb{R}^3$)

In this case, condition (ii) simply means what we called a 'regular' parametrisation.

Example 3.6 (irrational twist flow)

Define the map

$$\begin{aligned} \mathbb{R} &\to T^2 \\ t &\mapsto (e^{it}, e^{i\alpha t}) \end{aligned}$$

In this case, we see that the map is injective and an immersion, but not an embedding as the image is dense in T^2.

Is a submanifold the same as $f^{-1}(0)$ for some smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$, or more generally, $f^{-1}(p)$ for some $f : M \to N$.

In general, no. Since f is smooth, it is continuous, and so $f^{-1}(p)$ is a closed set. On the other hand,

Proposition 3.7. For every closed subset $E \subseteq \mathbb{R}^2$, we can find a smooth map $f : \mathbb{R}^2 \to \mathbb{R}$, such that $f^{-1}(0) = E$.

Proof. Omitted, optional exercise. □

Definition 3.8 (regular value)

Let $f : M \to Y$ be a smooth map between manifolds. Then $q \in Y$ is a *regular value* for f if for every $p \in f^{-1}(q)$, $(df)_p : T_pM \to T_qY$ is surjective.

Theorem 3.9 (preimage theorem). Suppose $f : M \to Y$ is a smooth map, $y \in Y$ is a regular value, with $f^{-1}(y)$ nonempty. Then $N = f^{-1}(y)$ is an embedded submanifold of M, with

$$\dim(N) = \dim(M) - \dim(Y)$$

Proof. Omitted. □

A fact from differential topology: Suppose M is a manifold, $N \subseteq M$ a subspace (with the subspace topology). Then there is at most one smooth structure on N making it into an embedded submanifold. Therefore, it makes sense for us to say N is or is not a submanifold.

The converse of the preimage theorem is only true locally.

Proposition 3.10. Let $N \hookrightarrow M$ be an embedded submanifold, $p \in N$. Then there exists an open neighbourhood $p \in U \subseteq M$, and $f : U \to \mathbb{R}^d$, $d = \dim(M) - \dim(N)$, with $0 \in \mathbb{R}^d$ is a regular value of f, and $f^{-1}(0) = U \cap f^{-1}(0)$.

Proof. Let $\varphi : U_0 \to \mathbb{R}^n$ be a chart on M near p, with $\varphi(p) = 0$, with local coordinates $x_1, \dots, x_n$. Let $\psi : V_0 \to \mathbb{R}^\ell$ be a chart on N, $\psi(p) = 0$ and with local coordinates $u_1, \dots, u_\ell$.

We may assume without loss of generality that $V_0 = U_0 \cap N$. In this case, $\iota : N \hookrightarrow M$ is expressed locally as

$$x_i = x_i(u_1, \dots, u_\ell)$$

Thus, the rank of the Jacobian matrix

$$\left(\frac{\partial x_i}{\partial u_j}\right)$$

is ℓ, by assumption. Without loss of generality,

$$\det\left(\frac{\partial x_i}{\partial u_j}\right)_{i,j=1}^{\ell} \neq 0$$

From the inverse function theorem from multivariate calculus, we have a local inverse

$$u_i = u_i(x_1, \dots, x_\ell)$$

near 0. Moreover, for $i > \ell$, then $x_i = x_i(u) = x_i(u(x_1, \dots, x_\ell)) = h_i(x_1, \dots, x_\ell)$.

Define $f_i(x) = x_i - h_i(x_1, \dots, x_\ell)$ for $\ell < i < n$. This gives the required map, with Jacobian at 0 given in block form as $\begin{pmatrix} * & I_d \end{pmatrix}$ Thus, 0 is a regular value. □

Example 3.11

Consider

$$N = \{(x_0 : x_1 : x_2) \in \mathbb{RP}^2 \mid x_2 = 0\}$$

We can see that $N \hookrightarrow \mathbb{RP}^2$, and $N \cong S^1$. However, it is impossible to write

$$N = f^{-1}(q)$$

for all $f : \mathbb{RP}^2 \to M$ where M is a 1-manifold, $q \in M$ a regular value.

To see this, suppose we found such an f. Then consider a chart Ψ around q on M, $\psi(q) = 0$. Then we have that

$$\psi \circ f : \mathbb{RP}^2 \supseteq U \to (-1, 1)$$

with $N \subseteq U \subseteq \mathbb{RP}^2$, U is open and connected. In this case, we have

$$N = \{p \mid (\psi \circ f)(p) = 0\}$$

As q is a regular value, there exists $p_+, p_- \in U$ such that

$$\psi(f(p_+)) > 0 \quad \text{and} \quad \psi(f(p_-)) < 0$$

But $U \setminus N \subseteq \mathbb{RP}^2 \setminus N$ is connected, since it is either an annulus or a disc.

Theorem 3.12 (Whitney embedding theorem)**.** For any manifold M^n, there exists an embedding $\psi : M \hookrightarrow \mathbb{R}^{2n}$.

On the examples sheet, we will show the result for $\mathbb{R}^N$, and with the additional assumption that M is compact. It's not too difficult to show that an embedding into $\mathbb{R}^{2n+1}$ exists, Reducing to $2n$ is the main difficulty in this theorem. Moreover, this result is optimal, for example by considering $\mathbb{RP}^2$ which embeds into $\mathbb{R}^4$ but not $\mathbb{R}^3$.

Lecture 8

Using what we have done so far, we can now prove theorem 2.22. That is, for matrix Lie groups, the map

$$\begin{aligned} \mathfrak{g} &\to \ell(G) \\ \xi &\mapsto X_\xi \end{aligned}$$

is a Lie algebra isomorphism.

Proof. We have shown before that $[X_\xi, X_\eta] = X_\zeta$ for some $\zeta \in \mathfrak{g}$. We would like to show that $\zeta = [\xi, \eta] = \xi\eta - \eta\xi$.

First consider the case when $G = \mathrm{GL}(n) \subseteq \mathrm{Mat}(n)$, which is an open subset. In this case, the Lie algebra is $\mathfrak{g} = \mathrm{Mat}(n)$. For $g = (g^i{}_j) \in G$, the left translation map is a linear function in the coordinates $g^i{}_j$. Thus, for $g, h \in \mathrm{GL}(n)$, we have that

$$\begin{aligned} (\mathrm{d}L_g)_h : \mathrm{Mat}(n) &\to \mathrm{Mat}(n) \\ A &\mapsto gA \end{aligned}$$

For $A = (A^i{}_j) \in \mathfrak{g}$, we have that

$$X_A(\mathfrak{g}) = \sum X^i_j(g) \frac{\partial}{\partial g^i{}_j}$$

with

$$X^i_j(g) = \sum_k g^i{}_k A^k{}_j$$

With this, the claim follows from a computation. That is, (using the summation convention),

$$\begin{aligned} g^i{}_k \left(A^k{}_j \frac{\partial}{\partial g^i{}_j} \left(g^\ell{}_p B^p{}_q\right) - B^k{}_j \frac{\partial}{\partial g^i{}_j} \left(g^\ell{}_p A^p{}_q\right) \right) \frac{\partial}{\partial g^\ell{}_q} &= g^i{}_k \left(A^k{}_j B^j{}_q \delta^i{}_\ell - B^k{}_j A^j{}_q \delta^i{}_\ell \right) \frac{\partial}{\partial g^\ell{}_q} \\ &= g^i{}_k (AB - BA)^k{}_q \frac{\partial}{\partial g^i{}_q} \end{aligned}$$

Using this, we see that

$$[X_A, X_B] = X_{[A,B]}$$

Now note that the change of basis of $\mathfrak{g}$ from the identity chart as above, to the log chart is given by

$$(\mathrm{d}\log)_I = \mathrm{id}$$

Thus, the basis are the same, and so the above computation is valid in the log chart as well.

Now consider a more general group satisfying our assumptions. The log chart assumption shows that $\iota : G \to \mathrm{GL}(n)$ is an immersion. Moreover, $L_g : G \to G$ is a restriction of $L_g : \mathrm{GL}(n) \to \mathrm{GL}(n)$, and its derivative the restriction of $\mathrm{d}L_g : \mathrm{Mat}(n) \to \mathrm{Mat}(n)$.

Using this, for $\xi \in \mathfrak{g}$, the left invariant vector field $X_\xi \in \ell(G)$ is a restriction of $X_\xi \in \ell(\mathrm{GL}(n))$. As such,

$$[X_\xi|_G, X_\eta|_G] = [X_\xi, X_\eta]|_G$$

We can use test functions which is constant in the variables which are transverse to G to see this. But we know that $[X_\xi, X_\eta] = X_{\xi,\eta}$ on $\mathrm{GL}(n)$, and so we have that

$$X_{[\xi,\eta]}|_G = [X_\xi|_G, X_\eta|_G] \in \ell(G)$$

□

4 Differential forms

Let M^n be a manifold, $p \in M$. We have defined the tangent space T_pM. Define the *cotangent space of M at p* to be

$$\mathrm{T}_p^*M = (\mathrm{T}_pM)^* = \left\{\text{linear maps } \mathrm{T}_pM \to \mathbb{R}\right\}$$

Choosing local coordinates x_i around p, we have a basis

$$\frac{\partial}{\partial x_i} \in \mathrm{T}_pM$$

for T_pM. This gives us a dual basis, which we will denote as

$$\mathrm{d}x_i \in \mathrm{T}_p^*M$$

That is,

$$\left\langle \mathrm{d}x_i, \frac{\partial}{\partial x_j} \right\rangle := \mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$$

For any $a \in \mathrm{T}_p^*M$, we have that

$$a = \sum_i a_i \mathrm{d}x_i$$

for $a_I \in \mathbb{R}$. Recall if y_i is a different choice of local coordinates, then

$$\frac{\partial}{\partial y_j} = \sum_i \frac{\partial x_i}{\partial y_j}\frac{\partial}{\partial x_i}$$

Then by standard linear algebra arguments,

$$\mathrm{d}x_i = \sum_j \frac{\partial x_i}{\partial y_j}\mathrm{d}y_j$$

Thus, for

$$a = \sum a_i \mathrm{d}x_i = \sum b_i \mathrm{d}y_i \in \mathrm{T}_p^*M$$

we have that

$$b_j = \sum_i a_i \frac{\partial x_i}{\partial y_j}$$

We can define the *cotangent bundle of M*, as

$$\mathrm{T}M = \bigsqcup_{p \in M} \mathrm{T}_p^*M$$

Theorem 4.1. T^*M is a smooth manifold, with $\dim(\mathrm{T}^*M) = 2\dim(M)$. Again, we have a projection map $\pi : \mathrm{T}^*M \to M$, with $\pi(p, a) = p$, which is a smooth map.

Proof. As for the tangent bundle. The only difference is the transformation law. □

Definition 4.2 (differential 1-form)
A *((smooth) differential) 1-form* α on M, is by definition a smooth map $\alpha : M \to \mathrm{T}^*M$, such that $\pi \circ \alpha = \mathrm{id}$. That is, for all $p \in M$, $\alpha(p) \in \mathrm{T}_p^*M$.

As for vector fields, in local coordinates x_i, we have that

$$\alpha|_U = \sum_i \alpha_i(p)\mathrm{d}x_i$$

where $\alpha_i \in C^\infty(U)$. This is equivalent to saying that

$$\langle \alpha, X \rangle \in C^\infty(M)$$

for all $X \in V(M)$. To see this, note that

$$\left\langle \alpha, \frac{\partial}{\partial x_i} \right\rangle = \alpha_i$$

We will now recall/define some multilinear algebra. For $r = 0, 1, 2, \ldots$, and a vector space V, define the *r-th exterior power of V^** as

$$\Lambda^r V^* = \left\{ \text{alternating multilinear maps } \underbrace{V \times \cdots \times V}_{r \text{ copies}} \to \mathbb{R} \right\}$$

In particular, if $r = 0$, by convention $\Lambda^0 V^* = \mathbb{R}$, and if $r = 1$, $\Lambda^1 V^* = V^*$. If $r > \dim(V)$, then $\Lambda^r V^* = 0$.

From this, we obtain the vector spaces $\Lambda^r \mathrm{T}_p^* M$, and we can define the *r-th exterior power of* $\mathrm{T}^* M$ as

$$\Lambda^r \mathrm{T}^* M = \bigsqcup_{p \in M} \Lambda^r \mathrm{T}_p^* M$$

Consider a smooth map $\alpha : M \to \Lambda^r \mathrm{T}^* M$, with $\pi \circ \alpha = \mathrm{id}$. We call α a *differential r-form*.

Lecture 9

This is a vector space, with dimension $\dim(\Lambda^r \mathrm{T}_p^* M) = \binom{n}{r}$. To see this, we can consider the basis

$$\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_r} \text{ with } 1 \le i_1 < \cdots < i_r \le n$$

By definition,

$$(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_r})(v_1, \ldots, v_r) := \det\big((\mathrm{d}x_{i_k}(v_\ell))_{k,\ell}\big)$$

In particular, we see that $\wedge$ is antisymmetric, and this extends by linearity and induction on p, q, to extend it to an assosiative bilinear product

$$\Lambda^p \mathrm{T}_x^* M \times \Lambda^q \mathrm{T}_x^* M \to \Lambda^{p+q} \mathrm{T}_x^* M$$

Notation 4.3 (multi-index notation)**.** If $I = (i_1, \ldots, i_r)$, then we write $|I| = r$, and

$$\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_r}$$

The transformation law from $\mathrm{d}x_I$ to $\mathrm{d}y_J$ is given by

$$\mathrm{d}y_J = \sum_{|I|=r} \left(\prod_{k=1}^{r} \frac{\partial y_{j_k}}{\partial x_{i_k}} \right) \mathrm{d}x_I$$

Using this, we can show that $\Lambda^r \mathrm{T}^* M$ is a smooth manifold, with dimension $n + \binom{n}{r}$. The projection map $\pi : \Lambda^r \mathrm{T}^* M \to M$ is smooth, and we call $\Lambda^r \mathrm{T}^* M$ the *bundle of r-forms*.

Definition 4.4 (smooth differential r-form)

A *((smooth) differential) r-form* is a section α of $\Lambda^r \mathrm{T}^* M$. The *degree* of α is r.

Note that α as above is smooth if and only if for any $X_1, \ldots, X_r$ vector fields on M, $\alpha(X_1, \ldots, X_n) = 0$. In local coordinates, we can write

$$\alpha = \sum_{|I|=r} \alpha_I \mathrm{d}x_I$$

where the α_I are smooth functions.

Definition 4.5 (space of r-forms)
For a manifold M, we write $\Omega^r(M)$ for the space of all r-forms on M. By convention, $\Omega^0(M) = C^\infty(M)$.

4.1 Orientability

Theorem 4.6. For any n-dimensional manifold, the following are equivalent:

(a) There exists a nowhere vanishing $\omega \in \Omega^n(M)$.

