Lie Algebras and their Representations

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Lecture 1

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1 Introduction

A *Lie group* is fundamentally a group, which also a (smooth) manifold. For example, $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}$.

Example 1.1
A prototypical example of a Lie group is the circle group S^1.

Let G be a Lie group. Then the *Lie algebra* of G is the tangent space at the identity e of G. That is,

$$\mathfrak{g} = \mathrm{T}_e G$$

$\mathfrak{g}$ is a vector space, with additional structure, which we will see later.

By taking a derivative, we turn the conjugation map

$$\begin{aligned} G &\to \mathrm{Aut}(G) \\ g &\mapsto g(\cdot)g^{-1} \end{aligned}$$

into a map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

called the *adjoint*. This gives a bilinear map

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ [x, y] &= \mathrm{ad}(x)(y) \end{aligned}$$

Example 1.2
If $G = \mathrm{GL}_n(\mathbb{R})$, then we have that $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) = \mathrm{Mat}_n(\mathbb{R})$, with

$$[x, y] = xy - yx$$

What do Lie algebras tell us about the structure of the Lie group G?

- We will define the *root system* of $\mathfrak{g}$, and this then tells us about commutator relations in G (see Carter's book).
- We will define the *Weyl group* of $\mathfrak{g}$. For example, the Weyl group of $\mathfrak{gl}_n(\mathbb{C})$ is isomorphic to S_n, there is an embedding $S_n \hookrightarrow \mathrm{GL}_n(\mathbb{C})$, vie permutation matrices. Let B denote the Borel subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$, then there exists a *Bruhat decomposition*

$$G = \bigsqcup_{w \in S_n} BwB$$

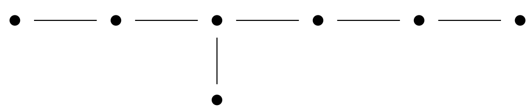
Lie algebras also give us information of the representation theory of G and $\mathfrak{g}$. For example, there exists a bijection

$$\{\text{finite dimensional } \mathbb{C}\text{-representations of } \mathrm{SL}_n(\mathbb{C})\} \leftrightarrow \{\text{finite dimensional } \mathbb{C}\text{-representations of } \mathfrak{sl}_n(\mathbb{C})\}$$

Moreover, we can describe the right hand side completely.

In addition, Lie algebras have applications in Algebraic Geometry, for example, we can use Lie algebras to build families of surfaces, of equivalenrly, algebraic curves (See book by Slodovy).

We will define the *Dynkin diagrams* of *semisimple Lie algebras*. For example,



is a Dynkin diagram of type E_7, and understanding the Dynkin diagram tells us about the singularities on the surfaces.

Moreover, Lie algebras also have applications in number theory, root systems/Weyl groups give the structure of groups over $\mathbb{Q}_p$, the p-adic integers (see paper on Moodle). Local Langlands correspondence predicts a relationship

$$\{\text{Galois theory of local fields}\} \leftrightarrow \{\text{complex Lie theory}\}$$

Finally, there any other applications, for example algebraic groups, quantum groups, theoretical physics, quantum mechanics.

2 Basic definitions and examples

Let k be a field. Most of the time, $k = \mathbb{C}$, but not always. We will sometimes point out how things can go wrong in characteristic p.

Definition 2.1 (Lie algebra)
A *Lie algebra* over k is a vector space $\mathfrak{g}$ over k, together with a bilinear pairing

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

1. $[xx] = 0$ for all $x \in \mathfrak{g}$,
2. the Jacobi identity
$$[x[yz]] + [y[zx]] + [z[xy]] = 0$$

Notation 2.2. Note that when clear, we will write $[xy] := [x, y]$.

Remark 2.3. In particular, we have antisymmetry, i.e.

$$[xy] = -[yx]$$

Definition 2.4 ((Lie) subalgebra)
A k-vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a *(Lie) subalgebra* if $\mathfrak{h}$ is closed under the Lie bracket of $\mathfrak{g}$. That is, for all $x, y \in \mathfrak{h}$, $[xy] \in \mathfrak{h}$.

Example 2.5
Let V be a finite dimensional k-vector space, then

1. Let $\mathfrak{gl}(V) = \mathrm{End}(V)$, with $[xy] = xy - yx$. If we choose a basis for V, then we can identify $\mathfrak{gl}(V)$ with $\mathrm{Mat}_n(k)$. In this case, we will write $\mathfrak{gl}_n$ or $\mathfrak{gl}_n(k)$.
2. Let
$$\mathfrak{sl}(V) = \{x \in \mathfrak{gl}(V) \mid \mathrm{tr}(x) = 0\}$$
This defines a subalgebra of $\mathfrak{gl}(V)$, called the *special linear Lie algebra*, with $\dim(\mathfrak{sl}(V)) = \dim(V)^2 - 1$. The standard basis of $\mathfrak{sl}(V)$ is given by
$$E_{i,j} \text{ for } i \neq j$$
$$E_{i,i} - E_{i+1,i+1}$$
We will often write $\mathfrak{sl}_n$ for this Lie algebra.

Lecture 2

Example 2.6 (continued) 3. Suppose $\text{char}(k) \neq 2$, and suppose V is endowed with a symmetric non-degenerate bilinear form

$$\langle \cdot, \cdot \rangle : V \times V \to k$$

Then define

$$\mathfrak{so}(V) = \{x \in \mathfrak{gl}(V) \mid \langle xv, w \rangle + \langle v, xw \rangle = 0 \text{ for all } v, w \in V\}$$

If $M \in \text{GL}(V)$ is such that $\langle v, w \rangle = v^{\mathsf{T}} M w$, so

$$\mathfrak{so}(V) = \left\{x \mid Mx + x^{\mathsf{T}} M = 0\right\}$$

We usually take

$$M = \begin{cases} \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix} & n = 2\ell \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix} & n = 2\ell + 1 \end{cases}$$

These are called the *orthogonal Lie algebra*, denoted $\mathfrak{so}_n$.

> **Remark 2.7.** In the case $n = 2$, let
>
> $$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
>
> viewed as matrices (in $\mathfrak{sl}_2(\mathbb{C})$). Indeed, this is the standard basis of $\mathfrak{sl}_2(\mathbb{C})$. Note
>
> $$[ef] = h$$
> $$[he] = 2e$$
> $$[hf] = -2f$$
>
> We'll see that (in some sense) the structure of all semisimple Lie algebras comes from $\mathfrak{sl}_2(\mathbb{C})$.

4. Again suppose $\text{char}(k) \neq 2$. Now suppose V is endowed with a non-degenerate skew-symmetric (or alternating) bilinear form $\langle \cdot, \cdot \rangle$, then

$$\mathfrak{sp}(V) = \{x \in \mathfrak{gl}(V) \mid \langle xv, w \rangle + \langle v, xw \rangle = 0\}$$

In coordinates, we take the form $\langle \cdot, \cdot \rangle$ to be the skew-symmetric form given by

$$M = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$$

where $n = 2\ell$. This is called the *symplectic Lie algebra*, denoted $\mathfrak{sp}_n$.

If we consider the Lie groups in the above as being defined by the equation

$$X^{\mathsf{T}} M X = M$$

For appropriate choices of M, we get the Lie groups SO_n, Sp_n. Differentiating this equation gives us the Lie algebras $\mathfrak{so}_n, \mathfrak{sp}_n$.

Exercise: Check that $\mathfrak{so}_n$ and $\mathfrak{sp}_n$ are Lie subalgebras of $\mathfrak{gl}_n$. It's not very hard to check this directly. On the other hand, we can also see that SO_n and Sp_n are subgroups, and so their tangent spaces are subspaces of $\mathfrak{gl}_n$, and hence their Lie algebras are subalgebras of $\mathfrak{gl}_n$.

Example 2.8 (continued) 5. Any vector space V is a Lie algebra with $[vw] = 0$ for all v, w. We call such Lie algebras *abelian*. It is named like this, since for *linear Lie algebras*, that is, any subalgebra of $\mathfrak{gl}(V)$ where V is finite dimensional, $[xy] = xy - yx = 0$ is true if and only if x and y commute.

6. $\mathfrak{b}_n$ is the *Borel algebra* of upper triangular matrices

$$\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$$

This is the Lie algebra associated to the Borel subgroup of upper triangular invertible matrices.

7. $\mathfrak{n}_n$ is the Lie algebra of strictly upper triangular matrices,

$$\begin{pmatrix} 0 & \cdots & * \\ & \ddots & \vdots \\ 0 & & 0 \end{pmatrix}$$

$\mathfrak{n}$ stands for *nilpotent*, see section 7.

3 Basic representation theory

Definition 3.1 (homomorphism, isomorphism)
A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is a *homomorphism* if

$$\varphi([xy]) = [\varphi(x), \varphi(y)]$$

for all $x, y \in \mathfrak{g}$. If φ is a linear isomorphism, we call φ an *isomorphism of Lie algebras*.

Definition 3.2 (representation)
A *representation* of $\mathfrak{g}$ is a Lie algebra homomorphism

$$\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$$

for some vector space V.

Notation 3.3. We also call V itself a *representation*, or a $\mathfrak{g}$-module. We write $\mathfrak{g} \circlearrowleft V$, and say $\mathfrak{g}$ acts on V. We will write

$$x \cdot v = xv := \varphi(x)(v)$$

The *dimension* of the representation is the dimension of V.

Example 3.4 1. Let $\dim(V) = 1$, then $\mathfrak{g} \circlearrowleft V$ via $xv = 0$. This is called the *trivial representation*.

2. $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$, then the natural inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ is called the *defining representation*.

3. Let $x \in \mathfrak{g}$, define $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_x(y) = [xy]$. The map

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}_x \end{aligned}$$

is called the *adjoint representation*.

Remark 3.5. Recall e, h, f from remark 2.7, the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$ has matrices

$$\mathrm{ad}_h = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix} \quad \mathrm{ad}_e = \begin{pmatrix} & -2 & \\ & & 1 \\ & & \end{pmatrix} \quad \mathrm{ad}_f = \begin{pmatrix} & & \\ -1 & & \\ & 2 & \end{pmatrix}$$

with respect to the basis $\{e, h, f\}$.

4. If V, W are representations of $\mathfrak{g}$, then so is their *direct sum* $V \oplus V$, via

$$x(v, w) = (xv, xw)$$

5. If V is a representation of $\mathfrak{g}$, then so is the *dual* V^*,

$$(xf)(v) = -f(xv)$$

   for all $x \in \mathfrak{g}, f \in V^*, v \in V$.

6. If V, W are representations of $\mathfrak{g}$, then so is the *homomorphisms* $\mathrm{Hom}(V, W)$, via

$$(x \cdot f)(v) = x \cdot f(v) - f(x \cdot v)$$

Definition 3.6 (equivariant, isomorphism)
If V, W are representations of $\mathfrak{g}$, then a linear map $\varphi : V \to W$ is called $\mathfrak{g}$-*equivariant* for all $x \in \mathfrak{g}, v \in V$,

$$x \cdot \varphi(v) = \varphi(x \cdot v)$$

We say V, W are *isomorphic* if there exists a $\mathfrak{g}$ equivariant isomorphism (of vector spaces) $V \to W$.

Definition 3.7 (subrepresentation, irreducible)
A *subrepresentation* $W \leq V$ is a subspace $xw \in W$ for all $x \in \mathfrak{g}, w \in W$. A non-zero representation V is *irreducible* or *simple* if the only subrepresentations of V are 0 and V.

<u>Exercise:</u> The trivial representation is irreducible, and so are the defining and the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$.

Lecture 3

Definition 3.8 (completely reducible, semisimple)
A representation V is *completely reducible*, or *semisimple* if it decomposes as the direct sum of irreducible representations.

<u>Exercise:</u> A representation V is completely reducible if and only if for every subrepresentation W of V, there exists another subrepresentation W' such that

$$V = W \oplus W'$$

Example 3.9
If V is a representation, $W \leq V$ a subrepresentation, then the *quotient* V/W is a representation of $\mathfrak{g}$, via

$$x(v + W) = (xv) + W$$

Remark 3.10. The isomorphism theorems hold for quotient representations as well.

4 Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

We'll see later on that the representation theory of general semisimple Lie algebras is "built up" from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. Moreover, it serves as a useful example to motivate various definitions.