(b) There exists a collection of coordinate charts $(\varphi_\alpha, U_\alpha)$ such that

- $\bigcup_\alpha U_\alpha = M$,
- for α, β with corresponding coordinates x_i, y_j, we have

$$\det\left(\frac{\partial y_j}{\partial x_i}\right) > 0$$

(c) $\Lambda^n T^* M$ is isomorphic to $M \times \mathbb{R}$.

Proof. **(a) $\implies$ (b):** From the transformation law and taking $r = n$, we see that

$$\begin{aligned} dx_1 \wedge \cdots \wedge dx_n &= \left(\sum_{i_1} \frac{\partial x_1}{\partial y_{i_1}} dy_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_n} \frac{\partial x_n}{\partial y_{i_n}} dy_{i_n}\right) \\ &= \left(\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \frac{\partial x_1}{\partial y_{\sigma(1)}} \cdots \frac{\partial x_n}{\partial y_{\sigma(n)}}\right) \\ &= \det\left(\frac{\partial x_i}{\partial y_j}\right) dy_1 \wedge \cdots \wedge dy_n \end{aligned}$$

Given ω as in (a), consider an open cover by coordinate neighbourhoods U_α, with all U_α connected. In this case,

$$\omega|_{U_\alpha} = f_\alpha(x) dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$$

We can assume without loss of generality that $f_\alpha > 0$ on each coordinate neighbourhood, but then

$$f_\alpha = \det\left(\frac{\partial x_i^\alpha}{\partial x_j^\beta}\right) f_\beta$$

and so the determinant must be positive.

(b) $\implies$ (a): We will assume the following theorem without proof:

Theorem 4.7. For every open cover $M = \bigcup_\alpha U_\alpha$, we can find a countable collection of smooth functions $\rho_i \in C^\infty(M)$, such that

(i) $\operatorname{supp}(\rho_i)$ is compact, and contained in some U_α,

(ii) the ρ_i are locally finite, that is, given $x \in M$, there exists a neighbourhood W_x of x, such that all but finitely many ρ_i are identically zero on W_x.

(iii) $\rho_i(x) \in [0, 1]$, and for any $x \in M$,

$$\sum_i \rho_i(x) = 1$$

We call $\{\rho_i\}$ a *partition of unity subordinate to* U_α.

For any α, define $\omega_\alpha = dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \in \Omega^n(U_\alpha)$. By passing to a subcollection, we may assume we only have countably many α, and say we have a parition of unity ρ_α subordinate to $\{U_\alpha\}$. Then

$$\rho_\alpha \omega_\alpha \in \Omega^n(M)$$

and using this, we can define

$$\omega = \sum \rho_\alpha \omega_\alpha$$

Since the sum is locally finite, ω is well defined. Moreover, since the ρ_α are nonnegative, and

$$\det\left(\frac{\partial x_i^\alpha}{\partial x_j^\beta}\right) > 0$$

ω never vanishes.

For **(c)** $\implies$ **(a)**, suppose we have an isomorphism $\Phi : \Lambda^n T^*M \to M \times \mathbb{R}$ of vector bundles, consider $f : M \to M \times \mathbb{R}$, given by $f(x) = (x, 1)$. Then $\Phi^{-1} \circ f \in \Omega^n(M)$ never vanishes.

Conversely, given a non-zero ω, notice for any $(p, \xi) \in \Lambda^n T^*M$, there exists a_ξ such that $\xi = a_\xi \omega(p)$. Thus, we can define

$$\Phi(p, \xi) = (\pi(\xi), a_\xi)$$

We leave as an exercise to check that Φ is a diffeomorphism. □

Definition 4.8 (orientable)
A manifold M satisfying any of the conditions in the above theorem is called *orientable*.

Suppose M is connected. then there are *two* choices of *orientation*, given by respectively:

(a) ω up to rescaling by a positive function,

(b) a choice of an open up to 'positive compatibility'.

(c) a choice of Φ up to compositions $(x, a) \mapsto (x, h(x)a)$, where $h \in C^\infty(M)$ is positive.

Definition 4.9 (oriented)
A manifold M with a choice of orientation is called *oriented*.

4.2 Exterior derivative

Consider $f \in C^\infty(M)$. Formally, it is a smooth map $f : M \to \mathbb{R}$ between manifolds, and so we have a differential

$$df_p : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$$

which depends on p in a smooth way. Then

$$df \in \Omega^1(M)$$

Remark 4.10. A coordinate $x_i : U \to \mathbb{R}$ is a smooth function, and so dx_i is a 1-form.

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Theorem 4.11. There exists a unique $\mathbb{R}$-linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$, determined by

(i) If $f \in \Omega^0(M) = C^\infty(M)$, then df is just the differential. That is,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

(ii) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge d\eta$

(iii) $d^2 = 0$.

The map d is called the *exterior derivative*.

Proof. Consider a coordinate neighbourhood $U \subseteq M$, with coordinates x_i. Consider $\omega \in \Omega^r(M)$, with

$$\omega|_U = f(x)\mathrm{d}x_I$$

We can assume this since d is linear. By (ii), (iii) and then (i),

$$\begin{aligned} \mathrm{d}(\omega|_U) = \mathrm{d}(f(x)\mathrm{d}x_I) &= \mathrm{d}f \wedge \mathrm{d}x_I + f(x)\sum_{k=1}^{r}\left(\cdots \wedge (-1)^{k-1}\mathrm{d}\mathrm{d}x_{i_k} \wedge \cdots\right) \\ &= \sum_{i=1}^{m}\frac{\partial f}{\partial x_i}\mathrm{d}x_i \wedge \mathrm{d}x_I \end{aligned} \tag{$*$}$$

Using this definition, (i) is clear, (ii) can be verified by a computation, and (iii) follows from the symmetry of mixed partial derivatives, since

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

we have that

$$\sum_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}\mathrm{d}x_i \wedge \mathrm{d}x_j = 0$$

Using this, we have shown uniqueness since it is well defined and unique with respect to anny choice of local coordinates. To show existence, we need to show that it is well defined with respect to a change of coordinates. But by $(*)$, $(\mathrm{d}\omega)_p$ is determined by ω restricted to any neighbourhood of p.

Thus, to show that d is well defined, it suffices to consider the intersection of coordinate neighbourhoods. Let y_i be the other coordinates. The same computation as above gives an operator δ. We want to show that $\delta = \mathrm{d}$. With this,

$$\delta(f(x)\mathrm{d}x_I) = \delta f \wedge \mathrm{d}x_I + \sum_k \left(\cdots \wedge (-1)^{k-1}\delta \mathrm{d}x_{i_k} \wedge \cdots\right)$$

Now $\delta f = \mathrm{d}f$ as the differential is well defined, and $\delta(x_{i_k}) = \mathrm{d}x_{i_k}$ as well. Thus, $\delta \mathrm{d}x_{i_k} = \delta^2 x_{i_k} = 0$. With this, we find that

$$\delta(f(x)\mathrm{d}x_I) = \mathrm{d}f \wedge \mathrm{d}x_I$$

as required. □

Definition 4.12 (pullback)

Let $f : M \to N$ be a smooth map between manifolds. Then we have a map

$$f^* : \mathrm{T}^*_{f(p)}N \to \mathrm{T}^*_p M$$

and more generally, we have

$$f^* : \Omega^r(N) \to \Omega^r(M)$$

which we call the *pullback map of* f. By definition,

$$(f^*\alpha)_p(v_1, \ldots, v_r) = \alpha_{f(p)}((\mathrm{d}f)_p v_1, \ldots, (\mathrm{d}f)_p v_r)$$

for every r-form α on N, $p \in M$, $v_1, \ldots, v_r \in \mathrm{T}_p M$.

Remark 4.13.

- f^* is defined for any smooth map. Compare this to the pushforward of vector fields, which is defined only when f is a diffeomorphism.
- $(f \circ g)^* = g^* \circ f^*$.
- $f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta)$. In particular, for any $h \in C^\infty(N)$, $f^*(h\alpha) = (h \circ f)f^*\alpha$. That is, f^* is $C^\infty(\cdot)$-linear.
- $\mathrm{d}(f^*\alpha) = f^*(\mathrm{d}\alpha)$. To see this, it suffices to do this in local coordinates, i.e. we can take M, N to be open subsets of Euclidean space. Moreover, we can assume $\alpha = \mathrm{d}y_j$, $v = \frac{\partial}{\partial x_k}$, since we know both operators are $\mathbb{R}$-linear, and we can use the Leibniz rule to extend to general r-forms. In this case, f becomes $y_j = y_j(x)$, and we have

that

$$(f^*\mathrm{d}y_j)\left(\frac{\partial}{\partial x_k}\right) = \mathrm{d}y_j\left((\mathrm{d}f)\frac{\partial}{\partial x_k}\right) = \mathrm{d}y_j\left(\sum_\ell \frac{\partial y_\ell}{\partial x_k}\frac{\partial}{\partial y_\ell}\right) = \frac{\partial y_j}{\partial x_k}$$

We see that f^* is determined by

$$f^*\mathrm{d}y_j = \sum_{k=1}^n \frac{\partial y_j}{\partial x_k}\mathrm{d}x_k = \mathrm{d}\big(y_j \circ f\big) = \mathrm{d}\big(f^* y_j\big)$$

4.3 de Rham cohomology

We will write d_r for $\mathrm{d} : \Omega^r(M) \to \Omega^{r+1}(M)$. That is, we have

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}_0} \Omega^1(M) \xrightarrow{\mathrm{d}_1} \Omega^2(M) \xrightarrow{\mathrm{d}_2} \cdots$$

If $\alpha = \mathrm{d}\beta$, then $\mathrm{d}\alpha = 0$. The converse may not hold. We can use this:

Definition 4.14 (exact form)
If $\alpha = \mathrm{d}\beta$ for some β, we say that α is *exact*. Equivalently, $\alpha \in \mathrm{im}(\mathrm{d}_r)$ for some r.

Definition 4.15 (closed form)
If $\alpha \in \ker(\mathrm{d}_r)$, then we say that α is *exact*.

Since $\mathrm{d}^2 = 0$, $\mathrm{im}(\mathrm{d}_{r-1}) \subseteq \ker(\mathrm{d}_r)$. Moreover, it is a vector subspace.

Definition 4.16 (de Rham cohomology)
The vector space

$$\mathrm{H}^r(M) = \frac{\ker(\mathrm{d}_r)}{\mathrm{im}(\mathrm{d}_{r-1})}$$

is called the *r-th de Rham cohomology group of M*. We write

$$\mathrm{H}^* = \bigoplus_r \mathrm{H}^r(M)$$

Proposition 4.17. If $f : M \to N$ is smooth, the pullback $f^* : \Omega^r(N) \to \Omega^r(M)$ descends to the quotient, i.e. we have a homomorphism

$$f^* : \mathrm{H}^r(N) \to \mathrm{H}^r(M)$$

Moreover, as for pullback of forms,

$$(g \circ f^*) = f^* \circ g^*$$

Proof. This follows from $f^*\mathrm{d} = \mathrm{d}f^*$ (i.e. f^* is a chain map). □

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In particular, if f is a diffeomorphism, then $f^* : \mathrm{H}^*(N) \to \mathrm{H}^*(M)$ is an isomorphism. The converse is false, for example

$$\mathrm{H}^*(\mathbb{RP}^3) \cong \mathrm{H}^*(S^3)$$

but they are diffeomorphic (not even homeomorphic, for example consider π_1).

Theorem 4.18 (de Rham).

$$\mathrm{H}^r_{\mathrm{dR}}(M) \cong \mathrm{H}^r(M; \mathbb{R})$$

where the right hand side is the singular cohomology of M as a topological space, with real coefficients.

If M is connected, then $\mathrm{H}^0(M) = \mathbb{R}$, as it is just the space of constant functions (i.e. functions with zero differential).

Theorem 4.19 (Poincaré lemma)**.** Suppose $U \subseteq \mathbb{R}^n$ is an open ball, $\alpha \in \Omega^k(U)$, for some $k > 0$, with $\mathrm{d}\alpha = 0$. Then α is exact, that is, we can find $\beta \in \Omega^{k-1}(U)$, with $\alpha = \mathrm{d}\beta$.

Corollary 4.20.

$$\mathrm{H}^k(U) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Sketch proof of theorem 4.19. The main idea is to invert the exterior derivative d, by constructing appropriate integral operators,

$$h_k : \Omega^k(U) \to \Omega^{k-1}(U)$$

with

$$h_{k+1} \circ \mathrm{d}_k + \mathrm{d}_{k-1} \circ h_k = \mathrm{id}_{\Omega^k(U)}$$

That is, h defines a homotopy between id and the zero map on the de Rham cochain complex. □

4.4 Integration

Throughout, assume M is a oriented n-manifold, with orientation given by positively oriented charts $(U_\alpha, \varphi_\alpha)$.

Let $\omega \in \Omega^n(M)$, and suppose $\mathrm{supp}(\omega)$ is compact[1]. If $\mathrm{supp}(\omega) \subseteq U_\alpha$ for some α, with local coordinates x_i, then we can write

$$\omega|_{U_\alpha} = f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

In this case, we define

$$\int_M \omega = \int_{\varphi_\alpha(U_\alpha)} f(x)\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

where the right hand side is the integral over a subset of $\mathbb{R}^n$, i.e. Riemann/Lebesgue integration.

To see that this is well defined, suppose $\mathrm{supp}(\omega) \subseteq U_\beta$, and U_β has local coordinates y_j, then

$$\omega|_{U_\beta} = h(y)\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$$

In this case,

$$\begin{aligned} \int_{\varphi_\beta(U_\beta)} h(y)\mathrm{d}y_1 \cdots \mathrm{d}y_n &= \int_{\varphi_\alpha(U_\alpha)} h(y(x)) \left| \det \left(\frac{\partial y_i}{\partial x_j} \right) \right| \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_{\varphi_\alpha(U_\alpha)} h(y(x)) \det \left(\frac{\partial y_i}{\partial x_j} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n \\ &= \int_{\varphi_\alpha(U_\alpha)} f(x)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \end{aligned}$$

For more general n-forms with compact support, use a parition of unity $\{\rho_i\}$ subordinate to a finite cover $\{U_i\}$ of $\mathrm{supp}(\omega)$ by coordinate patches $U_1, \ldots, U_N$. Let $U_0 = M \setminus \mathrm{supp}(\omega)$.

Definition 4.21 (integral of a differential form)

[1] This is different to lectures, but I'll use the definition that

$$\mathrm{supp}(\omega) = \overline{\{p \in M \mid \omega(p) \neq 0\}}$$

The *integral of* ω is

$$\int_M \omega = \sum_{i=1}^{N} \int_{U_i} \rho_i \omega$$

Basic properties:

- it is linear in ω,
- it is additive when we integrate over disjoint coordinate neighbourhoods,
- it is independent of the choice of a parition of unity. If $\{\tilde{\rho}_j\}$ is a different partition of unity, then

$$\sum_i \int_{U_i} \rho_i \omega = \sum_{i,j} \int_{U_i} \rho_i \tilde{\rho}_j \omega = \sum_{i,j} \int_{U_i \cap \tilde{U}_j} \rho_i \tilde{\rho}_j \omega$$

  Swapping ρ_i and $\tilde{\rho}_j$ gives the same result, and so we are done.

Theorem 4.22 (Stoke's for manifolds without boundary)**.** If $\eta \in \Omega^{n-1}(M)$ is compactly supported, then

$$\int_M \mathrm{d}\eta = 0$$

Note η being compactly supported implies $\mathrm{d}\eta$ is compactly supported, but the converse is false.

Proof. As above, let $U_1, \ldots, U_N$ be a positively oriented coordinate neighbourhoods covering $\operatorname{supp}(\eta)$, and let $U_0 = M \setminus \operatorname{supp}(\eta)$, which is positively oriented with respect to $U_1, \ldots, U_N$. Choose a partition of unity subordinate to $\{U_0, \ldots, U_N\}$. Then

$$\mathrm{d}\eta = \mathrm{d}\left(\sum_{i=1}^{N} \rho_i \eta\right) = \sum_{i=1}^{N} \mathrm{d}(\rho_i \eta)$$

By additivity, suffices to show

$$\int_M \mathrm{d}(\rho_i \eta) = 0$$

Fix i, and choose local coordinates x_k on U_i. Without loss of generality,

$$\rho_i \eta = h(x) \mathrm{d}x_2 \wedge \cdots \mathrm{d}x_n$$

We can do this as the integral is linear. Then

$$\mathrm{d}(\rho_i \eta) = \frac{\partial h}{\partial x_i} \mathrm{d}x_1 \wedge \cdots \mathrm{d}x_n$$

Integrating, choose R large enough such that

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathrm{d}(\rho_i \eta) &= \int_{\mathbb{R}^{n-1}} \left(\int_{x_1=-R}^{R} \frac{\partial h}{\partial x_1} \mathrm{d}x_1 \right) \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= \int_{\mathbb{R}^{n-1}} h(R, x_2, \ldots, x_n) - h(-R, x_2, \ldots, x_n) \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
&= 0
\end{aligned}$$

as we can assume $h(R, x_2, \ldots, x_n) = h(-R, x_2, \ldots, x_n) = 0$. □

Corollary 4.23 (integration by parts)**.** Assume α, β are differential forms on M, at least one is compactly supported, with $\deg(\alpha) + \deg(\beta) = \dim(M) - 1$. Then

$$\int_M \alpha \wedge \mathrm{d}\beta = (-1)^{\deg(\alpha)+1} \int_M (\mathrm{d}\alpha) \wedge \beta$$

Proof. Apply Stokes to $\eta = \alpha \wedge \beta$, which is compactly supported. $\square$

Lecture 12

5 Vector bundles

Definition 5.1 (vector bundle)
A *vector bundle E over a manifold B* is

1. a smooth manifold E,
2. a surjective submersion $\pi : E \to B$, i.e. π is smooth, and $(\mathrm{d}\pi)_p$ is surjective,

Moreover,

(i) there exists a (finite dimensional) vector space V, such that for all $p \in B$, $E_p = \pi^{-1}(p)$ is a vector space isomorphic to V,

(ii) for any $p \in B$, there exists an open neighbourhood U of p, and a diffeomorphism Φ_U making the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times V \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

commute.

(iii) for each $p \in U$, the restriction $\Phi_U : E_p \to \{p\} \times V \cong V$ is a linear isomorphism.

We call

- B the *base*,
- E the *total space*,
- π the *bundle projection*,
- V the *typical fibre*, if V is a real vector space, we call $E \to B$ a *real vector bundle*, and if V is a complex vector space, then we call $E \to B$ a *complex vector bundle*.
- $\dim(V)$ is the *rank of the vector bundle*[a],
- Φ_U a *local trivialisation over U*,
- U is a *trivialising neighbourhood*

[a]Note in the case of a complex vector bundle this depends on the real/complex dimension.

Remark 5.2. Note surjectivity of π follows from (i), and the fact that it is a submersion follows from (ii).

Definition 5.3 ((local) section)
A *section* of a vector bundle $E \to B$ is a smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$. A *local section* is a smooth map $s; U \to E$, with $\pi \circ s = \mathrm{id}_U$.

Example 5.4 (product)

If $E = B \times V$, $E \to B$ is a vector bundle, and the space of sections is just $C^\infty(B, V)$.

Example 5.5 ((co)tangent space)

TM and T^*M are real vector bundles of rank $n = \dim(M)$. The sections are vector fields and 1-forms respectively. In general, these are non-trivial, i.e. they are not products.

In general, $\Lambda^r T^*M$ is also a real vector bundle, and sections are r-forms. If TM is trivial, then so are the bundles of differential forms.

Example 5.6 (tautological vector bundle)

Over $\mathbb{RP}^n, \mathbb{CP}^n$ (or more generally, Grassmannians), we have the *tautological vector bundle*.

Say $B = \mathbb{CP}^n$, take

$$E = \bigsqcup_{\ell \text{ line in } \mathbb{C}^{n+1} \text{ through } 0} \ell$$

The bundle projection sends

$$\pi : E \ni \ell \mapsto \ell \in \mathbb{CP}^n$$

We can check that this is a rank 1 complex vector bundle (i.e. a complex line bundle).

5.1 Structure group and transition functions

Let $(U_\alpha, \Phi_\alpha), (U_\beta, \Phi_\beta)$ be local trivialisations of E. In this case, we have that for $b \in U_\alpha \cap U_\beta, v \in V$,

$$\Phi_\beta \circ \Phi_\alpha^{-1}(b, v) = (b, \psi_{\beta\alpha}(b)v)$$

where $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ is smooth.

Definition 5.7 (transition functions)

The $\psi_{\beta\alpha}$ are called the *transition functions*.

The transition functions satisfy the following:

- $\psi_{\alpha\alpha}(b) = \mathrm{id}$,
- $\psi_{\alpha\beta}(b)\psi_{\beta\alpha}(b) = \mathrm{id}$,
- $\psi_{\alpha\beta}(b)\psi_{\beta\gamma}(b)\psi_{\gamma\alpha}(b) = \mathrm{id}$

which we call the *cocycle conditions*.

Example 5.8 ((co)tangent bundle)

In the case of the (co)tangent bundles, with the charts defined earlier, the transition functions are given by the derivatives (or the dual matrix)

$$\left(\frac{\partial x_i}{\partial y_j}\right)_{i,j}$$

Proposition 5.9. The following data:

- base manifold B,
- trivialising neighbourhoods $\{U_\alpha\}_{\alpha \in A}$ covering B,
- maps $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$ smooth, satisfying the cocycle condition.

determines a vector bundle E over B, with typical fibre V.

The proof is called the Steenrod construction.

Proof. First we define the total space

$$E := \frac{\bigsqcup_{\alpha \in A}(U_\alpha \times V)}{\sim}$$

where $U_\alpha \times V \ni (b, v) \sim (b, \psi_{\beta\alpha}(b)v) \in U_\beta \times V$.

Without loss of generality, we may assume that the $U_\alpha \subseteq B$ are coordinate neighbourhoods, with charts φ_α Then

$$\widetilde{\varphi}_\alpha = (\varphi_\alpha, \mathrm{id}_V)$$

for each α is a chart on $U_\alpha \times V$. This implies that E is a manifold, as these form a smooth structure. The definitions of π and the existence of local trivialisations are clear. □

Definition 5.10 (*G*-structure, structure group)

Let $E \to B$ be a real vector bundle with typical fibre V. Let $G \leq \mathrm{GL}(V)$ be a subgroup. Suppose there exists local trivialisations (U_α, Φ_α) covering B, where the transition maps $\psi_{\alpha\beta}(b) \in G$ for all $b \in U_\alpha \cap U_\beta$. We call this a *$G$-structure on $E \to B$*, and that E has G as a *structure group*.

Example 5.11

If $G = \{\mathrm{id}\}$, then E has a global trivialisation $\Phi : E \to B \times V$.

If $G = \mathrm{GL}_+(V)$ is the subgroup of matrices with positive determinant. Then a G-structure on E is an orientation of E. If $E = TM$, then this determines an orientation of M.

Definition 5.12 (orientable, oriented)

A vector bundle $E \to B$ is *orientable* if it admits a $\mathrm{GL}_+(V)$ structure. $E \to B$ is *oriented* if we made a choice of orientation.

Lecture 13

Example 5.13 (orthogonal trivialisations)

Let E be a real vector bundle, if $G = O(V)$, then there exists an invariantly defined *inner product* on each fibre E_p. More precisely, this is obtained via a linear isometry $E_p \to (\mathbb{R}^k, \langle \cdot, \cdot \rangle)$. The respective $\Phi_{\alpha\beta}$ giving the $O(V)$ structure are then called *orthogonal (local) trivialisations*.

In particular, if $G = \mathrm{SO}(V) = \mathrm{GL}_+(V) \cap O(V)$, then this is equivalent to a choice of an inner product as well as an orientation on the fibres.

Example 5.14

Say now E is a real vector bundle with rank $2m$. Then we can consider $G = \mathrm{GL}(m, \mathbb{C}) \leq \mathrm{GL}(2m, \mathbb{R})$. For all $p \in B$, there exists $J_p \in \mathrm{GL}(E_p)$, with $J_p^2 = -\mathrm{id}_{E_p}$. Moreover, J_p depends smoothly on p. This corresponds to J_0 on $\mathbb{R}^{2n} = \mathbb{C}^n$, defined by multiplication by i. This naturally makes E into a complex vector bundle.

If $E = TM$, a $\mathrm{GL}(n, \mathbb{C})$-structure on M is called an *almost complex structure* on M.

Example 5.15

If $G = U(k) \subseteq \mathrm{GL}(k, \mathbb{C})$, we get a *unitary structure*, analogous to the orthogonal case. The trivialisations are called *unitary trivialisations*.

More generally, if G preserves a tensor T on V under the inclusion $G \to \mathrm{GL}(V)$ (or the appropriate action on the dual), then a G-structure induces a family T_p of tensors on E_p, which are equivalent to T under the trivialisations.

In general, existence of a G-structure for a given G can be a non-trivial problem. But for $G = O(V)$ and $G = U(V)$, the answer is yes.

5.2 Principal bundles

Let G be a Lie group, with identity element 1_G.

Definition 5.16 (smooth free right action)
A *smooth free right action* of G on a manifold P is a smooth map

$$\begin{aligned} P \times G &\to P \\ (p, h) &\mapsto p \cdot h \end{aligned}$$

where

(i) free: If $p \cdot h = p$, then $h = 1_G$,

(i) $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$.

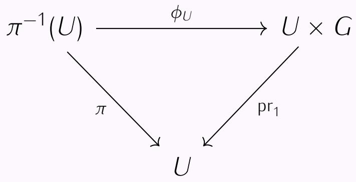
Remark 5.17. The fact that it is a right action implies that the map $p \mapsto ph$ is a diffeomorphism for all $h \in G$.

Definition 5.18 (principal bundle)
Let G be a Lie group. A *principal G-bundle* P over a manifold B is

1. a smooth manifold P, with a smooth free right action of G on P,
2. a surjective submersion $\pi : P \to B$,

Moreover,

(i) for any $b \in B$, there exists an open neighbourhood U of b and a diffeomorphism Φ_U making the diagram



commute.

(ii) Φ_U commutes with the action of G. That is,

$$\Phi_U(ph) = (b, gh) = (\pi(p), gh)$$

where $\Phi_U(p) = (b, g)$.

We call

Remark 5.19. From the definitions, it follows that the fibres $\pi^{-1}(p)$ are embedded submanifolds, diffeomorphic to G. By analogy, we will call P the total space, B the base and so on.

Remark 5.20 (Warning)**.** If $P \to B$ is a fibre bundle, where each fibre is a Lie group, P does not have to be a principal bundle, since we don't need to have a right action on each fibre.

Similar to the case of a vector bundle, we have: For every $b \in U_\alpha \cap U_\beta$, consider

$$\Phi_\beta \circ \Phi_\alpha^{-1}(b, g) = (b, \psi_{\beta\alpha}(b, g))$$

where

$$\psi_{\beta\alpha}(b, \cdot) : G \to G$$

The requirement that Φ_U commutes with the G-action gives that

$$\psi_{\beta\alpha}(b, gh) = \psi_{\beta\alpha}(b, g)h$$

for any $g, h \in G$. We will abuse notation, and write

$$\psi_{\beta\alpha}(b) := \psi_{\beta\alpha}(b, 1_G)$$

Using this, we get that

$$\psi_{\beta\alpha}(b, g) = \psi_{\beta\alpha}(b)g = L_{\psi_{\beta\alpha}(b)}g$$

With this, we have maps

$$\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$$

which are called the *transition functions*, and these satisfy the same cocycle conditions as for vector bundles.

Theorem 5.21. Given the data:

- base manifold B,
- trivialising neighbourhoods $\{U_\alpha\}$ covering B,
- maps $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ satisfying the cocycle conditions,

Let

$$P = \frac{\bigsqcup (U_\alpha \times G)}{\sim}$$

where $U_\alpha \times G \ni (b, h) \sim (b, \psi_{\beta\alpha}(b)h) \in U_\beta \times G$. Then P is a principal G-bundle, with projection map induced by the projections on each element of the disjoint union, with transition functions $\{\psi_{\beta\alpha}\}$. The right multiplication action is given by right multiplication on each element in the disjoint union.

Proof. Basically the same as the vector bundle case. ☐

Remark 5.22. If the G-action on a manifold P is smooth, free and proper, that is the map $P \times G \to P$ is proper, i.e. the preimage of a compact set is compact. Then we can make the orbit space P/G into a manifold, and we can define $B = P/G$ and π is the quotient map. This then defines a principal bundle $P \to B$.

Remark 5.23. Let G be a matrix Lie group. Suppose we have a vector bundle $E \to B$ with a G-structure, with transition maps $\psi_{\beta\alpha}$. From this we can obtain a principal G-bundle.

Conversely, if we have a principal G-bundle, and a smooth faithful representation $G \to \mathrm{GL}(V)$, we can make a vector bundle $E \to B$ using the transition functions.

We then say that E is *associated to the principal G-bundle* $P \to B$ (and vice versa).

Lecture 14

Example 5.24

If $E = \mathrm{T}M$, $\dim(M) = n$, taking $G = \mathrm{GL}(n, \mathbb{R})$, then the principal bundle P is called a *frame bundle*, and each fibre P_p is the set of a bases of the vector space T_pM, by considering the columnes of an invertible matrix.

If $G = \mathrm{O}(n)$< then P is called an *orthonormal frame bundle*, and each fibre is the set of orthonormal bases of T_pM.