Recall

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \,\middle|\, a, b, c \in \mathbb{C} \right\}$$

Example 4.1

We have

$$\mathfrak{b}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$$

which is the Borel subalgebra of 2×2 matrices. Let $V = \text{span}\left\{ v_! = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, and the defining representation of $\mathfrak{b}_2$. This is not completely reducible. To see this, set

$$V_1 = \text{span}\{v_1\}$$

and this does not have a complement.

Definition 4.2 (faithful)

A representation V of $\mathfrak{g}$ is *faithful* if the map

$$\mathfrak{g} \to \mathfrak{gl}(V)$$

is injective.

From now on, all Lie algebras and their representations will be over C, unless stated otherwise.

Let V be a representation of $\mathfrak{sl}_2(\mathbb{C})$. Recall the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{C})$. We know three representations of $\mathfrak{sl}_2(\mathbb{C})$ already.

dimension	name	action of h
1	trivial	0
2	defining	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
3	adjoint	$\begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}$

Definition 4.3 (weight space)

For $\lambda \in \mathbb{C}$, the λ-*weight space* of V is

$$V_\lambda = \{v \in V \mid hv = \lambda v\}$$

is the λ-eigenspace of h.

The following are vector space sums, not decompositions into subrepresentations.

- For the trivial representation,
$$V = V_0$$
- For the defining representation,
$$V = V_1 \oplus V_{-1}$$
- For the adjoint representation,
$$V = V_2 \oplus V_0 \oplus V_{-2}$$
where $V_2 = \langle e \rangle, V_0 = \langle h \rangle, V_{-2} = \langle f \rangle$.

For $v \in V_\lambda$,
$$h(ev) = (he)v = ([he] + eh)v = 2ev + \lambda ev = (\lambda + 2)ev$$
Hence $ev \in V_{\lambda+2}$. Similarly, $fv \in V_{\lambda-2}$.

That is, we have
$$\cdots \underset{f}{\overset{e}{\rightleftarrows}} V_{\lambda-2} \underset{f}{\overset{e}{\rightleftarrows}} V_\lambda \underset{f}{\overset{e}{\rightleftarrows}} V_{\lambda+2} \underset{f}{\overset{e}{\rightleftarrows}} \cdots$$

Definition 4.4 (highest weight vector)
A non-zero
$$v \in V_\lambda \cap \ker(e)$$
for some λ is called a *highest weight vector (of weight λ)*

Example 4.5
In the adjoint representation, e is a highest weight vector.

Lemma 4.6. Suppose $v \in V_\lambda$ is a highest weight vector. Then for all $n \geq 1$,
$$ef^n v = n(\lambda - n + 1)f^{n-1}v$$

Proof. Induction on n. For $n = 1$,
$$(ef)v = ([ef] + fe)v = (h + fe)v = \lambda v = 1(\lambda - 1 + 1)f^0 v$$
The inductive step follows similarly. □

Lemma 4.7. Suppose $v \in V_\lambda$ is a highest weight vector. Then
$$W = \operatorname{span}\{v, fv, f^2v, \dots\}$$
is a sub-representation of V,

Proof. Suffices to show that for $w = f^n v \in W$, then $ew, fw, hw \in W$. By definition, $fw \in W$ is obvious. $ew \in W$ follows by lemma 4.6, and the $n = 0$ case is just $ew = 0$.

For hw,
$$f^n v \in V_{\lambda-2n}$$
and so
$$hw = (\lambda - 2n)w \in W$$
□

Proposition 4.8. If V is finite dimensional, then a highest weight vector exists.

Proof. Choose any nonzero eigenvector v of h (always exists as we are working over $\mathbb{C}$). Consider

$$v, ev, e^2v, \dots$$

These are eigenvectors for h, with distinct eigenvalues. Hence the set

$$\{e^n v \mid e^n v \neq 0\}$$

is linearly independent. As V is finite dimensional, this set must be finite. Hence there must exists n such that

$$e^n v \neq 0 \quad \text{and} \quad e^{n+1} v = 0$$

Then $e^n v$ is a highest weight vector. □

Lemma 4.9. Suppose V is finite dimensional, and $v \in V_\lambda$ is a highest weight vector, then $\lambda \in \mathbb{Z}_{\geq 0}$.

Proof. Any non-zero vectors of the form $f^n v$ must be linearly independent, so there exists $n \geq 0$ such that $f^n v \neq 0, f^{n+1} v = 0$. By lemma 4.6,

$$0 = ef^{n+1}v = (n+1)(\lambda - n)f^n v$$

Hence we must have that $\lambda = n$, since $n + 1 \neq 0, f^n v \neq 0$. □

<u>Conclusion:</u> Suppose V is irreducible, of dimension $n+1$. Then by proposition 4.8, a highest weight vector $v \in V_\lambda$ exists. By lemma 4.7, we have a subrepresentation

$$W = \operatorname{span}\{v, fv, \dots\}$$

So by irreducibility,

$$\{v, fv, \dots, f^n v\}$$

is a basis, as the $f^i v$ are linearly independent, and we know from lemma 4.9 that $\lambda = n$.

Corollary 4.10. If V is an irreducible representation of $\mathfrak{sl}_2$, of dimension $n+1$, then there exists a basis $v_0, \dots, v_n$ of V, such that the actions are:

$$hv_i = (n-2i)v_i \qquad fv_i = \begin{cases} v_{i+1} & i+1 \leq n \\ 0 & i = n \end{cases} \qquad ev_i = \begin{cases} i(n-i+1)v_{i-1} & i-1 \geq 0 \\ 0 & i = 0 \end{cases}$$

In particular, there is a unique irreducible representation of $\mathfrak{sl}_2$ with dimension $n+1$ for all $n \geq 0$.

Lecture 4

Remark 4.11. Let V be the $n+1$ dimensional irreducible representation of $\mathfrak{sl}_2$, and let $v \in V$ be a highest weight vector. Note

$$\left(ef + fe + \frac{1}{2}h^2\right)(v) = \left(n + \frac{n^2}{2}\right)v$$

5 Irreducible modules for $\mathfrak{sl}_2$

Notation 5.1. We will write $V(n)$ for the $n+1$ dimensional irreducible representation of $\mathfrak{sl}_2$.

Definition 5.2 (weights)
Given a representation V for $\mathfrak{sl}_2$, the set

$$\{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\}$$

are the *weights* of V.

We will show *Weyl's theorem*.

Theorem 5.3. Every finite dimensional representation of $\mathfrak{sl}_2$ is completely reducible.

This result, along with corollary 4.10, implies that the action of h completely determines a finite dimensional representation of $\mathfrak{sl}_2$.

Example 5.4
Suppose V is a 5-dimensional representation of $\mathfrak{sl}_2$, and there exists $v \in V$ of weight 3. This means that by counting dimensions, the possible weights are $\{3, 1, -1, -3\}$, and $\{0\}$. Thus,

$$V \cong V(3) \oplus V(0)$$

We will need a few facts. Let $\mathfrak{g}$ be a Lie algebra, $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$, and suppose there exists $\sigma \in \mathfrak{gl}(V)$ commuting with $\varphi(x)$ for all $x \in \mathfrak{g}$. Then:
Fact 1:

$$\ker(\sigma - \lambda \operatorname{id}_V)$$

is a subrepresentation of V, for all $\lambda \in \mathbb{C}$. To see this, if $v \in V$ is such that $\sigma(v) = \lambda v$, then

$$\sigma(\varphi(x)v) = \varphi(x)\sigma(v) = \lambda\varphi(x)$$

Fact 2: If V is irreducible, then σ is a scalar multiple of id_V. That is, Schur's Lemma.

Definition 5.5 (Casimir element)
Let V be a finite dimensional representation of $\mathfrak{sl}_2$. Then

$$\Omega = ef + fe + \frac{1}{2}h^2 \in \mathfrak{gl}(V)$$

is called the *Casimir element (of $\mathfrak{sl}_2$)*.

In fact, Ω is *central*.

Lemma 5.6. If $\varphi : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ is a representation, then Ω commutes with $\varphi(x)$ for all $x \in \mathfrak{sl}_2$.

Proof. To show Ω is central, suffices to show $\Omega e = e\Omega, \Omega f = f\Omega, \Omega h = h\Omega$. Just compute. □

Corollary 5.7. If V is an irreducible finite dimensional representation of $\mathfrak{sl}_2$, then $\Omega \circlearrowleft V$ by a scalar.

Proof. By Schur's lemma and lemma 5.6. Moreover, the scalar is

$$\frac{n^2}{2} + n$$

□

Proof of theorem 5.3. Let $\varphi : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ be a finite dimensional representation of $\mathfrak{sl}_2$. Let $W \leq V$ be a subrepresentation. We need to find a subrepresentaion $U \leq V$, such that

$$V \cong W \oplus U$$

Case 1: W has codimension 1. So $V/W \cong V(0)$.

Subcase (i): W is trivial. In this case, $\dim(V) = 2$, and so we have a basis v_1, v_2 of V, with respect to which $\mathfrak{sl}_2$ acts on V, by matrices

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

We will show V is isomorphic to $\underbrace{V(0)}_{=W} \oplus \underbrace{V(0)}_{=U}$. Note that

$$\left[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \right] = 0$$

for all x, y. Since φ is a representation, it respects the Lie bracket. We must have that

$$\varphi(h) = [\varphi(e), \varphi(f)] = 0$$

and so

$$\varphi(e) = \frac{1}{2}[\varphi(h), \varphi(e)] = 0$$

and $\varphi(f) = 0$ similarly. Thus, the action of $\mathfrak{sl}_2$ is trivial.

Subcase (ii): $W = V(n)$ is irreducible, $n > 0$. We have the Casimir element $\Omega \in \mathfrak{gl}(V)$, and we will show

$$V = V(n) \oplus \ker(\Omega)$$

By Schur's lemma, and the fact that W is irreducible, and that Ω acts on V/W trivially, there is a basis for V, such that

$$\Omega = \begin{pmatrix} \lambda I & * \\ 0 & 0 \end{pmatrix}$$

Note here, since W is a subrepresentation, Ω restricts to an element of $\mathfrak{gl}(W)$, which is how we get the top left entry. W is non-trivial by assumption, and so $\ker(\Omega) \neq 0$, and $W \cap \ker(\Omega) = 0$. Hence

$$V = W \oplus \Omega$$

Subcase (iii): For a general W. We do this by induction on $\dim(V)$. If $\dim(V) = 1$, the result is clearly true. So we can assume $\dim(V) \geq 2$. Let $W' \leq W$ be a non-zero subrepresentation. As $\dim(W/W') < \dim(V)$, and $\operatorname{codim}_{V/W'}(W/W') = 1$, by induction, this implies that we have a splitting

$$\frac{V}{W'} = \frac{W}{W'} \oplus \frac{W''}{W'} \tag{1}$$

for some subspace $W'' \leq V$, with $W' \subseteq W''$, and W''/W' is a subrepresentation of V/W'. Moreover, W''/W' has dimension 1, and $\dim(W') < \dim(V)$. W'' being a subrepresentation of V follows from the fact that W''/W' is a subrepresentation. By induction again, there exists a subrepresentation $U \leq W''$ such that

$$W'' = W' \oplus U$$

We know that

$$V = W \oplus U$$

as $W \cap U = 0$ since eq. (1) is a direct sum, and so $W \cap U \leq W' \cap U = 0$. Using $\dim(U) = 1$, and counting dimensions we are done.

Case 2: Let W be arbitrary. Recall the action on $\operatorname{Hom}(V, W)$, is given by

$$(x\varphi)(v) = x\varphi(v) - \varphi(xv)$$

Define

$$\mathbb{V} = \{\psi \in \operatorname{Hom}(V, W) \mid \psi|_W = \lambda\, \mathrm{id}_W \text{ for some } \lambda\}$$

and we have a subspace

$$\mathbb{W} = \{\psi \in \mathbb{V} \mid \psi|_W = 0\} \leq \mathbb{V}$$

We lose one degree of freedom going from $\mathbb{V}$ to $\mathbb{W}$, and so $\operatorname{codim}_{\mathbb{V}}(\mathbb{W}) = 1$. Suppose $\psi|_W = \lambda \operatorname{id}_W$, $x \in \mathfrak{sl}_2$, $w \in W$, then

$$(x\psi)(w) = x\psi(w) - \psi(xw) = x(\lambda w) - \lambda(xw) = 0$$

So $\mathbb{V}$ is a subrepresentation of $\operatorname{Hom}(V, W)$, and so $\mathbb{W}$ is a subrepresentation as well. By case 1, there exists a one-dimensional subrepresentation $\mathbb{U} \leq \mathbb{V}$, such that

$$\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$$

Write $\mathbb{U} = \langle \gamma \rangle$, for some $\gamma \in \mathbb{V}$, and so $\gamma|_W = \lambda \operatorname{id}_W$, for some non-zero λ.