5.3 Hopf bundle

A Hopf bundle is an example of a tautological complex line bundle over $\mathbb{CP}^1$. Recall that the fibre over $(z_1 : z_2) \in \mathbb{CP}^1$ is the complex line spanned by (z_1, z_2) in $\mathbb{C}^2$. We will work out the transition functions, and appeal to the Steenrod construction.

Write $U_i = \{(z_1 : z_2) \in \mathbb{CP}^1 \mid z_i \neq 0\}$. $U_1 \cup U_2$ is an open cover of $\mathbb{CP}^1$. Write $z = z_2/z_1$ for the local coordinate on U_1, and $\zeta = z_1/z_2 = 1/z$ for the local coordinate on U_2.

Set

$$\begin{aligned} \Phi_1 : \pi^{-1}(U_1) &\to U_1 \times \mathbb{C} \\ (w, wz) &\mapsto ((1:z), w\sqrt{1+|z|^2}) \end{aligned}$$

and

$$\begin{aligned} \Phi_2 : \pi^{-1}(U_2) &\to U_2 \times \mathbb{C} \\ (\zeta w, w) &\mapsto ((\zeta:1), w\sqrt{|\zeta|^2+1}) \end{aligned}$$

We can compute

$$\Phi_1^{-1}((1:z), \widetilde{w}) = \left(\frac{\widetilde{w}}{\sqrt{1+|z|^2}}, \frac{\widetilde{w}}{\sqrt{1+|z|^2}} z \right)$$

If $z \neq 0$, then

$$\begin{aligned} \Phi_2 \circ \Phi_1^{-1}((1:z), \widetilde{w}) &= \Phi_2 \left(\frac{\widetilde{w}}{\zeta\sqrt{1+|\zeta|^{-2}}}\zeta, \frac{\widetilde{w}}{\zeta\sqrt{1+|\zeta|^{-2}}} \right) \\ &= \left((\zeta:1), \frac{|\zeta|}{\zeta}\widetilde{w} \right) \\ &= \left((1:z), \frac{z}{|z|}\widetilde{w} \right) \end{aligned}$$

Hence the transition function is

$$\psi_{21}((1:z)) = \frac{z}{|z|}$$

and

$$\psi_{12}((\zeta:1)) = \frac{|z|}{z} = \frac{\zeta}{|\zeta|}$$

Note that the cocycle conditions are trivial as we only have two transition functions. Note

$$\psi_{12}, \psi_{21} : U_1 \cap U_2 \to \mathrm{U}(1) = S^1 \subseteq \mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$$

With this, the Hopf bundle admits a U(1) structure. Explicitly,

$$\|(w, wz)\| = |w|\sqrt{1+|z|^2}$$

and we have a similar computation for $\pi^{-1}(U_2)$. If we now remove the zero section, then we have an isomorphism

$$E \setminus \{\text{image of zero section}\} \cong \mathbb{C}^2 \setminus \{(0,0)\})$$

which is the total space of an associated $\mathrm{GL}(1, \mathbb{C})$-bundle.

The associated principal $\mathrm{U}(1) = S^1$ bundle has total space

$$P = \{(w_1, w_2) \in \mathbb{C}^2 \mid |w_1|^2 + |w_2|^2 = 1\} = S^3$$

and the bundle projection will send $(w_1, w_2) \to (w_1 : w_2) \in \mathbb{CP}^1 \cong S^2$.

We can see that the Hopf bundle is not the trivial bundle, since S^3 and $S^2 \times S^1$ are not homeomorphic.

5.4 Pullback

Definition 5.25 (pullback)
Let $\pi : E \to B$ be a vector bundle. and suppose we have a smooth map $f : M \to B$. Then we can define the *pullback* f^*E of $E \to B$ as a vector bundle $\widetilde{\pi} : f^*E \to M$, such that

1. the typical fibres are the same,
2. there exists a smooth map F making the diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E \\ {\scriptstyle \widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f]{} & B \end{array}$$

commute, and $F : (f^*E)_p \to E_{f(p)}$ is a linear isomoprhism for all $p \in E$.

From the definition, it follows that if (Φ, U) is a trivialisation of π, then we have a trivialisation $\tilde{\Phi}$ with

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times V \\ {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle f \times \mathrm{id}_V} \\ \widetilde{\pi}^{-1}(\pi^{-1}(U)) & \xrightarrow[\tilde{\Phi}]{} & f^{-1}(U) \times V \end{array}$$

That is, we induce a local trivialisation of f^*E over $f^{-1}(U)$.

Example 5.26
If $B = \{\mathrm{pt}\}$, then $E \cong V$, and

$$\widetilde{\Phi} : x \mapsto (\tilde{\pi}(x), F(x)) \in M \times V$$

trivialises f^*E over M.

Example 5.27
Let $M = B \times X$, $f = \mathrm{pr}_1 : B \times X \to X$. Then f^*E is "trivial in the X-direction". More precisely, $f^*E = E \times X$, with bundle projection $\widetilde{\pi} = \pi \times \mathrm{id}_X$.

Example 5.28
In this case, if $M = \mathrm{pt}$, then

$$F : f^*E \cong C \hookrightarrow E$$

embeds V as a fibre $E_{f(\mathrm{pt})}$.

In general, f^*E is determined by pulling back transition functions, that is,

$$f^*\psi_{\beta\alpha} = \psi_{\beta\alpha} \circ f : f^{-1}(U_\alpha \cap U_\beta) \to \mathrm{GL}(V)$$

This gives an alternative definition of the pullback.

By replacing "vector bundle" with "principal G-bundle", V with G, $\mathrm{GL}(V)$ with G, we can define the pullback of a principal G-bundle.

In particular, if E is associated to P, then f^*E is associated to f^*P.

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5.5 Morphisms of (vector) bundles

Let $\pi : E \to B$, $\pi' : E' \to B'$ be vector bundles, with typical fibres V, V' respectively. Let $f : B \to B'$ be a smooth map.

Definition 5.29 (morphism of vector bundle)

A *vector bundle morphism covering* f is a smooth map $F : E \to E'$, such that

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow[f]{} & B' \end{array}$$

commutes, and the morphism on any fibre

$$F_p : E_p \to E'_{f(p)}$$

is a linear map.

Then for each local trivialisation (Φ, U) for $E \to B$, and (Φ', U') for $E' \to B'$, with $f(U) \subseteq U'$. Set

$$F_U = \Phi' \circ F|_{\pi^{-1}(U)} \circ \Phi^{-1}$$

Then

$$F_U(b, v) = (f(b), h(b)v)$$

where $h(b) \in L(V, V')$ a linear map, $b \mapsto h(b)$ smooth.

Example 5.30

If $\varphi : M \to N$ is a smooth map, then $\mathrm{d}\varphi : TM \to TN$ is a morphism of vector bundles covering φ.

Example 5.31

Let $\pi : E \to B$ be a vector bundle, $f : M \to B$ a smooth map, then the pullback $f^*E \to M$ gives us a morphism of vector bundles $f^*E \to E$ covering f.

Example 5.32

Suppose $B' = B$, and $f \in \mathrm{Diff}(B)$. Then if F is a morphism of vector bundles covering f, and each map on fibres is an isomorphism, then F is an *isomorphism of vector bundles covering* f.

If $f = \mathrm{id}$, $E = E'$, with $\pi' = \pi$, then we call F a *vector bundle automorphism of* E, and we denote the group of automorphisms as $\mathrm{Aut}(E)$. In this case, each $h(b) \in \mathrm{GL}(V)$, and so we have a smooth map $h : U \to \mathrm{GL}(V)$.

Example 5.33

If E admits a G structure for some $G \leq \mathrm{GL}(V)$, then

$$\mathrm{Aut}_G(E) = \{F \in \mathrm{Aut}(E) \mid F \text{ preserves the } G \text{ structure}\}$$

makes sense. In terms of local trivialisations, then the maps are $h : U \to G$ instead. In mathematical physics, we call $\mathcal{G} = \mathrm{Aut}_G(E)$ the *the group of gauge transformations*. For example, $G = \mathrm{U}(1), \mathrm{SU}(2), \mathrm{SU}(3), \mathrm{SO}(3)$ and so on.

6 Connections on vector bundles

Let $\pi : E \to B$ be a real vector bundle of rank m, i.e. with typical fibre $\mathbb{R}^m$. Let $s \in \Gamma(E)$ be a section, then locally,

$$s_U = \mathrm{pr}_2 \circ \Phi \circ (s|_U) : U \to \mathbb{R}^n$$

is a vector valued function. We would like to extend the differential calculus. Locally, we have

$$\mathrm{d}s_U : \mathrm{T}_b B \to \mathbb{R}^m$$

but we also have

$$\mathrm{d}s : \mathrm{T}_b B \to \mathrm{T}_{s(b)} E$$

In this case, $\dim(\mathbb{R}^m) = m$, but $\dim(\mathrm{T}_{s(b)}E) = \dim(B) + m$. Intuitively, we have extra dimensions coming from the base manifold B.

Throughout, we will fix notation:

- $\dim(B) = n$,
- $U \subseteq B$ is a coordinate neighbourhood and a trivialising neighbourhood for E, with local coordinates $(x^k)_{k=1}^n$, and with coordinates $(a^j)_{j=1}^m$ on the fibres.
- We will use the summation convention on $i, j \in \{1, \ldots, m\}$ and $k, \ell \in \{1, \ldots, n\}$, but not on Greek indices.

Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^m$ be a local trivialisation. $\pi^{-1}(b) = E_b$ is a submanifold, and we have local coordinates

$$\mathrm{T}_p E_b = \mathrm{span}\left\{\frac{\partial}{\partial a^j}\right\}$$

This allows us to identify

$$\mathrm{T}_p E = \mathrm{span}\left\{\frac{\partial}{\partial x^k}, \frac{\partial}{\partial a^j}\right\}$$

Definition 6.1 (vertical, horizontal subspace)
The subspace $\mathrm{Tv}_p E = \ker(\mathrm{d}\pi_p)$ is called the *vertical subspace* at $p \in E$.

A subspace $S_p \subseteq \mathrm{T}_p E$ is a *horizontal subspace* if

$$S_p \oplus \mathrm{Tv}_p E = \mathrm{T}_p E$$

In this case, we know that $\dim(S_p) = n$ for all p from linear algebra. However, it is not determined by π.

Any n-dimensional subspace of $\mathrm{T}_p E \cong R^{n+m}$ is given by

$$\bigcap_{i=1}^m \ker(\theta_p^i)$$

where the $\theta_p^1, \ldots, \theta_p^m \in \mathrm{T}_p^* E \cong (\mathbb{R}^{n+m})^*$ are linearly independent. In this case,

$$S_p = \left\{v \in \mathrm{T}_p M \mid \theta_p^i(v) = 0 \text{ for all } i\right\}$$

where

$$\theta_p^i = f_k^i \mathrm{d}x^k + g_j^i \mathrm{d}a^j$$

for constants f_k^i, g_j^i. Suppose $c \in \mathrm{T}_p E$ is vertical, that is,

$$c = c^j \frac{\partial}{\partial a^j}$$

(i.e. it has no $\frac{\partial}{\partial x^k}$ components). If $c \neq 0$, then there exists i such that

$$\theta_p^i(c) \neq 0$$

Equivalently,

$$g^i_j c^i \neq 0$$

That is, we require the matrix (g^i_j) to be invertible. Let its inverse be (h^i_j). Now consider

$$\tilde{\theta}^i = h^i_j \theta^j = \mathrm{d}a^i + e^i_k \mathrm{d}x^k$$

Letting p vary in $\pi^{-1}(U)$, we get functions $e^i_k : \pi^{-1}(U) \to \mathbb{R}$. We obtain

Proposition 6.2. Every field $S = \{S_p\}_{p \in E}$ of horizontal subspaces can be given in local trivialisations as

$$S_p = \bigcap_{i=1}^{m} \ker(\theta^i_p)$$

with

$$\theta^i_p = \mathrm{d}a^i + e^i_k(x, a)\mathrm{d}x^k$$

for some smooth $e^i_k : U \times \mathbb{R}^n \to \mathbb{R}$. If the e^i_k are smooth, we say that S is smooth.

Definition 6.3 (connection)
A smooth field $S = \{S_p\}_{p \in E}$ of subspaces is a *connection* on a vector bundle E if

1. all S_p are horizontal,
2. all the e^i_k are linear in $a \in \mathbb{R}^m$.

In this case, we can write

$$e^i_k(x, a) = \Gamma^i_{jk}(x) a^j$$

for some $\Gamma^i_{jk} : U \to \mathbb{R}$ smooth. Thus,

$$\theta^i = \mathrm{d}a^i + \Gamma^i_{jk}(x) a^j \mathrm{d}x^k = \mathrm{d}a^i + A^i_j a^j$$

where $A^i_j = \Gamma^i_{jk}(x)\mathrm{d}x^k$ is a matrix of differential 1-forms. We can then consider the matrix $A = (A^i_j)$, and we would like to consider the transformation law for it.

Suppose (U', Φ') is another trivialisation, with $U \cap U' \neq \varnothing$. We will use i', j' indices for coordinates with respect to U'. Write $\Psi^{-1} = (\Phi^{i'}_i)^m_{i,i'=1}$ for the transition function from Φ' to Φ.

Note that

$$\Psi^i_{i'} \Psi^{i'}_j = \delta^i_j$$

Now suppose

$$a^{i'} = \Psi^{i'}_i a^i$$

Then

$$\mathrm{d}a^{i'} = \left(\mathrm{d}\Psi^{i'}_i\right) a^i + \Psi^{i'}_i \mathrm{d}a^i$$

But we also have that

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$$\theta^{i'} = \mathrm{d}a^{i'} + A^{i'}_{j'} a^{j'}$$

and so

$$\theta^{i'} = (\mathrm{d}\Psi^{i'}_j) a^j + \Psi^{i'}_j \mathrm{d}a^j + A^{i'}_{j'} \mathrm{d}a^{j'}$$

Next, note that

$$\theta^i = \Psi^i_{i'} \theta^{i'} = \mathrm{d}a^i + (\Psi^i_{i'} \mathrm{d}\Psi^{i'}_j + \Psi^{i'}_i A^{i'}_{j'} \Psi^{j'}_j) a_j$$

Therefore,

$$A^i_j = \Psi^i_{i'} \mathrm{d}\Psi^{i'}_j + \Psi^{i'}_i A^{i'}_{j'} \Psi^{j'}_j$$

Writing $A^\Phi = (A^j_i)$ and so

$$A^{\Phi'} = A^{\Psi \circ \Phi} = (A^{j'}_{i'})$$

and so

$$A^{\Psi \circ \Phi} = \Psi A^\Phi \Psi^{-1} - (\mathrm{d}\Psi)\Psi^{-1} = \Psi A^\Phi \Psi^{-1} + \Psi \, \mathrm{d}\left(\Psi^{-1}\right) \tag{2}$$

Theorem 6.4. Every choice of (local) matrices of one-forms (A^i_j) satisfying eq. (2) defines a connection on E.

Remark 6.5. Note that eq. (2) shows that the (A^i_j) will *not* define a matrix valued 1-form, since it has a different transformation law.

One way to see this is that $A^{\Phi} = 0$ does not imply that $A^{\Psi\circ\Phi} = 0$.

Remark 6.6. We can consider Ψ as a local expression for an element of $\mathrm{Aut}(E)$, and then eq. (2) defines a natural action of $\mathrm{Aut}(E)$ on the space of all connections.

Consider local trivialisations $\Phi_\alpha, \Phi_\beta : \pi^{-1}(U) \to U \times V$. If we have $G_\alpha \in \mathrm{End}(V)$, with respect to the trivialisation Φ_α, then[2]

$$G_\beta = \Psi_{\beta\alpha} G_\alpha \Psi_{\alpha\beta}$$

This defines an action of $\mathrm{GL}(V)$ on $\mathrm{End}(V)$, and so we can use the Steenrod construction to get a vector bundle $\mathrm{End}(E)$, called the *endomorphism bundle of E*. The fibres are $\mathrm{End}(E)_b = \mathrm{End}(E_b) \cong \mathrm{End}(V)$.

With this, we have a well defined subset $\mathrm{GL}(E)$, where each typical fibre is $\mathrm{GL}(E)_p = \mathrm{GL}(E_p)$. **This is not a principal bundle.** For example, it has a global section $b \mapsto \mathrm{id}_{E_b}$, and it is not trivial. Moreover,

$$\Gamma(\mathrm{GL}(E)) = \mathrm{Aut}(E)$$

Hence elements of $\mathrm{Aut}(E)$ can also be considered as sections of $\mathrm{End}(E)$.

Similarly to the above, we can consider typical fibre

$$\{\mathbb{R}^n \times \cdots \times \mathbb{R}^n \to V \mid \text{ multilinear and alternating}\}$$

which gives the bundle of differential r-forms with values in fibres of E.

vector bundle	typical fibre	transition functions	space of sections
E	V	$v \mapsto \Psi_{\beta\alpha} v$	$\Gamma(E)$
$\mathrm{End}(E)$	$\mathrm{End}(V)$	$G \mapsto \Psi_{\beta\alpha} G \Psi_{\alpha\beta}$	$\Gamma(\mathrm{End}(E))$
$\mathrm{T}^*B \otimes E$	$L(\mathbb{R}^n, V) = (\mathbb{R}^n)^* \otimes V$	$v_k \mathrm{d}x^k \mapsto (\Psi_{\beta\alpha} v_k)\frac{\partial x^k}{\partial x^{k'}} \mathrm{d}x^{k'}$	$\Omega^1_B(E)$
$\Lambda^r(\mathrm{T}^*B) \otimes E$	$\Lambda^r(\mathbb{R}^n)^* \otimes V$	$v_K \mathrm{d}x^K \mapsto (\Psi_{\beta\alpha} v_K)\frac{\partial x^K}{\partial x^{K'}} \mathrm{d}x^{K'}$	$\Omega^r_B(E)$
$\mathrm{T}^*B \otimes \mathrm{End}(E)$	$(\mathbb{R}^n)^* \otimes \mathrm{End}(V)$	$G_k \mathrm{d}x^k \mapsto (\Psi_{\beta\alpha} G_k \Psi_{\alpha\beta})\frac{\partial x^k}{\partial x^{k'}} \mathrm{d}x^{k'}$	$\Omega^1_B(\mathrm{End}(E))$
$\Lambda^r\mathrm{T}^*B \otimes \mathrm{End}(E)$	$\Lambda^r(\mathbb{R}^n)^* \otimes \mathrm{End}(V)$		$\Omega^r_B(\mathrm{End}(E))$

where $\frac{\partial x^K}{\partial x^{K'}}$ is the appropriate change of coordinates for 1-forms. In particular, from eq. (2), we find that if $A, \widetilde{A}$ are 1-forms, then

$$A - \widetilde{A} = \Omega^1_B(\mathrm{End}(E))$$

and so we can think of the space of all connections on E as an affine space with underlying vector space $\Omega^1_B(\mathrm{End}(E))$.

Definition 6.7 (coariant derivative)

A *covariant derivative* on a real vector bundle $E \to B$ is an R-linear map $\mathrm{grad}^E : \Gamma(E) \to \Omega^1_B(E)$, with the Leibniz rule

$$\mathrm{grad}^E(fs) = \mathrm{d}f \otimes s + f\mathrm{grad}^E s$$

for $f \in C^\infty(B)$ and $s \in \Gamma(E)$.

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Example 6.8

Let A be a connection on E, given in local trivialisations by $A = (A^i_j)$, $A^i_j \in \Omega^1(U)$. Define

$$(d_A s)|_U = (\mathrm{d}s + As)|_U = (\mathrm{d}s^i + A^i_j s^j)^m_{i=1}$$

[2] No summation

where $s = (s^i)$. This is the *covariant derivative associated to the connection A*.

In parricular, if U is a coordinate neighbourhood, then

$$(\mathrm{d}_A s)^i = \left(\frac{\partial s^i}{\partial x^k} + \Gamma^i_{jk}(x) s^j \right) \mathrm{d}x^k$$

in terms of local coordinates x^k.

We need to check that this is defined independent of a choice of local trivialisation. Suppose (Φ', U') is another trivialisation, Let the transition function be $\psi : U \cap U' \to \mathrm{GL}(V)$ from Φ' to Φ. Then

$$\begin{aligned} s &= \psi s' \\ A &= \psi A' \psi^{-1} - (\mathrm{d}\psi)\psi^{-1} \end{aligned}$$

Substituting, locally

$$\begin{aligned} \mathrm{d}_A s &= \mathrm{d}s + As \\ &= \mathrm{d}(\psi s) + (\psi A' \psi^{-1} - (\mathrm{d}\psi)\psi^{-1})\psi s' \\ &= \psi \mathrm{d}s' + (\mathrm{d}\psi)s' + \psi A' S' - (\mathrm{d}\psi)s' \\ &= \psi(\mathrm{d}s' + A's') \\ &= \psi(\mathrm{d}_{A'} s') \end{aligned}$$

and so we have the correct transformation law for $\mathrm{d}_A s \in \Omega^1_B(E)$.

Suppose E has an inner product $\langle \cdot, \cdot \rangle$ on fibres, then a connection A on the real (resp. complex) vector bundle $E \to B$ is *orthogonal* (resp. *unitary*), if for any $s_1, s_2 \in \Gamma(E)$, then

$$\mathrm{d} \langle s_1, s_2 \rangle = \langle \mathrm{d}_A s_1, s_2 \rangle + \langle s_1, \mathrm{d}_A s_2 \rangle$$

where the right hand side is a one-form, and we take inner product on the E-component.

Theorem 6.9. Every covariant derivative grad^E on $E \to B$ is of the form $\mathrm{grad}^E = \mathrm{d}_A$ for some connection A.

Proof. First of all,

Claim 6.10. Every covariant derivative is a local operator. That is, for any $U \subseteq B$ open, if we have sections $s_1, s_2 \in \Gamma(E)$ with $s_1|_U = s_2|_U$, then

$$(\mathrm{grad}^E s_1)|_U = (\mathrm{grad}^E s_2)|_U$$

Proof. For all $b \in U$, choose $U_0 \subseteq U$, $b \in U_0$ with $U_0 \subseteq \overline{U_0} \subseteq U$. In particular, we have a smooth function α, with

$$\alpha = \begin{cases} 1 & \text{on } U_0 \\ 0 & \text{outside } U \end{cases}$$

and $0 \le \alpha \le 1$. With this, $\alpha(s_1 - s_2) = 0$. By linearity,

$$0 = \mathrm{grad}^E(\alpha(s_1 - s_2)) = \mathrm{d}\alpha \otimes (s_1 - s_2) + \alpha \mathrm{grad}^E(s_1 - s_2)$$

Since $s_1 - s_2$ vanishes at b, and $\alpha = 1$ at b, $\mathrm{grad}^E s_1 = \mathrm{grad}^E s_2$ at b. □

Therefore, it suffices to work in an arbitrary trivialisation over a coordinate neighbourhood U say. In this,

$$s = (s^1, \ldots, s^m) = s^i e_i$$

where $e_1, \ldots, e_m$ is the standard basis of $\mathbb{R}^m$, which defines local sections over U. Each $s^i : U \to \mathbb{R}$ is smooth. We define

$$\Gamma^i_{jk} = \left(\left(\mathrm{grad}^E e_j \right) \cdot \frac{\partial}{\partial x^k} \right)^i$$

which is a function $U \to \mathbb{R}$. Using this, we can compute

$$\begin{aligned}\mathrm{grad}^E s &= \mathrm{grad}^E(s^i e_i) \\ &= \left(\mathrm{d}s^i + s^j \Gamma^i_{jk} \mathrm{d}x^k\right) \otimes e_i \\ &= \mathrm{d}_A s\end{aligned}$$

The previous example shows that $A^i_j = \Gamma^i_{jk} \mathrm{d}x^k$ is a well defined connection, i.e. it has the correct transformation law. □

With this, we have obtained three different views of connections:

1. field of horizontal subspaces,
2. matrix valued 1-forms A^i_j, or scalars Γ^i_{jk},
3. a covariant derivative

Moreover, we can extend $\mathrm{d}_A : \Omega^r_B(E) \to \Omega^{r+1}_B(E)$, by requiring

$$\mathrm{d}_A(\sigma \wedge \omega) = (\mathrm{d}_A \sigma) \wedge \omega + (-1)^{\deg(\sigma)} \sigma \wedge \mathrm{d}\omega$$

In local trivialisations,

$$\begin{aligned}\mathrm{d}_A(s_I \mathrm{d}x^I) &= (\mathrm{d}_A s_I) \wedge \mathrm{d}x^I \\ &= \mathrm{d}\left(s_I \mathrm{d}x^I\right) + A \wedge (s_I \mathrm{d}x^I)\end{aligned}$$

Furthermore, we can extend $\mathrm{d}_A : \Omega^r_B(\mathrm{End}(E)) \to \Omega^{r+1}_B(\mathrm{End}(E))$, via

$$(\mathrm{d}_A C)s = \mathrm{d}_A(Cs) - C(\mathrm{d}_A s)$$

for $C \in \Gamma(\mathrm{End}(E)), s \in \Gamma(E)$. More generally,

$$(\mathrm{d}_A \mu) \wedge \sigma = \mathrm{d}_A(\mu \wedge \sigma) - (-1)^{\deg(\mu)} \mu \wedge \mathrm{d}_A \sigma$$

for any $\mu \in \Omega^p_B(\mathrm{End}(E))$, $q \in \Omega^q_B(E)$. Using this,

$$\mathrm{d}_A(\mu_1 \wedge \mu_2) = (\mathrm{d}_A \mu_1) \wedge \mu_2 + (-1)^{\deg(\mu_1)} \mu_1 \wedge (\mathrm{d}_A \mu_2)$$

Example 6.11

For $\mu \in \Omega^2_B(\mathrm{End}(E))$, the above implies that locally,

$$\mathrm{d}_A \mu = \mathrm{d}\mu + A \wedge \mu - \mu \wedge A$$

6.1 Curvature

Repeated applying the covariant derivative, we have

$$\Gamma(E) = \Omega^0_B(E) \xrightarrow{\mathrm{d}_A} \Omega^1_B(E) \longrightarrow \cdots \longrightarrow \Omega^{n-1}_B(E) \xrightarrow{\mathrm{d}_A} \Omega^n_B(E(\longrightarrow 0$$

For $s \in \Gamma(E)$, locally,

$$\begin{aligned}\mathrm{d}_A(\mathrm{d}_A s) &= \mathrm{d}(\mathrm{d}s + As) + A \wedge (\mathrm{d}s + As) \\ &= \mathrm{d}A \wedge s - A \wedge \mathrm{d}s + A \wedge \mathrm{d}s + A \wedge A \wedge s \\ &= (\mathrm{d}A + A \wedge A) \wedge s\end{aligned}$$

That is, it is the wedge of s with a two-form. Moreover, for any smooth function,

$$\mathrm{d}_A(\mathrm{d}_A(fs)) = f \mathrm{d}_A(\mathrm{d}_A s)$$

Hence it is $C^\infty(B)$-linear. Note that

$$(A \wedge A)^i_j = \Gamma^i_{pk} \mathrm{d}x^k \wedge \Gamma^p_{j\ell} \mathrm{d}x^\ell = \Gamma^i_{pk} \Gamma^p_{jl} \mathrm{d}x^k \wedge \mathrm{d}x^\ell$$

and so it need not vanish.

Definition 6.12 (curvature)
The form
$$F(A) = \mathrm{d}A + A \wedge A \in \Omega^2_B(\mathrm{End}(E))$$
is called the *curvature* of A.

Note that the curvature is independent of local trivialisations as the As are.

In coordinates,
$$F(A) = F(A)^i_{j,k\ell} \mathrm{d}x^k \wedge \mathrm{d}x^\ell$$
In particular, $F(A)^i_{j,k\ell} = -F(A)^i_{j,\ell,k}$, and the components are expressed in terms of Γ^i_{jk} and $\frac{\partial \Gamma^i_{jk}}{\partial x^\ell}$. Lecture 18

Definition 6.13 (flat)
A connection A is *flat* if $F(A) = 0$. A *flat vector bundle* $E \to B$ is a vector bundle with a choice of a flat connection.

Example 6.14
Let $E = B \times \mathbb{R}^m$, and we can choose
$$\mathrm{d}_A = \mathrm{d} : \mathbb{C}^\infty(B, \mathbb{R}^m) \to \Omega^1(B) \otimes \mathbb{R}^m$$
This connection is called the *trivial product connection*. The converse is only true locally (Examples Sheet 3 Q6).

Definition 6.15 (covariantly constant)
A section $s \in \Gamma(E)$ is *(covariantly) constant* with respect to a connection A, if $\mathrm{d}_A s = 0$.

Theorem 6.16 (second Bianchi identity). For any connection A on $E \to B$,
$$\mathrm{d}_A(F(A)) = 0$$

Proof. Let $s \in \Gamma(E)$, then
$$\mathrm{d}_A(F(A)s) = \mathrm{d}_A(F(A))s + F(A) \wedge \mathrm{d}_A s$$
On the other hand,
$$\mathrm{d}_A(F(A)s) = \mathrm{d}_A(\mathrm{d}_A\mathrm{d}_A s) = (\mathrm{d}_A\mathrm{d}_A)\mathrm{d}_A s = F(A) \wedge \mathrm{d}_A s$$
Hence $\mathrm{d}_A(F(A))s = 0$ for any section s, which is true if and only if $\mathrm{d}_A(F(A)) = 0$. □

7 Riemannian geometry

Definition 7.1 (Riemannian metric, Riemannian manifold)
A *Riemannian metric* g on M is a field of positive definition symmetric bilinear forms
$$g_p : \mathrm{T}_pM \times \mathrm{T}_pM \to \mathbb{R}$$
which is smooth in p. A *Riemannian manifold* is a pair (M, g) of a manifold M with a Riemannian metric g on M.