Claim 5.8. We have a vector space decomposition:

$$V = W \oplus \ker(\gamma)$$

Lecture 5

Proof of claim. By construction, $W \cap \ker(\gamma) = 0$, and by dimension counting, $\dim(V) = \dim(W) + \dim(\ker(\gamma))$, as $W = \operatorname{im}(\gamma)$. Since dimensions add up and the intersection is zero, we have a direct sum of vector spaces. □

Finally, it remains to show that $\ker(\gamma)$ is a subrepresentation of V. Let $v \in \ker(\gamma)$, $x \in \mathfrak{sl}_2$. Since $\mathbb{U}$ is one-dimensional of $\mathfrak{sl}_2$, it must be the trivial representation. Thus,

$$0 = (x\gamma)(v) = x\gamma(v) - \gamma(xv) = -\gamma(xv)$$

as $\gamma(v) = 0$. This means that $xv \in \ker(\gamma)$ as required.

□

In Humphreys' book §6.3, Humphreys proves theorem 5.3 for a general semisimple Lie algebra. Or see Henderson §5.2.1, §7.5.1.

Remark 5.9.

1. The proof only needed (in terms of representation theory)
   - existence of a Casimir element Ω,
   - the only one-dimensional representation is the trivial representation.
2. Complete reducibility is rare, it can fail for infinite dimensional Lie algebras, or simple Lie algebras in positive characteristic. For example, the adjoint representation of $\mathfrak{sl}_n(\mathbb{F}_p)$ on $\mathfrak{gl}_n(\mathbb{F}_p)$ is not completely reducible if $p \mid n$.

6 Tensor products

Given vector spaces V, W, with bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ respectively. We define the *tensor product* $V \otimes_{\mathbb{C}} W$ as the $\mathbb{C}$-vector space, with basis

$$\{v_i \otimes w_j\}_{i,j}$$

subject to the usual bilinearity conditions.

Definition 6.1 (tensor product of representations)

If V, W are representations of a Lie algebra $\mathfrak{g}$, then so is $V \otimes W$, with

$$x(v \otimes w) = (xv) \otimes w + v \otimes (xw)$$

Example 6.2

If V, W are $\mathfrak{sl}_2$ representations, $v \in V_\lambda$, $w \in W_\mu$, then

$$h(v \otimes w) = (\lambda + \mu)(v \otimes w)$$

That is,

$$v \otimes w = (V \otimes W)_{\lambda+\mu}$$

Thus, the weights of $V \otimes W$ are just $\lambda + \mu$, where λ is a weight for V, μ is a weight for W.

Example 6.3

For

$$V(2) \otimes V(2) = V(2)^{\otimes 2}$$

we have the weights:

	2	0	-2
2	4	2	0
0	2	0	-2
-2	0	-2	-4

and so

$$V(2)^{\otimes 2} = V(4) \oplus V(2) \oplus V(0)$$

In particular, if v_n is a highest weight vector in $V(n)$, then $v_n \otimes v_m$ is a highest weight vector for $V(n) \otimes V(m)$.

We would like a general formula for decomposing $V(n) \otimes V(m)$. The answer, as in Part II Representation theory, is a *Clebsch-Gordon* formula

$$V(n) \otimes V(m) = \bigoplus_{r=|n-m|, r \cong n-m \pmod 2}^{n+m} V(r)$$

We won't need this though.

Definition 6.4

The *n-th symmetric power* is

$$S^n V = \mathrm{Sym}^n(V) = \frac{V^{\otimes n}}{M_n}$$

where M_n is the span of

$$u_1 \otimes \cdots \otimes u_n - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$$

where $\sigma \in S_n$, and $u_i \in V$.

For example, M_2 is the span of $v \otimes w - w \otimes v$. In particular, note that M_n is a subrepresentation of $V^{\otimes n}$ of V. So $\mathrm{Sym}^n(V)$ is a representation of V.

Example 6.5

In $S^2 V$, $v \otimes w = w \otimes v$, and so $S^2 V$ has basis

$$v_i \otimes v_j$$

for $i \leq j$. Decomposing $S^2 V(2)$, we see that $e \otimes e \in S^2 V(2)$ is nonzero (note $V(2)$ is the adjoint representation), and so $V(4)$ is a subrepresentation of $S^2 V(2)$. In particular, we have a splitting

$$S^2 V(2) = V(4) \oplus V(0)$$

Definition 6.6

The *n-th exterior (or alternating) power* is

$$\Lambda^n V = \frac{V^{\otimes n}}{N_n}$$

where N_n is the span of

$$u_1 \otimes \cdots \otimes u_n$$

where $u_i \in V$, and $u_i = u_j$ for some $i \neq j$.

Again, N_n is a subrepresentation of $V^{\otimes n}$, and so $\Lambda^n V$ is a representation.

Example 6.7
With $n = 2$, $N_2 = \text{span}\{v \otimes v\}$.

Notation 6.8. We write (the coset of) $u_1 \otimes \cdots \otimes u_n$ in $\Lambda^n V$ as

$$u_1 \wedge \cdots \wedge u_n$$

Exercises:

1. Decompose $\Lambda^2 V(2) = V(2)$, with basis $e \wedge f, e \wedge h, h \wedge f$.
2. Find the dimensions of $S^n V$ and $\Lambda^n V$.

7 Results about semisimple Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$.

Definition 7.1 (ideal)
A subspace $I \subseteq \mathfrak{g}$ is an *ideal* of $\mathfrak{g}$ if

$$[xy] \in I$$

for all $x \in \mathfrak{g}, y \in I$.

Remark 7.2. Any ideal is automatically a subalgebra.

Suppose I is an ideal, then $\mathfrak{g}/I$ is a Lie algebra under

$$[x + I, y + I] = [x, y] + I$$

Moreover, I is an ideal if and only if it is a subrepresentation of the adjoint representation $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Example 7.3
The *centre* of $\mathfrak{g}$ is

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$$

and by definition, $Z(\mathfrak{g}) = \ker(\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}))$.

If $Z(\mathfrak{g}) = 0$, then $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is faithful, and thus we have an embedding

$$\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

and as such, $\mathfrak{g}$ can be regarded as a Lie subalgebra of $\mathfrak{gl}_n$, where $n = \dim(\mathfrak{g})$.

Theorem 7.4 (Ado). Suppose $\mathrm{char}(k) = 0$, then any finite dimensional Lie algebra $\mathfrak{g}$ embeds as a Lie subalgebra of $\mathfrak{gl}_m$ for some m.

Proof. Omitted. □

Note that the embedding need not be via the adjoint representation. In fact, this is true for $\mathrm{char}(k) = p > 0$, due to Iwasawa.

Example 7.5
The *derived subalgebra* of $\mathfrak{g}$ is

$$D(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] = \{[x, y] \mid x, y \in \mathfrak{g}\}$$

This is an ideal of $\mathfrak{g}$.

Recall $\mathfrak{sl}_n$ is a subalgebra of $\mathfrak{gl}_n$. In fact,

$$D(\mathfrak{gl}_n) = \mathfrak{sl}_n$$

Example 7.6
Suppose $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, then

$$\ker(\varphi) = \{x \in \mathfrak{g} \mid \varphi(x) = 0\}$$

is an ideal of $\mathfrak{g}$. In fact, every ideal arises in this way.

Lecture 6

Definition 7.7 (simple)
A Lie algebra $\mathfrak{g}$ is *simple* if $[\mathfrak{g}, \mathfrak{g}] \neq 0$, and the only ideals are 0 and $\mathfrak{g}$.

Example 7.8
We can show that $\mathfrak{sl}_n$ (for $n \geq 2$), $\mathfrak{so}_n$ (for $n \geq 2$) and $\mathfrak{sl}_{2\ell}$ (for $\ell \geq 1$) are simple.

Remark 7.9.
1. if $\mathfrak{g}$ is simple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.
2. if $\mathfrak{g}$ is simple, every representation of $\mathfrak{g}$ is either faithful, or the direct sum of trivial representations.
3. $\mathfrak{g}$ is simple if and only if the adjoint representation is irreducible.

Definition 7.10 (semisimple)
A Lie algebra $\mathfrak{g}$ is *semisimple* if it is the direct sum of simple ideals. That is, ideals which are simple as Lie algebras.

Example 7.11

$$\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$$

We will state a more 'standard' definition of semisimple Lie algebras, and show that these are equivalent.

Definition 7.12 ((lower) central series)
The *(lower) central series* of a Lie algebra $\mathfrak{g}$ is the sequence of subalgebras

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \cdots$$

with $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$. That is,

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq \cdots$$

Definition 7.13 (derived series, upper central series)
The *derived series*, or *upper central series* for a Lie algebra $\mathfrak{g}$ is the sequence

$$\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots$$

where $\mathfrak{g}^{(0)} = \mathfrak{g}$, and $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$. That is,

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq \cdots$$

Remark 7.14.
- $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$,
- $\mathfrak{g}^n$ and $\mathfrak{g}^{(n)}$ are ideals.

To see the second point, we induct on n. The case $n = 0$ is clear. Let $x \in \mathfrak{g}, y \in \mathfrak{g}^n$. Then

$$[x, y] \in \mathfrak{g}^{n-1}$$

since $\mathfrak{g}^{n-1}$ is an ideal, and $\mathfrak{g}^n \subseteq \mathfrak{g}^{n-1}$. As

$$\mathfrak{g}^n = \{[x, y] \mid x \in \mathfrak{g}, y \in \mathfrak{g}^{n-1}\}$$

this clearly contains $[x, y]$.

Example 7.15
1. if $\mathfrak{g}$ is simple, then $\mathfrak{g}^n = \mathfrak{g}^{(n)} = \mathfrak{g}$ for all n.
2. if $\mathfrak{g}$ is abelian, then $\mathfrak{g}^1 = \mathfrak{g}^{(1)} = 0$.
3. let $\mathfrak{n}_n \subseteq \mathfrak{gl}_n$ be the Lie algebra of strictly upper triangular $n \times n$ matrices. The central series for $\mathfrak{n}_n$ is

$$\left\{\begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix}\right\} \supseteq \left\{\begin{pmatrix} 0 & 0 & & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix}\right\} \supseteq \cdots \supseteq 0$$

Note $\mathfrak{n}_3$ is the *Heisenberg Lie algebra*.

Definition 7.16 (nilpotent)
If $\mathfrak{g}^n = 0$ for some n, then $\mathfrak{g}$ is called *nilpotent*.

Definition 7.17 (solvable, soluble)
If $\mathfrak{g}^{(n)} = 0$ for some n, then $\mathfrak{g}$ is called *solvable* (or *soluble* in BrE).

Exercise: Let $\mathfrak{b}_n \subseteq \mathfrak{gl}_n$ be the Borel Lie algebra of upper triangular matrices. Then $\mathfrak{b}_n$ is solvable but not nilpotent (for $n \geq 2$).

Note on the other hand that as $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$, nilpotent implies solvable.

The next result allows the theory of complex semisimple Lie algebras to go 'far' with minimal work.

Theorem 7.18 (Lie's theorem)**.** Let $k = \mathbb{C}$ (or an algebraically closed field with characteristic 0). Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra, and suppose $\mathfrak{g}$ is solvable. Then there exists a common eigenvector for all elements of $\mathfrak{g}$.

That is, there exists $v \in V$ non-zero, such that for all $x \in \mathfrak{g}$, $xv = \lambda_x v$ for some $\lambda_x \in \mathbb{C}$.

Proof. Omitted. See Humphreys Theorem 4.1. □

In particular, $\text{span}\{v\}$ defines a one-dimensional subrepresentation of V.

Corollary 7.19. There exists a basis for V such that every element is upper triangular.

In fact, using theorem 7.18 and induction on $\dim(V)$, we can show that there exists a chain of subspsaces

$$0 = V_0 \leq V_1 \leq \cdots \leq V_n = V$$

with $\dim(V_i) = i$, and $\mathfrak{g} \cdot V_i \subseteq V_i$. By considering a basis for V_i, we get the corollary. Therefore, we can consider $\mathfrak{g} \subseteq \mathfrak{b}_n$ as a subalgebra of the upper triangular matrices. To fill in the details here, the base case $\dim(V) = 1$ is trivial. Now suppose the result holds for all representations W with $\dim(W) = n$. Let $\dim(V) = n + 1$. By Lie's theorem, we have a one-dimensional subrepresentation U. Now consider V/U, which has dimension n. Hence by induction, there exists a chain of subspaces

$$0 = W_0 \leq W_1 \leq \cdots \leq W_n = W = V/U$$

with $\dim(W_i) = i$ and $\mathfrak{g} \cdot W_i \subseteq W_i$. By the correspondence theorem, say $W_i = V_1/U$. Then we obtain a chain

$$0 = V_0 \leq V_1 \leq \cdots \leq V_n = V$$

with the desired properties.

<u>Aside:</u> We call the sequence

$$0 = V_0 \leq V_1 \leq \cdots \leq V_n$$

a *maximal flag*, and there is interesting geometry related to this.

One application of Lie's theorem is when we have the adjoint representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, since subrepresentations correspond to ideals. Thus, we have a sequence

$$0 = \mathfrak{g}_0 \leq \cdots \leq \mathfrak{g}_n = \mathfrak{g}$$

of ideals of $\mathfrak{g}$, with $\dim(\mathfrak{g}_i) = i$.

Proposition 7.20. Suppose I, J are ideals of $\mathfrak{g}$. Then

(i) if $\mathfrak{g}$ is solvable, then any subalgebra or quotient of $\mathfrak{g}$ is solvable.

(ii) if I is solvable, and $\mathfrak{g}/I$ is solvable, then so is $\mathfrak{g}$.

(iii) if I, J are solvable, then so is $I + J$.

Proof. (i) is clear from definitions. For (ii), choose n such that $(\mathfrak{g}/I)^{(n)} = 0$. Then this forces $\mathfrak{g}^{(n)} \subseteq I$. But then we have $\mathfrak{g}^{(n+m)} \subseteq I^{(m)}$ for each $m \geq 0$. Since we know that I is solvable, we are done.

For (iii), note that

$$\frac{I + J}{J} \cong \frac{I}{I \cap J}$$

and the right hand side is solvable, by (i), and J is solvable by assumption, and so by (ii), $I + J$ is solvable. □

Definition 7.21 (radical)

The *radical* of $\mathfrak{g}$ is $\operatorname{Rad}(\mathfrak{g})$ is the maximal solvable ideal of $\mathfrak{g}$. That is, it is the sum of all solvable ideals of $\mathfrak{g}$.

Definition 7.22 (trace form)

Suppose $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation of $\mathfrak{g}$. Then the *trace form* of V (or φ) is the symmetric bilinear form

$$\begin{aligned}(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{C} \\ (x, y) &= \operatorname{tr}(\varphi(x)\varphi(y))\end{aligned}$$

Exercise: We have the invariance relation $([x, y], z) = (x, [y, z])$. This is essentially just the cyclic property of trace.

Definition 7.23 (Killing form)

The *Killing form* $K(\cdot, \cdot)$ is the trace form of ad. That is,

$$K(x, y) = \operatorname{tr}(\operatorname{ad}(x) \operatorname{ad}(y)) = (x, y)_{\operatorname{ad}}$$

Lecture 7

Theorem 7.24 (Cartan-Killing criterion). For a finite dimensional Lie algebra $\mathfrak{g}$, the following are equivalent:

(i) $\mathfrak{g}$ is semisimple,

(ii) $\operatorname{Rad}(\mathfrak{g}) = 0$,

(iii) the Killing form of $\mathfrak{g}$ is non-degenerate.

Remark 7.25. $\operatorname{Rad}(\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})) = 0$, since a suitable ideal of $\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ would lift to give an ideal J of $\mathfrak{g}$, containing $\operatorname{Rad}(\mathfrak{g})$, with $J/\operatorname{Rad}(g)$ solvable. Hence J is solvable, and $J \subseteq \operatorname{Rad}(\mathfrak{g})$.

Using this, $\mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ is semisimple.

Theorem 7.26 (Levi's theorem). Let k be a field with $\operatorname{char}(k) = 0$, and $\mathfrak{g}$ a finite dimensional Lie algebra over k. Then there exists a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$, with $\mathfrak{g}' \cap \operatorname{Rad}(\mathfrak{g}) = 0$, and as vector spaces,

$$\mathfrak{g} = \mathfrak{g}' \oplus \operatorname{Rad}(\mathfrak{g})$$

and $\mathfrak{g}'$ is isomorphic to $\operatorname{Rad}(\mathfrak{g})$, and thus semisimple. That is, the short exact sequence

$$0 \longrightarrow \operatorname{Rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\operatorname{Rad}(\mathfrak{g}) \longrightarrow 0$$

splits. This is called the *Levi decomposition of* $\mathfrak{g}$, and $\mathfrak{g}'$ the *Levi subalgebra of* $\mathfrak{g}$.

Not proven in the course. See Fulton-Harris appendix E. □

Lemma 7.27. Let $\mathfrak{g}$ be a Lie algebra,

(i) if I is an ideal of $\mathfrak{g}$, then so is $[I, I]$,

(ii) $\mathrm{Rad}(\mathfrak{g}) = 0$ if and only if $\mathfrak{g}$ has no non-trivial abelian ideals.

Proof. For (i), if $x, y \in I, z \in \mathfrak{g}$, we need to show that

$$[z, [x, y]] \in [I, I]$$

Using the Jacobi identity,

$$[z, [x, y]] = -[x, [y, z]] - [y, [x, z]] \in [I, I]$$

as I is an ideal.

For (ii), it is clear that any abelian ideal is solvable. Conversely, if I is solvable, then the last non-zero term in the derived series of I is abelian. □

Notation 7.28. Define

$$\mathfrak{g}^{\perp} = \{x \in \mathfrak{g} \mid K(x, y) = 0 \text{ for all } x \in \mathfrak{g}\}$$

Lemma 7.29. $\mathfrak{g}^{\perp}$ is an ideal.

Proof. For $x \in \mathfrak{g}^{\perp}, y, z \in \mathfrak{g}$, then $K([x, y], z) = K(x, [y, z]) = 0$ □

Lemma 7.30. Let I be an ideal of $\mathfrak{g}$, and let K_I denote the Killing form of I. Then

$$K_I(x, y) = K(x, y)$$

for all $x, y \in I$.

Proof. Choose a basis for I, and extend it to a basis of $\mathfrak{g}$. Given $x, y \in I$, with respect to this basis,

$$\mathrm{ad}(x) = \begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix}$$

where $A = (\mathrm{ad}(x))|_I$, and similarly for $\mathrm{ad}(y)$. Set $B = (\mathrm{ad}(y))|_I$. Then

$$K_I(x, y) = \mathrm{tr}(AB) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = K(x, y)$$

Note here that

$$\begin{pmatrix} A & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & * \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} AB & * \\ 0 & 0 \end{pmatrix}$$

□

Theorem 7.31 (Cartan's criterion (for solvability))**.** Suppose $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$, for V a finite dimensional vector space over $\mathbb{C}$. $\mathfrak{g}$ is solvable if and only if $\mathrm{tr}(xy) = 0$ for all $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$.

In other words, $\mathfrak{g}^{(1)} \leq \mathfrak{g}^{\perp}$.

Proof. See Humphreys 4.3 using Lie's theorem and the Jordan decomposition. For the Jordan decomposition, see §8. □

Corollary 7.32. (i) if $\mathfrak{g} = \mathfrak{g}^{\perp}$, then $\mathfrak{g}$ is solvable.

(ii) if $\mathfrak{g}$ is simple, then $\mathfrak{g}^{\perp} = 0$,

(iii) $\mathfrak{g}^{\perp}$ is solvable for any finite dimensional Lie algebra $\mathfrak{g}$.

Proof. (i) Consider the adjoint $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. The image is $\mathrm{ad}(\mathfrak{g}) = \mathfrak{g}/\ker(\mathfrak{g}) = \mathfrak{g}/Z(\mathfrak{g})$. $Z(\mathfrak{g})$ is solvable, since it is abelian, and by assumption $\mathfrak{g} = \mathfrak{g}^\perp$. By theorem 7.31, $\mathrm{ad}(\mathfrak{g})$ is solvable, and so $\mathfrak{g}$ is solvable.

(ii) Since $\mathfrak{g}^\perp$ is an ideal, and $\mathfrak{g}$ is simple, $\mathfrak{g}^\perp = 0$ or $\mathfrak{g}^\perp = \mathfrak{g}$. In the second case, by (i) $\mathfrak{g}$ is solvable, contradicting the fact that $\mathfrak{g}$ is simple, as $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

(iii) By lemma 7.30, $(\mathfrak{g}^\perp)^\perp = \mathfrak{g}^\perp$, so by (i), $\mathfrak{g}^\perp$ is solvable. □

With this, we can now prove the Cartan-Killing criterion.

Proof of theorem 7.24. First we show (ii) $\implies$ (iii). In this case, $\mathfrak{g}$ is solvable, and so $\mathfrak{g}^\perp \leq \mathrm{Rad}(\mathfrak{g}) = 0$. Thus K is non-degenerate.

For (iii) $\implies$ (ii), let A be an abelian ideal of $\mathfrak{g}$. We will show that $A \subseteq \mathfrak{g}^\perp$. Choose $a \in A, y \in \mathfrak{g}$. Choose a basis for A, and extend it to a basis of $\mathfrak{g}$. That is,

$$\mathrm{ad}(a) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ad}(y) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

Computing $\mathrm{tr}(\mathrm{ad}(a)\,\mathrm{ad}(y))$, we find that it is zero. Thus $A = 0$. Therefore, $\mathrm{Rad}(\mathfrak{g})$ is non-zero, then $\mathfrak{g}^\perp$ is non-zero.

Now assume (ii) and (iii) hold. If $\mathfrak{g}$ is simple, then we are done. If not, choose a minimal non-zero ideal I. Let

$$\mathfrak{g}_I = \{x \in \mathfrak{g} \mid K(x, y) = 0 \text{ for all } y \in I\}$$

This is an ideal of $\mathfrak{g}$.

Claim 7.33. $\mathfrak{g} = \mathfrak{g}_I \oplus I$.

Proof. Since I is simple (by minimality), and non-abelian (by (ii)),

$$I \cap \mathfrak{g}_I \subseteq I^\perp = 0$$

Consider the map

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^* \longrightarrow I^*$$

given by

$$x \mapsto K(x, \cdot)$$

The kernel of this map is $\mathfrak{g}_I$. Thus, $\mathfrak{g}/\mathfrak{g}_I \cong I^* \cong I$ as vector spaces. □

Repeat this argument with $\mathfrak{g}_I$. That is, choosing some minimal ideal of $\mathfrak{g}_I$. We can do this as any ideal of $\mathfrak{g}_I$ is an ideal of $\mathfrak{g}$, and so $\mathrm{Rad}(\mathfrak{g}_I) = 0$.

Claim 7.34. $(\mathfrak{g}_I)^\perp = 0$

Proof. Since if $x \in (\mathfrak{g}_I)^\perp$, then $x \in \mathfrak{g}^\perp$. More precisely, let $x \in (\mathfrak{g}_I)^\perp$, $y \in \mathfrak{g}$. We can write $y = y_1 + y_2$, where $y_1 \in \mathfrak{g}_I, y_2 \in I$. Then

$$K(x, y) = K(x, y_1) + K(x, y_2)$$

The first term vanishes as $x \in (\mathfrak{g}_I)^\perp$, and the second term vanishes as $x \in \mathfrak{g}_I$. □

This proves (i). To see this, we note that $(\mathfrak{g}_I)^\perp = 0$ implies that the Killing form of $\mathfrak{g}_I$ is non-degenerate. Thus (by induction on the dimension), $\mathfrak{g}_I$ is semisimple. It remains to show that any ideal of $\mathfrak{g}_I$ is an ideal of $\mathfrak{g}$. Let $J \subseteq \mathfrak{g}_I$ be an ideal, $x \in J, y \in \mathfrak{g}$. As above, write $y = y_1 + y_2$, with $y_1 \in \mathfrak{g}_I, y_2 \in I$. Then

$$[x, y] = [x, y_1] + [x, y_2]$$

But $[x, y_1] \in J$ as J is an ideal, and $[x, y_2] \in \mathfrak{g}_I \cap I = 0$.