Equivalently, we can define it as a section of $S^2 T^*M$ which is positve definite on each fibre. In terms of vector fields, let X, Y be vector fields on M, then $g(X, Y) : M \to \mathbb{R}$ is smooth. On each coordinate neighbourhood U with coordinates x^i, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \in C^\infty(U)$$

Note symmetry becomes $g_{ij} = g_{ji}$. In local coordinates, we can write

$$g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

Formally,

$$\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{2}\left(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i\right)$$

Example 7.2

If $r = r(u, v)$ is a parametrisation for a surface in $\mathbb{R}^3$, then we have the first fundamental form

$$E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$$

where $E = g_{11}, F = g_{12} = g_{21}, G = g_{22}$.

Theorem 7.3. Every manifold M admits a Riemannian metric.

Proof. Every vector bundle admits an inner product, using a partition of unity. Apply the result to TM. □

If $F : M \to N$ is a smooth map of manifolds, g a Riemannian metric on N, we can define

$$F^*g(v, w) = g(\mathrm{d}F(v), \mathrm{d}F(w))$$

which is a symmetric bilinear form, which is nonnegative definite. If F is an immersion then F^*g is a Riemannian metric. For example, if $M \subseteq N$ is a submanifold, we have a metric given by restriction.

Definition 7.4 (connection)

A *connection* on a manifold M is a connection on $TM \to M$.

Recall local coordinates x^i on $U \subseteq M$ gives us a trivialisation of TM, with

$$TM|_U = TU \cong U \times \operatorname{span}\left\{\frac{\partial}{\partial x_i}\right\}$$

The transition functions are determined by Jacobian matrices

$$\psi^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}$$

Note we use the convention $\psi^{-1} = (\psi^{i'}_i)$. The coefficients Γ^i_{jk} of a connection on M are called *Christoffel symbols*. The transformation law for the connection 1-forms A^i_j gives

$$\Gamma^i_{jk} = \Gamma^{i'}_{j'k'}\psi^i_{i'}\psi^{j'}_j\frac{\partial x^{k'}}{\partial x^k} + \psi^i_{i'}\frac{\partial \psi^{i'}_j}{\partial x^k}$$

Thus for a connection on a manifold,

$$\Gamma^i_{jk} = \Gamma^{i'}_{j'k'}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^{k'}}{\partial x^k} + \frac{\partial x^i}{\partial x^{i'}}\frac{\partial^2 x^{i'}}{\partial x^j \partial x^k}$$

Now if

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{kj}$$

then the $\widetilde{\Gamma}$ have the correct transformation law, and so they give us a well defined connection.

Definition 7.5 (torsion)
Define the *torsion* of the connection as

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} \in \Omega^1_M(\mathrm{End}(\mathrm{T}M))$$

Formally, in local coordinates we should write

$$T^i_{jk}\mathrm{d}x^k$$

which is a well defined 1-form. If $X, Y \in V(M)$, then consider

$$T(X,Y) = T^i_{jk}X^jY^k\frac{\partial}{\partial x_i} \in V(M)$$

By construction, $T(X,Y) = -T(Y,X)$. Thus, we can also write

$$T \in \Omega^2_M(\mathrm{T}M)$$

Definition 7.6 (symmetric)
A connection A on M is *symmetric*, or *torsion-free* if $T = 0$. In local coordinates,

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

Denote the covariant derivative on M by $D : V(M) = \Omega^0_M(\mathbb{R}M) \to \Omega^1_M(\mathrm{T}M)$. If $\alpha \in \Omega^1(\mathrm{T}M), X \in V(M)$, then $\alpha(X) \in \Omega^0(\mathrm{T}M) = V(M)$. We will write

$$D_XY = (DY)(X)$$

and so we have a map $D_X : V(M) \to V(M)$. In local coordinates,

$$(D_XY)^i\frac{\partial}{\partial x_i} = \left(X^j\frac{\partial Y^i}{\partial x^j} - \Gamma^i_{jk}Y^jX^k\right)\frac{\partial}{\partial x_i}$$

Proposition 7.7. A connection D is symmetric if and only if

$$D_XY - D_YX = [X,Y]$$

for all vector fields X, Y.

Proof. Stare at the above line. □

Theorem 7.8 (Levi-Civita connection)**.** On each Riemannian manifold (M, g), there exists a unique connection D such that

1. D is orthogonal, that is for all $X, Y, Z \in V(M)$,

$$Zg(X,Y) = g(D_ZX,Y) + g(X,D_ZY)$$

2. D is symmetric.

We call D the *Levi-Civita connection of g*.

Lecture 19

Note that for a vector field X, we have a derivation D_X, which is determined by the following properties:

(a) $D_X Y$ is linear in Y,

(b) $D_X(hY) = (Xh)Y + hD_X Y$,

(c) $D_{fX}Y = fD_X Y$,

Proof. **Step 1: Uniqueness.** We will show that the Γ^i_{jk} are uniquely determined in each trivialisation. First,

$$D\frac{\partial}{\partial x_i} = \Gamma^p_{ik}\frac{\partial}{\partial x^p} \otimes \mathrm{d}x^k$$

We will write $\partial_i := \frac{\partial}{\partial x^i}$. Using (i), set $X = \partial_i, Y = \partial_j, Z = \partial_k$, we get

$$\partial_k g_{ij} = \Gamma^p_{ik} g_{pj} + \Gamma^p_{jk} g_{ip} \tag{i}$$

We can apply a cyclic permutation of i, j, k to get

$$\partial_j g_{ki} = \Gamma^p_{kj} g_{pi} + \Gamma^p_{ij} g_{kp} \tag{ii}$$

$$\partial_j g_{jk} = \Gamma^p_{ji} g_{pk} + \Gamma^p_{ki} g_{jp} \tag{iii}$$

Let $(g^{iq}) = (g_{iq})^{-1}$ be the inverse matrix. Then for example,

$$\Gamma^p_{jk} g_{pq} g^{iq} = \Gamma^p_{jk} g_{pq} \delta^i_p = \Gamma^i_{jk}$$

Now consider (i) + (ii) − (iii). We get (using the fact that g and Γ) are symmetric,

$$\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk} = 2\Gamma^p_{jk} g_{pi}$$

and so

$$\Gamma^i_{jk} = \frac{1}{2} g^{iq}\left(\frac{\partial g_{qj}}{\partial x^k} + \frac{\partial g_{qk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^q}\right) \tag{3}$$

Hence there is at most once choice of the Christoffel symbols, given by eq. (3). We can also write this in a coordinate-free manner, as

$$g(D_X Y, Z) = \frac{1}{2}\left(Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y])\right) \tag{†}$$

Step 2: Existence. We will check properties (a), (b), (c) as above for D_X, which is determined by (†). (a) is clear.

For (c), let $f \in C^\infty(M)$, then recall $[fX, Z] = (fX)Z - Z(fX) = f[X,Z] - (Zf)X$. With this,

$$\begin{aligned} g(D_{fX}Y, Z) &= \frac{1}{2}\big(fXg(Y,Z) + Y(fg(Z,X)) - Z(fg(X,Y)) - fg(Y,[X,Z]) + (Zf)g(X,Y) \\ &\quad - fg(Z,[Y,X]) + (Yf)g(Z,X) + fg(X,[Z,Y])\big) \\ &= fg(D_X Y, Z) \\ &= g(fD_X Y, Z) \end{aligned}$$

For (b), for $h \in C^\infty(M)$,

$$\begin{aligned} g(D_X(hY), Z) &= \frac{1}{2}\big(X(hg(Y,Z)) + (hY)g(Z,X) - Z(hg(X,Y)) - hg(Y,[X,Z]) - g(Z,[hY,X]) + g(X,[Z,hY])\big) \\ &= \frac{1}{2}\big((Xh)g(Y,Z) + h(Xg(Y,Z) + Yg(X,X) - Zg(X,Y)) - (Zh)g(X,Y) \\ &\quad - h(g(Y,[X,Z]) + g(Z,[X,Y] - g(X,[Z,Y]))) + (Xh)g(Z,Y) + (Zh)h(X,Y)\big) \\ &= (Xh)g(Y,Z) + hg(D_X Y, Z) \end{aligned}$$

Hence we have that

$$g(D_X(hY), Z) = g((Xh)Y + hD_X Y, Z)$$

and so

$$D_X(hY) = (Xh)Y + hD_X Y$$

Hence D is a well defined covariant derivative, and so the Levi-Civita connection exists. ☐

7.1 Geodesics

Let $\gamma = \gamma(t) : I \to M$ be a smooth curve, $I \subseteq R$ an interval. Let $E \to M$ be a vector bundle, with typical fibre V and connection $A = (\Gamma^i_{jk})$.

Definition 7.9 (lift)
A curve $\gamma_E : I \to E$ is a *lift* of γ if $\pi \circ \gamma_E = \gamma$.

If $\gamma(t) = (x^k(t))$ in local coordinates, then

$$\gamma_E(t) = (x^k(t), a^i(t)) \in U \times V = \pi^{-1}(U)$$

Definition 7.10 (horizontal lift)
A lift γ_E is *horizontal* if

$$\dot{\gamma}_E(t) \in S_{\gamma_E(t)}$$

for all t. That is, it is in the horizontal subspace of $\mathrm{T}_{\gamma_E(t)}E$ with respect to A.

Equivalently,

$$\theta^i(\dot{\gamma}_E(t)) = 0 \text{ for } i = 1, \dots, m = \mathrm{rank}(E)$$

Moreover,

$$\theta^i = \mathrm{d}a^i + \Gamma^i_{jk} a^j \mathrm{d}x^k$$

and so

$$\left(\mathrm{d}a^i + \Gamma^i_{jk} a^j \mathrm{d}x^k\right)\left(\dot{x}^\ell \frac{\partial}{\partial x^\ell} + \dot{a}^q \frac{\partial}{\partial a^q}\right) = 0$$

Equivalently,

$$\dot{a}^i + \Gamma^i_{jk} a^j \dot{x}^k = 0$$

This gives a system of linear ODEs for $a^1, \dots, a^m$. From ODE theory, this exists a unique solution on any interval $I \subseteq R$, given initial condition $a^i(0)$. Hence horizontal lift always exists, and is unique, once we fix the initial condition $\gamma_E(0) \in E_{\gamma(0)}$.

Lecture 20

If $E = \mathrm{T}M$, then γ has a *canonical lift* given by $a^j(t) = \dot{x}^j(t)$.

Definition 7.11 (geodesic)
Let (M, g) be a Riemannian manifold, a curve $\gamma : I \to M$ is a *geodesic* if the canonical lift of γ is horizontal with respect to the Levi-Civita connection.

In a local coordinate trivialisation, we have that

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

This is a non-linear second order ODE for $x(t) = (x^k(t))$, and so there exists $\varepsilon > 0$, such that there exists a unique solution for $x(t)$ with $|t| < \varepsilon$, given initial conditions $x(0) = p \in M$, $\dot{x}(0) = a \in \mathrm{T}_pM$. Denote this geodesic as

$$\gamma_p(t, a)$$

From standard ODE theory, the solutions are smooth with respect to the initial conditions, and so γ is a smooth function in $(p, q) \in \mathrm{T}M$.

Proposition 7.12. Given any smooth curve γ on M, with $\dot{\gamma}(0) \neq 0$. Then there exists an open neighbourhood U of $\gamma(0)$, and a vector field X on U, such that X extends $\dot{\gamma}$. More precisely,

$$X(\gamma(t)) = \dot{\gamma}(t)$$

Moreover, for any such extensions Y, Z of $\dot\gamma$. For any connection $\widehat{D}$ on M, we have

$$\widehat{D}_Y Y|_{\gamma(t)} = \widehat{D}_Y Z|_{\gamma(t)} \widehat{D}_Z Z|_{\gamma(t)}$$

Then the expression

$$\widehat{D}_{\dot\gamma}\dot\gamma|_{\gamma(t)} = \widehat{D}_Y Y|_{\gamma(t)}$$

is well defined.

Note that in local coordinates, if $\gamma(t) = (x^k(t))$, then

$$\widehat{D}_{\dot\gamma}\dot\gamma = \ddot{x}^i + \widehat{\Gamma}^i_{jk}\dot{x}^j\dot{x}^k$$

Corollary 7.13. γ is a geodesic curve if and only if $D_{\dot\gamma}\dot\gamma = 0$, where D is the Levi-Civita connection.

Proof of proposition 7.12. Since $\dot\gamma(0) \neq 0$, without loss of generality $\dot{x}^1(0) \neq 0$. So there exists a smooth local inverse $t = t(x^1)$ for $\left|x^1\right| < \delta$ where $\delta > 0$. With this, we get that

$$x^i = x^i(x^1) \text{ for } i = 2, \dots, m$$

Set $X(x^1, x^2, \dots, x^n) = \dot\gamma(t(x^1))$. Recall that

$$\widehat{D}_Z Y = \left(Z^\ell \partial_\ell Y^i + \widehat{\Gamma}^i_{jk} Y^j Z^k\right)\partial_i$$

Let $p = \gamma(t_0) = (x^i(t_0))$, $Y(p) = Z(p) = \gamma(t_0)$. Clearly the second term only depends on the point p. For the first term,

$$(Z^\ell \partial_\ell Y^i)|_p = (\dot{x}^\ell \partial_\ell Y^i)|_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} Y^i(\gamma(t)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \dot{x}^i(t) = \ddot{x}^i(t_0)$$

With this,

$$(\widehat{D}_Z Y)(p) = (\ddot{x}(t_0) + \widehat{\Gamma}^i_{jk}\dot{x}^j(t_0)\dot{x}^k(t_0))\partial_i = \widehat{D}_\gamma \gamma|_{\gamma(t_0)}$$

□

Proposition 7.14. Suppose γ is a geodesic, then $|\dot\gamma(t)|_g = \text{const}$ for all t (assuming the domain of γ is connected).

Proof. For any $p = \gamma(t)$, there exists an extension X of $\dot\gamma$ in a neighbourhood U of p. We can write

$$\begin{aligned} \dot\gamma \cdot g(\dot\gamma, \dot\gamma) &= X \cdot g(\dot\gamma, \dot\gamma) \\ &= g(D_{\dot\gamma}\dot\gamma, \dot\gamma) + g(\dot\gamma, D_{\dot\gamma}\dot\gamma) \\ &= 0 \end{aligned}$$

as γ is a geodesic. But the first term is

$$x^i \frac{\partial}{\partial x^i}(g(\dot\gamma, \dot\gamma)) = \frac{\mathrm{d}}{\mathrm{d}t}|\dot\gamma(t)|_g^2$$

by the chain rule. Hence $|\dot\gamma|_g$ is constant. □

Example 7.15

Consider $\mathbb{R}^n$ with the standard Euclidean metric $g_{ij} = \delta_{ij}$. This means that $\Gamma^i_{jk} = 0$, and so the geodesic equation becomes

$$\ddot{x}^i \equiv 0$$

and so $\dot{x}^i$ is constant. That is, $\gamma_p(t, a) = p + at$ is a straight line.