Finally, to show (i) $\implies$ (ii), write

$$\mathfrak{g} = \bigoplus_j I_j$$

where the I_j are simple ideals. Let $\pi_j : \mathfrak{g} \to I_j$ denote the projection.

Lecture 8

Claim 7.35. If J is an ideal of $\mathfrak{g}$, then $\pi_j(J)$ is an ideal of I_j.

Now if A is an abelian ideal of $\mathfrak{g}$, then $\pi_j(A)$ is an abelian ideal of I_j, and so $\pi_j(A) = 0$ for all j. With this, $A = 0$. ☐

Theorem 7.36 (Weyl). Any finite dimensional representation of a semisimple Lie algebra is completely reducible.

Proof. Almost the same as for $\mathfrak{sl}_2(\mathbb{C})$, as in theorem 5.3. The main ingredient follows from a Casimir element. ☐

Exercise: Any ideal or quotient of a semisimple Lie algebra is semisimple. For this, note that the decomposition

$$\mathfrak{g} = I \oplus \mathfrak{g}_I$$

holds for any ideal. In particular, the Killing form of I is non-degenerate. Moreover, $\mathfrak{g}/I \cong \mathfrak{g}_I$ is isomorphic to an ideal of $\mathfrak{g}$, which is semisimple.

In fact, I is a sum of the I_j.

For the Casimir element, let $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be an irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. Without loss of generality, assume φ is faithful (if not, we can consider $\mathfrak{g}/\ker(\varphi)$). We know by theorem 7.24 that the trace form $(\cdot, \cdot)_V$ is non-degenerate. Choose a basis $x_1, \dots, x_n$ for $\mathfrak{g}$. With respect to the trace form, we have a dual basis $y_1, \dots, y_n$ for $\mathfrak{g}$. That is,

$$(x_i, y_j)_V = \delta_{ij}$$

Definition 7.37 (Casimir element)

Define the *Casimir element associated with* φ

$$\Omega_\varphi = \sum_i \varphi(x_i)\varphi(y_i)$$

Remark 7.38.

- $\Omega_\varphi \in \mathfrak{gl}(V)$,
- Ω_φ commutes with $\varphi(x)$ for all $x \in \mathfrak{g}$. In particular, by Schur's lemma, Ω_φ is a scalar multiple of id_V, and

$$\mathrm{tr}\left(\Omega_\varphi\right) = \sum \mathrm{tr}(\varphi(x_i)\varphi(y_i)) = \dim(\mathfrak{g})$$

  From this, we also see that Ω_φ is independent of the choice of basis of $\mathfrak{g}$ which we chose.

Example 7.39

If $\mathfrak{g} = \mathfrak{sl}_2 \leq \mathfrak{gl}_2$, let $V = \mathbb{C}^2$ and $\varphi = \mathrm{id}$ is the defining representation. Recall the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Some easy linear algebra gives a dual basis with respect to the trace form, which is $\{f, \frac{1}{2}h, e\}$ (in the same order). With this,

$$\Omega_\varphi = ef + fe + \frac{1}{2}h^2 = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

which is the same as the one we obtained earlier.

8 Jordan decomposition

Two observations:

1. If $\mathfrak{g}$ is a simple Lie algebra, $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation, then $\varphi(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$. This is because $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, and so

$$\varphi(\mathfrak{g}) = \varphi([\mathfrak{g}, \mathfrak{g}]) = [\varphi(\mathfrak{g}), \varphi(\mathfrak{g})] \leq [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$$

2. Recall from Linear algebra that if $x \in \mathfrak{gl}(V)$, then there exists a basis of V such that x a block diagonal matrix, with Jordan blocks of the form

$$\begin{pmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

Definition 8.1 (nilpotent, semisimple)

We say $x \in \mathfrak{gl}(V)$ is *nilpotent* if $x^n = 0$ for some n. We say x is *semisimple* if the roots of its minimal polynomial are distinct, that is, it is diagonalisable.

Proposition 8.2 (Jordan decomposition). If $x \in \mathfrak{gl}(V)$, where V is finite dimensional. Then

(i) there exists unique elements $x_s, x_n \in \mathfrak{gl}(V)$, where x_s is semisimple and x_n is nilpotent, with

$$x = x_s + x_n$$

and $[x_s, x_n] = 0$.

(ii) there exists polynomials $p_s, p_t \in \mathbb{C}[t]$, without constant terms, such that $x_s = p_s(x)$ and $x_n = p_n(x)$. In particular, x_s and x_n will commute with any $y \in \mathfrak{gl}(V)$ with $[x, y] = 0$.

(iii) if $A \leq B \leq V$ are subspaces, and $x(B) \subseteq A$, then $x_s(B) \subseteq A$ and $x_n(B) \subseteq A$.

The decomposition $x = x_s + x_n$ is called the *(additive) Jordan(-Chevalley) decomposition of* x. x_s and x_n are called the *semisimple part* and the *nilpotent part* of x respectively.

Proof. Routine linear algebra. See Humphreys §4.2. □

Example 8.3

If x is represented by a single Jordan block

$$x = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

then the Jordan decomposition is

$$x_s = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ & & \ddots \\ & & \lambda \end{pmatrix} \qquad x_n = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

i.e. the diagonal and nilpotent parts.

Why is this valuable?

Let V be a finite dimensional vector space. We can consider the adjoint representation $\mathrm{ad} : \mathfrak{gl}(V) \to \mathfrak{gl}(\mathfrak{gl}(V))$. If $x \in \mathfrak{gl}(V)$ is semisimple, then so is $\mathrm{ad}(x)$. Similarly, if x is nilpotent, then so is $\mathrm{ad}(x)$.

Lemma 8.4. Let $x \in \mathfrak{g} \leq \mathfrak{gl}(V)$, where V is finite dimensional. Let $x = x_s + x_n$ be its Jordan decomposition. Then $\mathrm{ad}(x) = \mathrm{ad}(x_s) + \mathrm{ad}(x_n)$ is the Jordan decomposition of $\mathrm{ad}(x) \in \mathfrak{gl}(\mathfrak{g})$.

Proof. $\mathrm{ad}(x_s)$ and $\mathrm{ad}(x_n)$ are semisimple and nilpotent respectively, they commute since

$$[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = \mathrm{ad}([x_s, x_n]) = \mathrm{ad}(0) = 0$$

Thus, by uniqueness in proposition 8.2, the Jordan decomposition must be as stated. □

Theorem 8.5. Suppose $\mathfrak{g}$ is a semisimple Lie algebra, which is a subalgebra of $\mathfrak{gl}(V)$. Let $x \in \mathfrak{g}$, then $x_s, x_n \in \mathfrak{g}$.

Proof. Let

$$N(\mathfrak{g}) = \{y \in \mathfrak{gl}(V) \mid [y, z] \in \mathfrak{g} \text{ for all } z \in \mathfrak{g}\}$$

be the *normaliser of* $\mathfrak{g}$ *in* $\mathfrak{gl}(V)$.

Claim 8.6. (i) $N(\mathfrak{g})$ is a subalgebra of $\mathfrak{gl}(V)$, containing $\mathfrak{g}$ as an ideal.

(ii) $x_s, x_n \in N(\mathfrak{g})$.

Proof. (i) is clear from the definition of the normaliser. For (ii), let $z \in \mathfrak{g}$, we have that

$$[x_s, z] = \mathrm{ad}(x_s)(z) = \mathrm{ad}(x)_s(z)$$

By proposition 8.2 (ii), this is in $\mathfrak{g}$, as $\mathrm{ad}(x)_s$ is a polynomial in $\mathrm{ad}(x)$ with no constant term. But for $z \in \mathfrak{g}$, as $x \in \mathfrak{g}$, $\mathrm{ad}(x)(z) = [x, z] \in \mathfrak{g}$. □

Let W be an irreducible subrepresentation of V, and define

$$\mathfrak{g}_W = \{y \in \mathfrak{gl}(V) \mid yw \in W \text{ for all } w \in W \text{ and } \mathrm{tr}(y|_W) = 0\}$$

Claim 8.7. $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}_W$.

Proof. W is a subrepresentation of $\mathfrak{g}$, and so it is stabilised by $\mathfrak{g}$, and also the image of $\mathfrak{g}$ in $\mathfrak{gl}(W)$, say $\bar{\mathfrak{g}}$, is also semisimple.

Lecture 9

With this, $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] = \bar{\mathfrak{g}}$, and so every element of $\bar{\mathfrak{g}}$ is a sum of commutators, and all of the traces are zero, and so $\mathfrak{g} \leq \mathfrak{g}_W$. □

Using this, $\mathrm{tr}(x|_W) = 0$. Note x_s, x_n are polynomials in x, and so they stabilise everything that x does. Moreover, $\mathrm{tr}(x_n|_W) = 0$ as $x_n|_W$ is nilpotent. Thus, $\mathrm{tr}(x_s|_W) = \mathrm{tr}(x|_W) - \mathrm{tr}(x_n|_W) = 0$. Using this, $x_s, x_n \in \mathfrak{g}_W$ for all W.

To finish, define

$$\mathfrak{g}' = N(\mathfrak{g}) \cap \bigcap_{W \leq V \text{ irred subrep}} \mathfrak{g}_W$$

Claim 8.8. $\mathfrak{g} = \mathfrak{g}'$.

Proof. Since $\mathfrak{g}' \le N(\mathfrak{g})$, $\mathfrak{g}$ is an ideal of $\mathfrak{g}'$. Then $\mathfrak{g}$ is a subrepresentation of $\mathfrak{g}'$ under the adjoint action of $\mathfrak{g}$. By Weyl's theorem[1], $\mathfrak{g}' = \mathfrak{g} \oplus U$ as representations. It suffices to show $U = 0$.

Choose $u \in U$, then as $\mathfrak{g}$ is an ideal, $[u, \mathfrak{g}] \subseteq \mathfrak{g}$. But $\mathrm{ad}(\mathfrak{g})(U) \subseteq U$, and so $[u, \mathfrak{g}] \subseteq U$. Hence $[u, \mathfrak{g}] = 0$, and so u commutes with every element of $\mathfrak{g}$. Using this, u is a $\mathfrak{g}$-endomorphism $V \to V$, and so it stabilises every irreducible subrepresentation W.

By Schur's lemma, $u|_W = \lambda \, \mathrm{id}_W$ for some scalar $\lambda \in \mathbb{C}$. But $\mathrm{tr}(u|_W) = 0$ since $u \in \mathfrak{g}_W$, and so $\lambda = 0$. But every representation splits as a direct sum of irreducibles, and so u must be zero. □

By the above, we see that x_s, x_n is in each set on the right hand side, and so $x_s, x_n \in \mathfrak{g}$. □

For $\mathfrak{g} \le \mathfrak{gl}(V)$ a semisimple Lie algebra, we can define an *abstract Jordan decomposition*

$$\mathrm{ad}(x) = \mathrm{ad}(x)_s + \mathrm{ad}(x)_n$$

Since ad is faithful, as $\mathfrak{g}$ is semisimple, $\mathfrak{g}$ is isomorphic to $\mathrm{ad}(\mathfrak{g}) \le \mathfrak{gl}(\mathfrak{g})$. By theorem 8.5, $\mathrm{ad}(x)_s, \mathrm{ad}(x)_n \in \mathrm{ad}(\mathfrak{g})$, and so there exists $x_s, x_n \in \mathfrak{g}$ such that $x = x_s + x_n$.

Suppose $\mathfrak{g} \le \mathfrak{gl}(V)$ for some V, with $x = x_s + x_n$. Then since $\mathrm{ad}(x_n) = \mathrm{ad}(x)_n$ and $\mathrm{ad}(x_s) = \mathrm{ad}(x)_s$, the abstract Jordan decomposition is just the usual Jordan decomposition.

Corollary 8.9. Let $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be any representation of a semisimple Lie algebra $\mathfrak{g}$. Choose $x \in \mathfrak{g}$, and let it have a Jordan decomposition $x = x_s + x_n$, then

$$\varphi(x_s) = \varphi(x)_s \quad \text{and} \quad \varphi(x_n) = \varphi(x)_n$$

defines a Jordan decomposition of $\varphi(x)$.

Proof. See Corollary 5.11 of the notes by David Stuart on Moodle. It needs semisimplicity, and the fact that we are working over $\mathbb{C}$. It fails if we work over a field with positive characteristic. □

10 Cartan subalgebras and root space decompositions

In this section, $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over $\mathbb{C}$.

Definition 10.1 (toral subalgebra)

A subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is *toral* if

1. $\mathfrak{t}$ is abelian,
2. $\mathrm{ad}(x)$ is semisimple for all $x \in \mathfrak{t}$.

A maximal toral subalgebra is called a *maximal torus*, or a *Cartan subalgebra* (CSA).

To justify the terminology, note that a connected abelian Lie group is isomorphic to $\mathbb{R}^k \times T^\ell$, and so a connected *compact* abelian Lie group is a torus.

Remark 10.2. Many authors, including Humphreys define Cartan subalgebras as a nilpotent subalgebra which equals its normaliser in $\mathfrak{g}$. That is,

$$\mathfrak{t} = \{x \in \mathfrak{g} \mid [x, \mathfrak{t}] \subseteq \mathfrak{t}\}$$

This is equivalent to our definition.

Example 10.3

If $\mathfrak{g} \le \mathfrak{sl}_n, \mathfrak{gl}_n$, with $\mathfrak{t}$ being the set of diagonal matrices, then $\mathfrak{t}$ is a maximal torus. It is true for $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$ as well.

[1] applied to the representation $\mathfrak{g}'$ of the semisimple Lie algebra $\mathfrak{g}$

Lemma 10.4. Let $t_1, \dots, t_n \in \mathrm{End}(V)$ which pairwise commute, and are all semisimple. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, define

$$V_\lambda = \{v \in V \mid t_i v = \lambda_i v \text{ for all } i\}$$

That is, the simultaneuous eigenspaces for the t_i. Then

$$V = \bigoplus_{\lambda \in \mathbb{C}^n} V_\lambda$$

Proof. By induction on n. $n = 1$ is true by definition. For $n > 1$, we know by induction that

$$V = \bigoplus_{\lambda' \in \mathbb{C}^{n-1}} V_{\lambda'}$$

for the action of $t_1, \dots, t_{n-1}$. Then since the t_i commute,

$$t_n(V_{\lambda'}) \subseteq V_{\lambda'}$$

for all λ'. By decomposing each $V_{\lambda'}$ in terms of t_n eigenspaces, we are done. □

Lemma 10.5. Any $\mathfrak{g}$ contains a Cartan subalgebra.

Proof. Needs Engel's theorem (Examples sheet 2), and Zorn's lemma. See David Stuart's notes. □

Recasting lemma 10.4, suppose we have $\mathfrak{h} \leq \mathfrak{gl}(V)$ with a basis of commuting semisimple elements $t_1, \dots, t_n$. Take $\lambda \in \mathbb{C}^n$, this corresponds the element of $\mathfrak{h}^*$, given by

$$t_i \mapsto \lambda_i$$

Then

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$$

In our situation, fix a Cartan subalgebra $\mathfrak{t} \leq \mathfrak{g}$, then

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{t}^*} \mathfrak{g}_\lambda$$

where

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [t, x] = \lambda(t)x \text{ for all } x \in \mathfrak{t}\}$$

Lecture 10

Definition 10.6 (root)
Let

$$\Phi = \{\alpha \in \mathfrak{t}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$$

The elements of Φ are the *roots of* $\mathfrak{g}$ *with respect to* $\mathfrak{t}$. If $\alpha \in \Phi$, then $\mathfrak{g}_\alpha$ is called a *root space*.

With this, we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

which is the *root space decomposition*, or the *Cartan decomposition* of $\mathfrak{g}$.

Proposition 10.7. (i) For all $\alpha, \beta \in \mathfrak{t}^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

(ii) If $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$, then $\mathrm{ad}(x)$ is nilpotent.

(iii) If $\alpha + \beta \neq 0$, then $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for all $\alpha, \beta \in \mathfrak{g}^*$.

Proof. For (i), let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$, $t \in \mathfrak{t}$. We have the Jacobi identity:

$$[t,[x,y]] = -[x,[y,t]] - [y,[t,x]] = [x,[t,y]] - \alpha(t)[y,x] = \beta(t)[x,y] - \alpha(t)[y,x] = (\alpha+\beta)(t)[x,y]$$

Note Fulton-Harris calls this the *fundamental calculation*.

For (ii), use (i) and the fact that $\mathfrak{g}$ is finite dimensional.

For (iii), if $\alpha+\beta \neq 0$, we can find $t \in \mathfrak{t}$ such $(\alpha+\beta)(t) \neq 0$. Fix such a $t \in \mathfrak{t}$, $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then

$$\alpha(t)K(x,y) = K([t,x],y) = -K([x,t],y) = -K(x,[t,y]) = -\beta(t)K(x,y)$$

and so $(\alpha+\beta)(t)K(x,y) = 0$. But by assumption $\alpha(t)+\beta(t) \neq 0$, and so $K(x,y) = 0$. □

Corollary 10.8. (i) The Killing form restricted to $\mathfrak{g}_0$ is non-degenerate.

(ii) If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

Proof. For (i), if $z \in \mathfrak{g}_0$, with $K(z,x) = 0$ for all $x \in \mathfrak{g}_0$, then by (iii), we know that $\mathfrak{g}_0$ is orthogonal to all $\mathfrak{g}_\alpha$ with $\alpha \neq 0$. If $x \in \mathfrak{g}$, we can write it as

$$x = x_0 + \sum_{\alpha\in\Phi} x_\alpha$$

with $x_\alpha \in \mathfrak{g}_\alpha$. Using this, we see that $K(z,x) = 0$ for all $x \in \mathfrak{g}$. By non-degeneracy of the Killing form (as $\mathfrak{g}$ is semisimple), we must have that $z = 0$.

For (ii), the proof is similar. □

Proposition 10.9.

$$\mathfrak{g}_0 = \mathfrak{t}$$

Proof. See Humphreys §8.2. □

Corollary 10.10. The Killing form is non-degenerate when restricted to $\mathfrak{t}$. In particular, the map

$$\begin{aligned} \mathfrak{t} &\to \mathfrak{t}^* \\ t &\mapsto K(t,\cdot) \end{aligned}$$

is an isomorphism of vector spaces. We denote the inverse map as $\lambda \mapsto t_\lambda$, where t_λ is called the *coroot associated to* λ, defined by

$$K(t_\lambda, x) = \lambda(x)$$

for all $x \in \mathfrak{g}$.

Example 10.11

For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathfrak{t} = \text{span}\{h\}$. Define $\alpha \in \mathfrak{t}^*$ by $\alpha(h) = 2$. Then $\mathfrak{g}_\alpha = \text{span}\{e\}$, and $\mathfrak{g}_{-\alpha} = \text{span}\{f\}$. With this,

$$\mathfrak{sl}_2 = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

Example 10.12

For $\mathfrak{g} = \mathfrak{sl}_3$, the Cartan subalgebra is $\mathfrak{t} = \text{span}\{h_1, h_2\}$, where

$$h_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

Let $\alpha_i \in \mathfrak{t}^*$ be such that

$$\alpha_i \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} = a_i$$

Then

$$\mathfrak{sl}_3 = \mathfrak{t} \oplus \mathfrak{g}_{\alpha_1-\alpha_2} \oplus \mathfrak{g}_{\alpha_1-\alpha_3} \oplus \mathfrak{g}_{\alpha_2-\alpha_3} \oplus \mathfrak{g}_{\alpha_2-\alpha_1} \oplus \mathfrak{g}_{\alpha_3-\alpha_1} \oplus \mathfrak{g}_{\alpha_3-\alpha_2}$$

where

$$\mathfrak{g}_{\alpha_i-\alpha_j} = \text{span}\{e_{i,j}\}$$

We can decompose the adjoint in a similar fashion. Moreover, this generalises to $\mathfrak{g} = \mathfrak{sl}_n$, with $\mathfrak{t}$ being the diagonal.

Similarly, the diagonal matrices form the Cartan subalgebra of $\mathfrak{so}_n, \mathfrak{sp}_{2n}$.

Proposition 10.13. Let $\alpha \in \Phi$ be a root, $e_\alpha \in \mathfrak{g}_\alpha$, then there exists $f_\alpha \in \mathfrak{g}_{-\alpha}$, such that

$$\mathfrak{m}_\alpha = \text{span}\{e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]\} \cong \mathfrak{sl}_2$$

We call $e_\alpha, f_\alpha, h_\alpha$ an $\mathfrak{sl}_2$-*triple*.

We're saying that every semisimple Lie algebra is "made up from $\mathfrak{sl}_2$s".

Note that if $t \in \mathfrak{t}$ and it satisfies $\alpha(t) = 0$, for all $\alpha \in \Phi$, then $t = 0$, since if $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$ non-zero, then

$$0 = \alpha(t)x = [t, x]$$

Since a toral subalgebra is by definition abelian, this holds for all of $\mathfrak{g}$, and so $t \in Z(\mathfrak{g})$.

Lemma 10.14. Φ spans $\mathfrak{t}^*$.

Proof. If not, then there exists $t \neq 0$ such that $\alpha(t) = 0$ for all $\alpha \in \Phi$. □

Lemma 10.15. $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one-dimensional.

Proof. Take $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\alpha$, then $[x, y] \in \mathfrak{t}$. Let $t \in \mathfrak{t}$, so

$$K([x, y], t) = K(x, [y, t]) = -K(x, [t, y]) = \alpha(t)K(x, y)$$

Hence $[x, y] = K(x, y)t_\alpha$. With this,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \text{span}\{t_\alpha\}$$

and so the dimension is at most one. But by non-degeneracy, we can find x, y such that $K(x, y) \neq 0$, and so the dimension is one. □

Lemma 10.16. $\alpha(t_\alpha) \neq 0$.

Proof. Since the Killing form is non-degenerate, and rescaling is possible, we may choose $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) = 1$. Then

$$\begin{aligned} [x, y] &= t_\alpha \\ [t_\alpha, x] &= \alpha(t_\alpha)x \\ [t_\alpha, y] &= -\alpha(t_\alpha)y \end{aligned}$$

Thus, the space $\mathfrak{h} = \text{span}\{x, y, t_\alpha\}$ is a subalgebra of $\mathfrak{g}$. Suppose $\alpha(t_\alpha) = 0$. Then, $[\mathfrak{h}, \mathfrak{h}] = \text{span}\{t_\alpha\}$, and so $\mathfrak{h}$ is solvable. Now consider the adjoint representation

$$\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

This shows that $\mathfrak{h}$ embeds as a solvable Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. By Lie's theorem, we can assume $\text{ad}(\mathfrak{h})$ is a subset of the space of upper triangular matrices. With this, $\text{ad}(t_\alpha) = [\text{ad}(x), \text{ad}(y)]$ is strictly upper triangular. Hence $\text{ad}(t_\alpha)$ is nilpotent, but $\text{ad}(t_\alpha)$ is semisimple[2], and so $\text{ad}(t_\alpha) = 0$. Thus,

$$t_\alpha \in Z(\mathfrak{g}) = 0$$

Contradiction, as $\alpha \neq 0$ implies $t_\alpha \neq 0$. □

Lemma 10.17. $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$.