Example 7.16

If we consider S^n with the round metric, given by the embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. For $p \in S^n$ a unit vector in $\mathbb{R}^{n+1}$, then we have a natural identification

$$\mathrm{T}_p S^n = \mathrm{span}\{p\}^\perp$$

Choose $a \in \mathrm{T}_p S^n$. Then consider the plane

$$\mathcal{P} = \mathrm{span}\{p, a\}$$

Consider reflection T in the plane $\mathcal{P}$. This induces a diffeomorphism on S^n and preserves the metric (i.e. it is an isometry). Then T preserves the Γ^i_{jk}, and by construction it fixes p and $a \in \mathrm{T}_p M$. Thus it preserves all of the data defining $\gamma_p(t, a)$. By uniqueness, we must have that $T(\gamma_p(t, a)) = \gamma_p(t, a)$. Hence $\gamma_p(t, a)$ must lie in $\mathcal{P} \cap S^n$. That is, it is a great circle on S^n.

Example 7.17

Now consider geodesics on parametrised surfaces $\Sigma \subseteq \mathbb{R}^3$. Suppose we have a regular parametrisation $r(u, v)$. Then g is the first fundamental form, with coefficients E, F, G. Then Γ^i_{jk} can be computed from the Gauss-Weingarten formulae. See examples sheet 3 question 10.

Let (M, g) be a Riemannian manifold, and fix $p \in M$. Consider $\gamma(t, a) = \gamma_p(t, a)$. Let $\lambda in \mathbb{R}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(\lambda t, a) = \lambda \dot{\gamma}(\lambda t, a)$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\gamma(\lambda t, a) = \lambda^2 \ddot{\gamma}(\lambda t, a)$$

From this, we see that $\gamma(\lambda t, a)$ is also a solution to the geodesic equations. With this, we have that

$$\gamma(\lambda t, a) = \gamma(t, \lambda a)$$

Remark 7.18. For all $a \in \mathrm{T}_p M$, we can find $\varepsilon = \varepsilon_a > 0$ such that for any $|a| < \varepsilon_a$,

$$\gamma(s, a) = \gamma(1, sa)$$

But the map $a \mapsto \varepsilon_a$ can be chosen to be continuous, and so by compactness of the unit sphere in $\mathbb{R}^n$, there exists $\varepsilon > 0$ such that for all $|a|_g < \varepsilon$, $\gamma(1, a)$ is defined.

Lecture 21

Definition 7.19 (exponential map)

Let (M, g) be a Riemannian manifold. The the *exponential map* at p is the map

$$\exp_p : \mathrm{T}_p M \to M$$
$$\exp_p(a) = \gamma_p(1, a)$$

Note that $\exp_p(a)$ is well defined for (and potentially *only for*) $a \in T_p M$ with $|a|_g < \varepsilon$. In this case, $\exp_p$ is smooth by standard ODE theory.

Proposition 7.20.

$$(\mathrm{d}\exp_p)_0 = \mathrm{id}_{\mathrm{T}_pM}$$

Note here we identify $\mathrm{T}_0\mathrm{T}_pM$ with T_pM. Intuitively, consider the Taylor expansion.

Proof. Clearly $\exp_p(0) = p$. When $|a|_g < \varepsilon$, $\exp(a)$ is well defined, and we have that

$$\gamma_p(t, a) = \gamma_p(1, ta)$$

for $|t| \leq 1$. Then

$$\begin{aligned}(\mathrm{d}\exp_p)_0(a) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_p(ta) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma_p(1, ta) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma_p(t, a) \\ &= \dot{\gamma}_p(0, a) \\ &= a\end{aligned}$$

□

Corollary 7.21. For some $r_0 > 0$, the exponential map

$$\exp_p : B_g(0, r_0) \to U$$

is a diffeomorphism onto its image U, a neighbourhood of p. In this case, $B_g(0, r_0)$ is the radius r_0 open ball with respect to the Riemannian metric g.

Proof. Inverse function theorem. □

Using this, $(\exp_p)^{-1}$ defines a chart around p. The respeective local coordinates are called the *geodesic (normal) coordinates*. In these local coordinates,

$$\exp_p^{-1}(\gamma_p(t, a)) = ta \tag{4}$$

and so the geodesics through p are represented by straight lines through 0. We call these *radial geodesics*.

On $B_g(0, \varepsilon)$, we have polar coordinates, given by

$$\begin{aligned}(0, \varepsilon) \times S^{n-1} &\to B_g(0, \varepsilon) \\ (r, x) &= rx\end{aligned}$$

Using this, we obtain a local parametrisation given by

$$f(r, x) = \exp(rx)$$

which is the *geodesic polar coordinates*. For r fixed, define the *geodesic sphere*

$$\Sigma_r = f(\{r\} \times S^{n-1}) \subseteq M$$

Lemma 7.22 (Gauss). $\gamma_p(t, a)$ meets Σ_r orthogonally for every $r < \varepsilon$, $a \in \mathrm{T}_pM$. Thus, locally

$$g = \mathrm{d}r^2 + h(r, v)$$

where $h(r, v) = g|_{\Sigma_r}$.

Proof. Choose a vector field $X \in V(S^{n-1})$, where $S^[n-1]$ will be the unit sphere in T_pM with respect the inner product induced by g. Then extend X to a vector field X on $B^*_g(0,1)$[3], by choosing X to be independent of the radius.

Let $\tilde{X}(r, v) = rX(v)$. This is still defined on $B^*_g(0,1)$. Let

$$Y(f(r, v)) = (\mathrm{d}\exp_p)_{rv}\tilde{X}(r, v)$$

then Y is a vector field on a punctured neighbourhood U of p.

First, consider the vector field $\frac{\partial}{\partial r}$. In this case,

$$\frac{\partial}{\partial r} = \frac{1}{|a|_g}\dot{\gamma}_p(t, a)$$

from eq. (4). Now consider $\dot{\gamma}_p(t, a)$, for $|a|_g = 1$ varying over $S^{n-1} \subseteq T_pM$, with $|t| < \varepsilon$. This defines a vector field on U. Also,

$$\frac{\mathrm{d}}{\mathrm{d}t} g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial t}\right) = \frac{\mathrm{d}}{\mathrm{d}t} g(\dot{\gamma}, \dot{\gamma}) = 0$$

and so taking the limit $t \to 0$ we see that

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1$$

Thus, it remains to show that

$$g\left(Y, \frac{\partial}{\partial r}\right) = 0$$

First, consider

$$\begin{aligned} D_{\dot{\gamma}}Y - D_Y\dot{\gamma} &= (\mathrm{d}f)\left(D_{\frac{\partial}{\partial r}}\tilde{X} - D_{\tilde{X}}\frac{\partial}{\partial r}\right) \\ &= (\mathrm{d}f)\frac{\partial}{\partial r}\tilde{X} \\ &= (\mathrm{d}f)\frac{\tilde{X}}{r} \\ &= (\mathrm{d}f)\frac{Y}{r} \end{aligned}$$

With this,

$$\begin{aligned} \frac{\partial}{\partial r} g\left(Y, \frac{\partial}{\partial r}\right) &= g\left(D_{\dot{\gamma}}Y, \dot{\gamma}\right) + \underbrace{g(Y, D_{\dot{\gamma}}\dot{\gamma})}_{=0 \text{ as } \gamma \text{ geodesic}} \\ &= g\left(D_{\dot{\gamma}}\dot{\gamma} + \frac{Y}{r}, \dot{\gamma}\right) \\ &= \frac{1}{r}g(Y, \dot{\gamma}) \end{aligned}$$

as

$$g(D_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) = \frac{1}{2}Yg(\dot{\gamma}, \dot{\gamma}) = 0$$

If we set $G = g(Y, \frac{\partial}{\partial r})$, then

$$\frac{\mathrm{d}}{\mathrm{d}r}G = \frac{G}{r}$$

G is linear in r, and taking the limit as $r \to 0$, we get

$$\lim_{r \downarrow 0} \frac{\mathrm{d}}{\mathrm{d}r}G = \lim_{r \downarrow 0} g\left(X, \frac{\partial}{\partial r}\right) = 0$$

since

$$\lim_{v \to 0}(\mathrm{d}\exp_p)_v = \mathrm{id}$$

which is an isometry. □

[3] Punctured unit ball

7.2 Curvature

Definition 7.23 (Riemann cuvrature)
The *(full) Riemann curvature* of a metric g on a manifold M is $R = R(g)$, which is the curvature of the Levi-Civita connection on M.

In particular,

$$R \in \Omega^2_M(\mathrm{End}(\mathrm{T}M))$$

we take the sign convention that

$$R(g) = -D \circ D$$

Then in local coordinates

$$R = \frac{1}{2} R^i_{j,k\ell} \mathrm{d}x^k \wedge \mathrm{d}x^\ell$$

We call $R^i_{j,k\ell}$ the *Riemann cuvrature tensor*.

For example, if $X, Y \in V(M)$, we obtain $R(X, Y) \in \Gamma(\mathrm{End}(\mathrm{T}M))$. Locally, if $X = X^k \partial_k$, $Y = Y^\ell \partial_\ell$, then

$$R(X, Y) = \left(R^i_{j,k\ell} X^k Y^\ell\right)^i_j$$

Set

$$R_{k\ell} = R(\partial_k, \partial_\ell) \in \mathrm{End}(\mathrm{T}_p M)$$

Then

$$R(X, Y) = X^k Y^\ell R_{k\ell}$$

In a local coordinate and trivialising neighbourhood, $D = \mathrm{d} + A$, where $A = A_k \mathrm{d}x^k$. Write

$$D_k = D_{\partial k} = \partial k + A_k$$

Then for any vector field $Z \in V(M)$,

$$(-D \circ D)Z = RZ = (R_{k\ell} \mathrm{d}x^k \wedge \mathrm{d}x^\ell)Z$$

But

$$R_{k\ell} Z = D_\ell D_k Z - D_k D_\ell Z$$

Thus,

$$R_{k\ell} = -[D_k, D_\ell]$$

With this,

$$R^i_{j,k\ell} = \left((D_\ell D_k - D_k D_\ell)\partial_j\right)^i$$

Write $D_X = X^k D_k$, then

$$\begin{aligned} -[D_X, D_Y] &= -[X^k D_k, Y^\ell D_\ell] \\ &= -X^k(\partial_k Y^\ell) D_\ell - X^k Y^\ell D_k D_\ell + Y^k(\partial_k X^\ell) D_\ell + Y^k X^\ell D_\ell D_k \\ &= X^k Y^\ell R_{k\ell} - [X, Y]^\ell D_\ell \end{aligned}$$

With this, we have shown

Lemma 7.24.

$$R(X, Y) = D_{[X,Y]} - [D_X, D_Y]$$

Lecture 22

Sometimes it is convenient to consider

$$R_{ij,k\ell} = g_{iq} R^i_{j,k\ell}$$

In a coordinate free fashion,

$$(X, Y, Z, T) \mapsto g(R(X, Y)Z, T)$$

where $X, Y, Z, T \in \mathrm{T}_p M$. Thus,

$$R_{ij,k\ell} = g\left(R_{k\ell} \partial_j, \partial_i\right)$$

Proposition 7.25 (symmetries of the curvature tensor)**.** We have

(i) $R_{ij,\ell k} = -R_{ij,k\ell} = R_{ji,k\ell}$,

(ii) (*first Bianchi identity*) $R^i_{j,k\ell} + R^i_{k,\ell j} + R^i_{\ell,jk} = 0$,

(iii) $R_{ij,k\ell} = R_{k\ell,ij}$

Proof. For (i), the first equality holds by properties of 2-forms. For the second equality, consider

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x_k} &= g(D_k\partial_i, \partial_j) + g(\partial_i, D_k\partial_j)\\ \frac{\partial^2 g_{ij}}{\partial x^\ell \partial x^k} &= g(D_\ell D_k\partial_i, \partial_j) + g(D_k\partial_i, D_\ell\partial_j) + g(D_\ell\partial_i, D_k\partial_j) + g(\partial_i, D_\ell D_k\partial_j)\end{aligned}$$

But we know that g_{ij} is smooth, and so the partial derivatives commute. Hence

$$\begin{aligned}0 &= \frac{\partial^2 g_{ij}}{\partial x^\ell \partial x^k} - \frac{\partial^2 g_{ij}}{\partial x^k \partial x^\ell}\\ &= g([D_\ell, D_k]\partial_i, \partial_j) + g(\partial_i, [D_\ell, D_k]\partial_j)\\ &= g(R_{k\ell}\partial_i, \partial_j) + g(\partial_i, R_{jk}\partial_j)\\ &= R_{ji,k\ell} + R_{ij,k\ell}\end{aligned}$$

as required.

For (ii), consider

$$R^i_{j,k\ell} + R^i_{k,\ell j} + R^i_{\ell,jk} = \left(D_\ell D_k\partial_j - D_k D_\ell\partial_j + D_j D_\ell\partial_k - D_\ell D_k\partial_k + D_k D_j\partial_\ell - D_j D_k\partial_\ell\right)^i$$

We claim that

$$(D_\ell D_k\partial_j)^i = (D_\ell D_j\partial_k)^i$$

This follows by (i). A similar computation for the other terms shows that the sum is zero. Note we also use that

$$(D_k\partial_j)^q = \Gamma^q_{jk} = \Gamma^q_{kj} = (D_j\partial_k)^q$$

For (iii), first note that using the metric, we have that

$$R_{ij,k\ell} + R_{ik,\ell j} + R_{i\ell,jk} = 0$$

See online notes for octahedron trick. □

Corollary 7.26.

$$R_{ij,k\ell} + R_{ik,\ell j} + R_{i\ell,jk} = 0$$

Corollary 7.27. At each $p \in M$, $(R_{ij,k\ell})$ defines a symmetric bilinear form on $\Lambda^2 T_pM$.

It is useful to extract from $(R^i_{j,k\ell})$ simpler objects, with less components.

Definition 7.28 (Ricci curvature)

The *Ricci curvature* of a metric g is

$$\begin{aligned}\mathrm{Ric}_p &: T_pM \times T_pM \to \mathbb{R}\\ \mathrm{Ric}_p(X, Y) &= \mathrm{tr}(v \mapsto R(X, v)Y)\end{aligned}$$

for $X, Y, v \in T_pM$.