Proof. If $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ with $K(x, y) \neq 0$, then for all $z \in \mathfrak{g}_\alpha$,

$$[[x, y], z] = K(x, y)[t_\alpha, z] = K(x, y)\alpha(t_\alpha)z$$

□

Lecture 11

Proof of proposition 10.13. Take $e_\alpha \in \mathfrak{g}_\alpha$, and find $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$K(e_\alpha, f_\alpha) = \frac{2}{\alpha(t_\alpha)}$$

Define

$$h_\alpha = \frac{2}{K(t_\alpha, t_\alpha)} t_\alpha$$

We can check that this satisfies the $\mathfrak{sl}_2$ relations.

$$[e_\alpha, f_\alpha] = K(e_\alpha, f_\alpha)t_\alpha = h_\alpha$$
$$[h_\alpha, e_\alpha] = \frac{2}{\alpha(t_\alpha)}[t_\alpha, e_\alpha] = 2e_\alpha$$
$$[h_\alpha, f_\alpha] = -2f_\alpha$$

and so $\mathfrak{m}_\alpha \cong \mathfrak{sl}_2$. □

Exercise: Show that weights add. If $\mathfrak{g}$ is semisimple, with root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$$

and V, W representations of $\mathfrak{g}$, with weight spaces V_α, W_α. Then

1. $\mathfrak{g}_\alpha \cdot V_\beta \subseteq V_{\alpha+\beta}$
2. $V_\alpha \otimes W_\beta \subseteq (V \otimes W)_{\alpha+\beta}$.

Lemma 10.18.
1. if V is a finite dimensional representation of $\mathfrak{g}$, then $V|_{\mathfrak{m}_\alpha}$ is a finite dimensional representation of $\mathfrak{m}_\alpha$,
2. for $\beta \in \Phi$, or $\beta = 0$, let

$$V = \bigoplus_{c \in \mathbb{C}} \mathfrak{g}_{\beta \oplus c\alpha}$$

where we sum over $c \in \mathbb{C}$ such that $\beta + c\alpha \in \Phi$. This is a representation of $\mathfrak{m}_\alpha$ under the adjoint action.

We call V the *α-root string through β*

Proof. (i) is true by generic facts about restrictions. For (ii), it follows by proposition 10.7. □

[2] This follows from $t_\alpha \in \mathfrak{t}$.

Proposition 10.19. Let $\alpha \in \Phi$. Then the root spaces $\mathfrak{g}_{\pm\alpha}$ are 1-dimensional. Moreover, the if $c\alpha \in \Phi$, for some $c \in \mathbb{C}$, then $c = \pm 1$.

Proof. Suppose $c\alpha \in \Phi$, then h_α takes $c\alpha(h_\alpha) = 2c$ as an eigenvalue. The eigenvalues of h_α are integers, and so either $c \in \mathbb{Z}$, or $c \in \mathbb{Z} + \frac{1}{2}$.

Write

$$V = \mathfrak{t} \oplus \bigoplus_{c\alpha \in \Phi} \mathfrak{g}_\alpha$$

Let $K = \ker(\alpha) \leq \mathfrak{t}$. We can check that $K + \mathfrak{m}_\alpha$ is an $\mathfrak{m}_\alpha$-subrepresentation of V. By Weyl's theorem, as a representation of $\mathfrak{m}_\alpha$,

$$V = K \oplus \mathfrak{m}_\alpha \oplus W$$

where W is a complementary subrepresentation. Suppose either of the conclusions in the statement are false. Then $W \neq 0$. Let $W_0 \leq W$ be an irreducible subrepresentation. We know $W_0 \cong V(s)$ for some s. Then W_0 has a highest weight vector w_0, with $w_0 \in \mathfrak{g}_{c\alpha}$ for some c, and

$$[h_\alpha, v_0] = sv_0$$

Case 1: s is even. In this case, 0 is an eigenvalue of h_α. Let e be the eigenvector. But the zero eigenspace of $\mathfrak{h}_\alpha$ on V is $\mathfrak{t}$, which is contained in $K \oplus \mathfrak{m}_\alpha$. Thus, $e \in (K \oplus \mathfrak{m}_\alpha) \cap W_0 = 0$. Contradiction.

Aside: If 2α is a root, then h_α has $2\alpha(h_\alpha) = 4$ as an eigenvalue, but the eigenvalues of h_α on $K \oplus m_\alpha$ are $0, 2, -2$. So the only way this could happen is if W contains an irreducible subrepresentaion $V(s)$, where s is even. With this, if α is a root, then 2α is not a root.

Case 2: s is odd. In this case, 1 is an eigenvalue of h_α. As $\alpha(h_\alpha) = 2$, this means that $\frac{1}{2}\alpha$ is a root. But then by the above,

$$\alpha = 2 \cdot \frac{1}{2}\alpha$$

is not a root. Contradiction. $\square$

Exercise: We have a canonical identification $\mathfrak{m}_\alpha = \mathfrak{m}_{-\alpha}$ and $h_\alpha = h_{-\alpha}$.

Proposition 10.20. Let $\alpha, \beta \in \Phi$ such that $\alpha \neq \pm\beta$. Then

(i) $\beta(h_\alpha) \in \mathbb{Z}$, and we call these the *Cartan integers*.

(ii) there exists integers $p, q \geq 0$ such that if $r \in \mathbb{Z}$, then

$$\beta + r\alpha \in \Phi \iff -p \leq r \leq q$$

Moreover, $p - q = \beta(h_\alpha)$.

(iii) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

Proof. For (i), consider

$$V = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}_{\beta + r\alpha}$$

and let $\mathfrak{m}_\alpha$ act on V by the adjoint action. Let

$$q = \max\{r \in \mathbb{Z} \mid \beta + r\alpha \in \Phi\}$$

In particular, $\beta + q\alpha$ is a root. Choose $v \in \mathfrak{g}_{\beta + q\alpha}$ non-zero. Then

$$[e_\alpha, v] \in \mathfrak{g}_{\beta + (q+1)\alpha} = 0$$

and

$$[h_\alpha, v] = (\beta + q\alpha)(h_\alpha)(v) \in \operatorname{span}\{v\}$$

Thus, v is a highest weight vector, with weight $(\beta + q\alpha)(h_\alpha)$. With this, by $\mathfrak{sl}_2$ representation theory,

$$\beta(h_\alpha) + q\alpha(h_\alpha) \in \mathbb{Z}_{\geq 0}$$

and so

$$\beta(h_\alpha) + 2q \in \mathbb{Z}_{\geq 0}$$

which means that $\beta(h_\alpha) \in \mathbb{Z}$.

For (ii), recall from lemma 4.7 that

$$W = \operatorname{span}\{v, fv, f^2v, \dots\}$$

is an irreducible representation of V, and so h_α acts by

$$\begin{pmatrix} (\beta + q\alpha)(h_\alpha) & & & \\ & (\beta + (q-1)\alpha)(h_\alpha) & & \\ & & \ddots & \\ & & & -(\beta + q\alpha)(h_\alpha) \end{pmatrix}$$

In particular,

$$W = \bigoplus_{r=-p}^{q} \mathfrak{g}_{\beta + r\alpha}$$

for some p, where

$$p = \min\{r \in \mathbb{Z} \mid \beta - r\alpha \in \Phi\}$$

Suppose $W' \leq V$ is a proper subrepresentation. Then W' contains a highest weight vector $w \in \mathfrak{g}_\gamma$, for some γ. Then

$$0 \leq \gamma(h_\alpha) < -(\beta + q\alpha)(h_\alpha) \leq 0$$

Contradiction. Finally,

$$(\beta - p\alpha)(h_\alpha) = -(\beta + q\alpha)(h_\alpha)$$

and rearranging,

$$\beta(h_\alpha) = p - q$$

Lecture 12

For (iii), we already know that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Thus, if $\alpha + \beta$ is not a root we are done. On the other hand, if $\alpha + \beta$ is a root, choose $v \in \mathfrak{g}_\beta$ non-zero. Suppose $[e_\alpha, v] = 0$, then v is a highest weight vector for v. Contradiction. Thus, $[e_\alpha, v] \neq 0$, and $\mathfrak{g}_{\alpha+\beta}$ is spanned by it, as it is one-dimensional. □

Definition 10.21 (reflection)

For $\alpha \in \Phi$, define the *reflection at α* by

$$w_\alpha : \mathfrak{t}^* \to \mathfrak{t}^*$$
$$w_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha$$

Corollary 10.22 (of proposition 10.20)**.** $w_\alpha(\Phi) = \Phi$.

Proof. Let $\beta \in \Phi$, let p, q be as in proposition 10.20. We need to show that $\beta - \beta(h_\alpha)\alpha \in \Phi$. We have that

$$\beta - \beta(h_\alpha)\alpha = \beta - (p - q)\alpha$$

But

$$-p \leq -(p - q) \leq q$$

and so this lives in the root string. □

image

w_α is the reflection in the *root hyperplane*

$$H_\alpha = \{\lambda \in \mathfrak{t}^* \mid \lambda(h_\alpha) = 0\}$$

and this reflection preserves Φ. We will now define a root system as something with the nice properties of Φ, and we'll show that there is a correspondence

$$\{\text{root systems}\} \leftrightarrow \{\text{semisimple Lie algebras}\}$$

11 Root systems

11.1 Roots in Euclidean space

Recall that Φ spans $\mathfrak{t}^*$.

Proposition 11.1. Define a bilinear form on $\mathfrak{t}^*$ by

$$\langle \lambda, \mu \rangle = K(t_\lambda, t_\mu)$$

where K is the Killing form[a], $\lambda, \mu \in \mathfrak{t}^*$. Then

(i) If $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle \in \mathbb{Q}$.

(ii) If $\alpha_1, \ldots, \alpha_\ell$ is a basis of $\mathfrak{t}^*$, and $\beta \in \Phi$, then $\beta = \sum_i c_i \alpha_i$, with $c_i \in \mathbb{Q}$. That is,

$$\dim_{\mathbb{Q}}(\Phi) = \dim_{\mathbb{C}}(\mathfrak{t})$$

(iii) $\langle \cdot, \cdot \rangle$ is positive definite on $\text{span}_{\mathbb{Q}}(\Phi)$.

[a] of $\mathfrak{g}$ restricted to $\mathfrak{t}$

Proof. See Grojnowski notes, Proposition 4.7. For (i), note

$$\beta(h_\alpha) = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$$

□

11.2 Abstract root systems

Let $(E, (\cdot, \cdot))$ be a real Euclidean space. If $\alpha \in E$ is non-zero, define

$$\alpha^\vee : E \to \mathbb{R}$$
$$\alpha^\vee(\lambda) = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}$$

and we define

$$w_\alpha : E \to E$$
$$w_\alpha(\lambda) = \lambda - \alpha^\vee(\lambda)$$

Geometrically, this is reflection in the hyperplane with normal α.

Definition 11.2 (root system)
A finite subset $\Phi \subseteq E$ is a *root system* if

1. $0 \notin \Phi$, Φ spans E,
2. if $\alpha, \beta \in \Phi$, then $\beta^\vee(\alpha) \in \mathbb{Z}$,
3. if $\alpha \in \Phi$, then $w_\alpha(\Phi) = \Phi$,
4. if $\alpha, c\alpha \in \Phi$, then $c = \pm 1$.

Each $\alpha \in \Phi$ is called a *root*.

Remark 11.3. Removing 4. gives a *"non-reduced" root system*.

Notation 11.4. If $\mu \in E, \lambda \in E^*$, we will write

$$\langle \mu, \lambda \rangle = \lambda(\mu)$$

and so $\langle \beta, \alpha^\vee \rangle = \alpha^\vee(\beta)$.

This may seem opposite to the usual convention, but using the canonical isomorphism $E \to E^*$ given by the Riesz representation theorem, the ordering "doesn't matter".

Example 11.5

If $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{t} \leq \mathfrak{t}$ a Cartan subalgebra, Φ is the set of roots associated to Φ, then Φ is a root system in $\operatorname{span}_{\mathbb{R}}(\Phi)$.

Definition 11.6 (rank)

The *rank* of a root system (Φ, E) is $\dim_{\mathbb{R}}(E)$.

Definition 11.7 (isomorphism)

Given root systems $(\Phi, E), (\Phi', E')$, an *isomorphism* is a linear isomorphism $\rho : E \to E'$, with

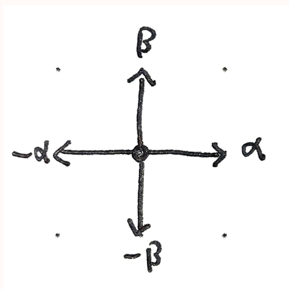
1. $\rho(\Phi) = \Phi'$,
2. $\langle \rho(\alpha), \rho(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle$ for all $\alpha, \beta \in \Phi$.

Example 11.8 (rank 1)

In this case, we only have the A_1 root system, $E = \mathbb{R}, (x, y) = xy$, and $\Phi = \{\alpha, -\alpha\}$ for some $\alpha \neq 0$, $\langle \alpha, \alpha^\vee \rangle = 2$.