In local coordinates, $\mathrm{Ric} = (\mathrm{Ric}_{ij})$, then

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}_{ij} X^i Y^j$$

In terms of the full Riemann curvature tensor,

$$\mathrm{Ric}_{ij} = R^q_{i,jq} = g^{pq} R_{pi,jq}$$

Using the last expression, we see that Ric_p is a symmetric bilinear form on $\mathrm{T}_p M$.

Definition 7.29 (scalar curvature)
The *scalar curvature* of a metric g is $s = \mathrm{scal}(g) \in C^\infty(M)$, it is the trace of Ric with respect to the metric g.

Explicitly, if $g_{ij} = \delta_{ij}$, then

$$s = \sum_i \mathrm{Ric}_{ii}$$

In general,

$$s = g^{ij} \mathrm{Ric}_{ij} = g^{ij} g^{pq} R_{pi,jq} = g^{jk} R^\ell_{j,k\ell}$$

Lemma 7.30 (from representation theory). Consider the "space of curvature tensors" $(R_{ij,k\ell})$, i.e. tensors satisfying the symmetries of the Riemann curvature tensor. This space decomposes when $\dim(M) \geq 5$ into irreducible representations of $\mathrm{SO}(n)$. Recall we have that at each $p \in M$, $R \in \mathrm{Sym}^2(\Lambda^2(T_p^* M))$. Apply trace as in the definition on scalar curvature, we get

$$\mathrm{Sym}^2(\Lambda^2 \mathrm{T}_p^* M) \xrightarrow{\mathrm{Tr}} \mathrm{Sym}^2(\mathrm{T}_p^* M) \xrightarrow{\mathrm{tr}} \mathbb{R}$$

where $\mathrm{Ric} = \mathrm{Tr}(R), \mathrm{scal} = \mathrm{tr}(\mathrm{Ric})$. Then

$$\mathrm{Sym}^2(\Lambda^2 \mathrm{T}_p^* M) = \ker(\mathrm{Tr}) \oplus \ker(\mathrm{tr}) \oplus \mathbb{R}$$

as irreducible representations.

Proof. Omitted. □

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Note that we have that $g, \mathrm{Ric}(g) \in \Gamma(\mathrm{Sym}^2 \mathrm{T}^* M)$, and so the equation

$$\mathrm{Ric}(g) = \lambda g$$

makes sense. If the equation holds, then

$$\mathrm{Ric}_0 = \ker(\mathrm{tr}) = 0$$

In this case, (M, g) is called an *Einstein manifold*. If $\lambda = 0$, then $\mathrm{Ric}(g) = 0$ and we call (M, g) *Ricci-flat*.

Next, we can consider the scalar curvature in low dimension.

- In dimension 2, consider an embedded surface in $\mathbb{R}^3$. The Riemannian metric corresponds to the first fundamental form. Recall

  $$s = g^{ij} g^{pq} R_{pi,jq}$$

  Then

  $$\frac{1}{2} s = \frac{R_{12,21}}{EG - F^2}$$

  If we write the second fundamental form as

  $$L du^2 + M du dv + N dv^2$$

where $L = \langle r_{uu}, n \rangle$, n is the unit normal. Recall that the Gaussian curvature

$$K = \frac{LN - M^2}{EG - F^2}$$

only depends on E, F, G and their first derivatives, by the Theorema Egregium. In fact,

$$LN - M^2 = R_{12,21}$$

and so

$$s = 2K$$

- in dimension 3, $R(g)$ is determined by $\operatorname{Ric}(g)$. That is, the map Tr is a linear isomorphism. See examples sheet 4 question 6.
- in dimension 4, consider the *Weyl curvature*

$$W(g) = \ker(\operatorname{Tr}) = W_+ \oplus W_-$$

  This can be related to examples sheet 4 question 8, for example by splitting into self-dual and anti-self-dual parts.

7.3 The Laplace-Beltrami operator

Throughout, assume (M, g) is an oriented Riemannian manifold, $\dim(M) = n$, and suppose $\varepsilon \in \Omega^n(M)$ is an orientation form.

For all $x \in M$, apply Gram-Schmidt to the $\{\frac{\partial}{\partial x_i}\}$ to obtain on any open neighbourhood of x, a local orthonormal frame field $e_1, \ldots, e_n$. We can assume this is positively oriented, that is,

$$\varepsilon(e_1, \ldots, e_n) > 0$$

Let $\omega_1, \ldots, \omega_n$ be the corresponding dual coframe field. That is, each $\omega_i \in \Omega^1(U)$, and $\omega_i(e_j) = \delta_{ij}$. In this case,

$$\omega_1 \wedge \cdots \wedge \omega_n = a(x)\varepsilon$$

where $a(x) > 0$. With respect to the inner product on the dual space, the ω_i form an orthonormal basis as well. We can extend this to a unique inner product on $\Lambda^p T_x^* M$, by setting

$$\left\{\omega_{i_1} \wedge \cdots \omega_{i_p} \mid i_1 < \cdots < i_p\right\}$$

to be an orthonormal basis. In this case,

$$\left\langle \alpha_1 \wedge \cdots \wedge \alpha_p, \beta_1 \wedge \cdots \wedge \beta_p \right\rangle = \det\left(\left\langle \alpha_i, \beta_j \right\rangle\right)$$

if α_i, β_i are one-forms.

Suppose ω_i' is another positively oriented orthonormal coframe field. Then

$$\omega_1' \wedge \cdots \wedge \omega_n' = \det(\Phi) \cdot \omega_1 \wedge \cdots \wedge \omega_n$$

where Φ is the change of basis matrix. Since Φ is orthogonal, and has positive determinant, $\det(\Phi) = 1$. In particular, the local n-forms

$$\omega_1 \wedge \cdots \wedge \omega_n$$

patch together over (M, g) to a well defined non-vanishing n-form ω_g. We call this the *volume form* of (M, g).

> **Definition 7.31** (Hodge star)
> The *Hodge star* on M is the linear map
>
> $$* : \Lambda^p \mathrm{T}_x^* M \to \Lambda^{n-p} \mathrm{T}_x^* M$$
>
> such that
>
> $$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \omega_g$$

For uniqueness, the Hodge star is defined by its action on a basis. For example

$$*(\omega_1 \wedge \cdots \wedge \omega_p) = \omega_{p+1} \wedge \cdots \wedge \omega_n$$

using orthogonality. For example, $*1 = \omega_g$ and $*\omega_g = 1$. A computation shows that

Lemma 7.32.

$$*^2 = (-1)^{p(n-p)} \operatorname{id}_{\Lambda^p T_x^* M}$$

Definition 7.33 (codifferential)
Define the *codifferential*

$$\delta = (-1)^{n(p+1)+1} * \mathrm{d} * : \Omega^p(M) \to \Omega^{p-1}(M)$$

for $p > 0$. We define δ on 0-forms to be zero.

Definition 7.34 (Laplace-Beltrami)
The *Laplace-Beltrami operator* is

$$\Delta = \delta \mathrm{d} + \mathrm{d}\delta : \Omega^p(M) \to \Omega^p(M)$$

Proposition 7.35.

$$\int_M \langle \mathrm{d}\alpha, \beta \rangle_g \, \omega_g = \int_M \langle \alpha, \delta\beta \rangle_g \, \omega_g$$

for all $\alpha \in \Omega^{p-1}(M), \beta \in \Omega^p(M)$ compactly supported.

Proof. By Stokes,

$$\int_M \mathrm{d}(\alpha \wedge *\beta) = 0$$

But

$$\mathrm{d}(\alpha \wedge *\beta) = (\mathrm{d}\alpha) \wedge *\beta + (-1)^{p-1} \alpha \wedge \mathrm{d}(*\beta)$$

Now note that

$$- * \delta\beta = (-1)^{n(p+1)} * * \mathrm{d}(*\beta) = (-1)^{n(p-1)+(n-p+1)(p-1)} \mathrm{d}(*\beta) = (-1)^{p-1} \mathrm{d}(*\beta)$$

With this, we see that

$$\mathrm{d}(\alpha \wedge *\beta) = \langle \mathrm{d}\alpha, \beta \rangle \, \omega_g - \langle \alpha, \delta\beta \rangle \, \omega_g$$

□

Remark 7.36. Define the L^2 inner product of p-forms as

$$\langle\langle \xi, \eta \rangle\rangle_{M,g} = \int_M \langle \xi, \eta \rangle_g \, \omega_g$$

and so we have that

$$\langle\langle \mathrm{d}\alpha, \beta \rangle\rangle_{M,g} = \langle\langle \alpha, \delta\beta \rangle\rangle_{M,g}$$

So δ is the formal L^2-adjoint of d.

Corollary 7.37. $\Delta^* = \Delta$ is formally self-adjoint.

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Definition 7.38 (harmonic)
Define
$$\mathcal{H}^p(M) = \ker(\Delta : \Omega^p(M) \to \Omega^p(M))$$
for the space of *harmonic p-forms*.

Remark 7.39. $*\Delta = \Delta *$, and so the Hodge star is a well defined map
$$* : \mathcal{H}^p(M) \to \mathcal{H}^{n-p}(M)$$

Proposition 7.40. Suppose M is compact. Then for $\alpha \in \Omega^p(M)$, $\Delta\alpha = 0$ if and only if $\mathrm{d}\alpha = 0$ and $\delta\alpha = 0$.

Proof. The 'if' direction is obvious. Now suppose $\Delta\alpha = 0$, then
$$0 = \langle\langle \Delta\alpha, \alpha \rangle\rangle = \langle\langle \mathrm{d}\delta\alpha + \delta\mathrm{d}\alpha, \alpha \rangle\rangle = \left\|\delta\alpha\right\|^2 + \left\|\mathrm{d}\alpha\right\|^2$$
and so $\delta\alpha = 0$ and $\mathrm{d}\alpha = 0$. □

Corollary 7.41. Let $f \in C^\infty(M)$, M compact and connected. If f is harmonic then f is constant.

Note thet the result is false if M is not compact, for example $e^x \cos(y)$ on $\mathbb{R}^2$.

Theorem 7.42 (Hodge decomposition)**.** Let M be a compact, oriented, Riemannian manifold, $0 \leq p \leq \dim(M)$, and we have that

1. $\mathcal{H}^p$ is a finite dimensional vector space,
2. we have the L^2-orthogonal decompositions

$$\begin{aligned}\Omega^p(M) &= \mathcal{H}^p(M) \oplus \Delta\Omega^p(M)\\ &= \mathcal{H}^p(M) \oplus \mathrm{d}\delta\Omega^p(M) \oplus \delta\mathrm{d}\Omega^p(M)\\ &= \mathcal{H}^p(M) \oplus \mathrm{d}\Omega^{p-1}(M) \oplus \delta\Omega^{p+1}(M)\end{aligned}$$

Corollary 7.43. Let M be as in the Hodge decomposition. Then for all $a \in \mathrm{H}^p_{\mathrm{dR}}(M)$ there exists a unique $\alpha \in \mathcal{H}^p(M)$ such that $[\alpha] = a$.

Proof. First we show uniqueness. If we have $\alpha_1, \alpha_2 \in \mathcal{H}^p$, with $\alpha_1 = \alpha_2 + \mathrm{d}\beta$. With this,
$$\left\|\mathrm{d}\beta\right\|^2 = \langle\langle \alpha_1 - \alpha_2, \mathrm{d}\beta \rangle\rangle = \langle\langle \delta\alpha_1 - \delta\alpha_2, \beta \rangle\rangle = 0$$
and so $\beta = 0$.

For existence, given a class a, it will be represented by a closed form $\widetilde{\alpha}$. Now
$$\widetilde{\alpha} = \alpha + \mathrm{d}\beta + \delta\gamma$$
by the Hodge decomposition, with α harmonic. Thus, we must have that $\mathrm{d}\delta\gamma = 0$. Now
$$0 = \langle\langle \mathrm{d}\delta\gamma, \gamma \rangle\rangle = \left\|\delta\gamma\right\|$$
and so $[\widetilde{\alpha}] = [\alpha]$. □

Corollary 7.44. We have a linear isomorphism of vector spaces

$$\begin{aligned} \mathcal{H}^p(M) &\to H^p_{\mathrm{dR}}(M) \\ \alpha &\mapsto [\alpha] \end{aligned}$$

The proof of the Hodge decomposition is quite involved. We will however make some remarks about the proof.

The main argument concerns $\Omega^p(M) = \mathcal{H}^p(M) \oplus \Delta\Omega^p(M)$. We can think of this as the existence of a solution

$$\Delta\omega = \alpha \tag{5}$$

for some fixed $\alpha \in \Omega^p$. If ω is a solution, then

$$\langle\langle \Delta\omega, \varphi \rangle\rangle = \langle\langle \alpha, \varphi \rangle\rangle$$

for all $\varphi \in \Omega^p$. Define

$$\begin{aligned} \ell : \Omega^p(M) &\to \mathbb{R} \\ \ell_\omega(\beta) &= \langle\langle \omega, \beta \rangle\rangle \end{aligned}$$

ℓ defines a bounded linear map, since by Cauchy-Schwarz

$$|\ell_\omega(\beta)| \leq ||\omega|| \left\|\beta\right\|$$

Moreover,

$$\ell_\omega(\Delta\varphi) = \langle\langle \omega, \Delta\varphi \rangle\rangle = \langle\langle \Delta\omega, \varphi \rangle\rangle = \langle\langle \alpha, \beta \rangle\rangle$$

Thus, we define a *weak solution* of eq. (5) is a bounded linear map $\ell : \Omega^p(M) \to \mathbb{R}$ with

$$\ell(\Delta\varphi) = \langle\langle \alpha, \varphi \rangle\rangle$$

for all $\varphi \in \Omega^p$. For this, we will require

Theorem 7.45 (elliptic regularity)**.** Every weak solution of eq. (5) is of the form

$$\ell(\beta) = \langle\langle \omega, \beta \rangle\rangle$$

for some $\omega \in \Omega^p$. That is, every weak solution comes from a weak solution.

Theorem 7.46 (compactness)**.** If a sequence $\alpha_n \in \Omega^p(M)$ is such that $\left\|\alpha_n\right\|, \left\|\Delta\alpha_n\right\|$ are bounded, then α_n contains a Cauchy subsequence.

The fact that the space of harmonic forms is finite dimensional follows. If not, then it would contradict compactness.

We can write

$$\Omega^p = \mathcal{H}^p \oplus (\mathcal{H}^p)^\perp$$

where $(\mathcal{H}^p)^\perp$ is a closed subspace. By self-adjointness, $\Delta\Omega^p \subseteq (\mathcal{H}^p)^\perp$. Thus, it suffices to show that eq. (5) has a weak solution when $\alpha \perp \mathcal{H}^p$.

For $\eta \in \Omega$, put

$$\ell(\Delta\eta) = \langle\langle \alpha, \eta \rangle\rangle$$

We can show that ℓ extends from $\Delta\Omega^p$ to a bounded linear functional on p-forms, using Hahn-Banach.

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