Example 11.9 ($A_1 \times A_1$)

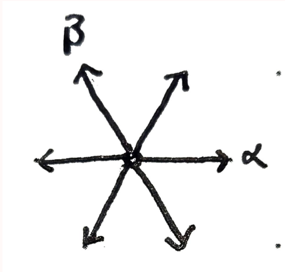
$E = \mathbb{R}^2$, $\Phi = \{\pm\alpha, \pm\beta\}$, given by the standard basis vectors $\pm e_1, \pm e_2$.



Example 11.10 (A_2)

In this case,

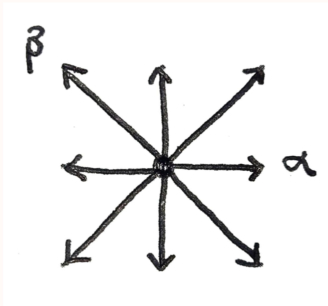
$$\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$$



Example 11.11 (B_2)

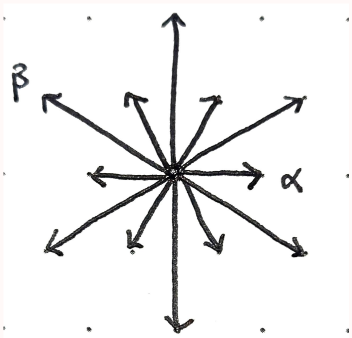
In this case,

$$\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$$



Example 11.12 (G_2)

In this case, the root lattice is:



Returning to Lie algebras,

Example 11.13

$$\mathfrak{sl}_2 = \langle h \rangle \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

with $\alpha(h) = 2$. Taking h as the generator for the Cartan subalgebra, $\mathfrak{sl}_2$ has root system A_1.

Analogously, $\mathfrak{sl}_3$ has root system A_2, if we set

$$h_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad h_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

Then $\alpha(h_1) = 2, \alpha(h_2) = -1, \beta(h_1) = -1, \beta(h_2) = 2$.

Finally, $\mathfrak{sp}_4$ and $\mathfrak{so}_5$ have root system B_2.

Definition 11.14 (Weyl group)

The *Weyl group* of (Φ, E) is a the subgroup W of $\mathrm{GL}(E)$ generated by the w_α.

Note that W is finite. To see this, by definition each w_α acts as a permutation on Φ. As such, we have an embedding of W into $\mathrm{Sym}(\Phi)$, which is finite. The map is an injection as $\mathrm{span}_{\mathbb{R}}(\Phi) = E$, and so if two reflections agree on Φ, then they agree on all of E.

Lecture 13

Example 11.15

1. For A_1, $W \cong C_2$,
2. For A_2, $W \cong D_6 \cong S_3$,
3. For B_2, $W \cong D_8$,
4. For G_2, $W \cong D_{12}$,
5. For $A_1 \times A_1$, $W \cong V_4 = C_2 \times C_2$.

Definition 11.16

If $(\Phi_1, E_1), (\Phi_2, E_2)$ are root systems, then $(\Phi_1 \times \{0\} \cup \{0\} \times \Phi_2, E_1 \oplus E_2)$ is also a root system.

Any root system which can be written in this form, with Φ_1, Φ_2 non-empty, is called *reducible*. Otherwise, it is *irreducible*.

Remark 11.17. By abuse of notation, sometimes we will write it as $(\Phi_1 \sqcup \Phi_2, E_1 \oplus E_2)$.

Example 11.18

$A_1 \times A_1$ is reducible, A_1, A_2, B_2, G_2 are irreducible.

Example 11.19

If Φ corresponds to a Cartan subalgebra $\mathfrak{t}$ in a semisimple Lie algebra $\mathfrak{g}$, then Φ is irreducible when $\mathfrak{g}$ is indecomposable.

Lemma 11.20 (finiteness). If Φ is a root system, $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$, then

$$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$$

Proof. $(\alpha, \beta) = \|\alpha\| \|\beta\| \cos(\theta)$, where θ is the angle between α and β. So

$$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = \frac{4(\alpha, \beta)^2}{\|\alpha\|^2 \|\beta^2\|} = 4\cos^2(\theta) \in \mathbb{Z}$$

Now $\cos^2(\theta) \in [0, 1]$, and so $\cos^2(\theta) \in \{0, 1/4, 1/2, 3/4, 1\}$. But they are not parallel, and so $\cos^2(\theta) \neq 1$. $\square$

In particular, this puts constraints on the angles, and the ratios of lengths.

- 0 corresponds to $\theta = \pi/2$, and so there is no constraint on the lengths,
- 1 corresponds to $\theta = \pi/3$, and the ratio of lengths is 1 (i.e. they have the same length),
- 2 corresponds to $\theta = \pi/4$, and the ratio of lengths is $\sqrt{2}$,
- 3 corresponds to $\theta = \pi/6$, and the ratio of lengths is $\sqrt{3}$.

Corollary 11.21. If Φ is a root system, and α, β are roots, then

$$\langle \alpha, \beta^\vee \rangle \in \{0, \pm 1, \pm 2, \pm 3\}$$

Exercise: The only rank 2 systems are, up to isomorphism, $A_1 \times A_1, A_2, B_2, G_2$. Use the angles and length ratios from finiteness.

Corollary 11.22. If Φ is an irreducible root system, then $||\alpha||^2$ can take at most two values as $\alpha \in \Phi$ varies.

Proof. Exercise. Suppoe not, then we get a contradiction due to the fact that the $\langle \alpha, \beta^\vee \rangle$ are integers. □

Definition 11.23 (simply laced)
An irreducible root system Φ is *simply-laced* if all the roots are of the same length.

Example 11.24
A_1, A_2 are simply laced, B_2, G_2 are not.

Exercise: If Φ is simply laced, then (Φ, E) is isomorphic to a root system (Φ', E'), where $\langle \alpha, \beta^\vee \rangle \in \{0, \pm 1\}$ for all $\alpha, \beta \in \Phi', \alpha \neq \pm\beta$. This follows from the length ratio constraints above.

12 Weyl chambers and root bases

Throughout, (Φ, E) is a root system.

For a root $\alpha \in \Phi$, we have the *root hyperplane*

$$H_\alpha = \{\lambda \in E \mid \langle \lambda, \alpha^\vee \rangle = 0\}$$

corresponding to α. The connected components of

$$E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$$

are called the *Weyl chambers*.

A subset $\Delta = \{\alpha_1, \ldots, \alpha_\ell\} \subseteq \Phi$ is called a *root basis*, or a *base*, if

1. Δ is a basis for E,
2. if $\alpha \in \Phi$, with

$$\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$$

then all of the c_i have the same sign (or are zero). i.e. $c_i c_j \geq 0$.

Definition 12.1 (simple roots, positive and negative roots)
The elements of Δ are called *simple roots*. If

$$\alpha = \sum_i c_i \alpha_i$$

with all $c_i \geq 0$, then we call α a *positive root*, denoted $\alpha > 0$ (or $\alpha \succ 0$). If all $c_i \leq 0$, we call α a *negative root*.

The set of all positive roots is denoted Φ^+, and $\Phi^- = -\Phi^+$ the set of negative roots.

Note that $H_\alpha = H_{-\alpha}$. In particular, the Weyl chambers are all of the form

$$C_\varepsilon = \left\{\lambda \in E \mid \varepsilon_\alpha \left\langle \lambda, \alpha^\vee \right\rangle > 0\right\}$$

where $\varepsilon_\alpha \in \{\pm 1\}$. Note on the other hand not all choices of (ε_α) give a Weyl chamber.

Remark 12.2. Δ defines a partial order on E, by saying

$$\mu < \lambda \iff \lambda - \mu \text{ is a sum of positive root or } \lambda = \mu$$

Lemma 12.3. Let W be the Weyl group of (Φ, E). Then if Δ is a base, and $w \in W$, then $w(\Delta)$ is a base.

Proof. We know that w is invertible, so $w(\Delta)$ is a basis for E, and as w acts on Φ, $w(\Delta) \subseteq \Phi$. If $\alpha \in \Phi$ is a root, with

$$\alpha = \sum_i c_i \alpha_i$$

with all $c_i \geq 0$ (without loss of generality). Then

$$w(\alpha) = \sum_i c_i w(\alpha_i)$$

with all $c_i \geq 0$. □

That is, the Weyl group acts on the set of root bases. It remains to show how to construct a root basis.

- Choose $\gamma \in E \setminus \bigcup_\alpha H_\alpha$ (i.e. γ in a Weyl chamber). Define

  $$\Phi_\gamma^+ = \{\alpha \in \Phi \mid \left\langle \gamma, \alpha^\vee \right\rangle > 0\}$$

  and $\Phi_\gamma^- = -\Phi_\gamma^+$. Note $\Phi = \Phi_\gamma^+ \cup \Phi_\gamma^-$.
- Define

  $$\Delta_\gamma = \{\alpha \in \Phi_\gamma^+ \mid \alpha \neq \beta_1 + \beta_2 \text{ for all } \beta_1 + \beta_2 \in \Phi_\gamma^+\}$$

Theorem 12.4. (i) Δ_γ is a root basis,

(ii) every root basis is of this form Δ_γ for some γ in a Weyl chamber.

Proof. For (i),

Claim 12.5. If $\alpha, \beta \in \Delta_\gamma$, then $\alpha - \beta \notin \Delta_\gamma$.

Proof. Suppose $\alpha, \beta \in \Delta_\gamma$. Without loss of generality $\alpha - \beta \in \Phi_\gamma^+$. Otherwise, take $\beta - \alpha$. Then

$$\alpha = (\alpha - \beta) + \beta$$

Contradicting the definition. □

Lecture 14

Claim 12.6. If $\alpha, \beta \in \Delta_\gamma$ are distinct, then $\langle \alpha, \beta^\vee \rangle = 0$.

Proof. Recall from lemma 11.20 that

$$\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$$

Suppose $\langle \alpha, \beta^\vee \rangle > 0$. Without loss of generality, assume $\langle \alpha, \beta^\vee \rangle = 1$. Otherwise, consider $\langle \beta, \alpha^\vee \rangle$. Now

$$w_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha - \beta \in \Delta_\gamma$$

But w_β preserves Δ_γ, contradicting claim 12.5. □

Claim 12.7. Let $\Delta_\gamma = \{\alpha_1, \dots, \alpha_\ell\}$, $\alpha \in \Phi_\gamma^+$,

$$\alpha = \sum_{i-1}^{n} c_i \alpha_i$$

then $c_i \geq 0$ for all i.

Proof. Suppose not. Choose an α which cannot be written this way, and with (α, γ) minimal. By construction, $\alpha \notin \Delta_\gamma$, hence $\alpha = \beta_1 + \beta_2$ where $\beta_1, \beta_2 \in \Phi_\gamma^+$. With this, $(\alpha, \gamma) = (\beta_1, \gamma) + (\beta_2, \gamma)$. By definition, $\beta_i \in \Phi_\gamma^+$, and so $(\beta_i, \gamma) > 0$, hence $(\alpha, \gamma) > (\beta_i, \gamma)$. With this, β_1, β_2 can be written as a $\mathbb{Z}_{\geq 0}$-linear combination of the α_i. But then so can $\alpha = \beta_1 + \beta_2$. Contradiction. □

If we worked with Φ_γ^- instead, this means that every element in Φ_γ^- is a non-positive linear combination instead. Also, Δ_γ spans E as Φ spans E.

Claim 12.8. Δ_γ is a linearly independent set.

Proof. Suppose for some $c_i \in \mathbb{R}$, $\sum_i c_i \alpha_i = 0$. Without loss of generality, we can assume $c_i \geq 0$ for $1 \leq i \leq m$ and $c_i \leq 0$ for $m + 1 \leq i \leq \ell$. Set

$$v = \sum_{i=1}^{m} c_i \alpha_i = -\sum_{j=m+1}^{\ell} c_j \alpha_j$$

Now consider

$$(v, v) = -\sum_{i=1}^{m} \sum_{j=m+1}^{\ell} \underbrace{c_i c_j}_{\leq 0} \underbrace{(\alpha_i, \alpha_j)}_{\leq 0} \leq 0$$

Hence $v = 0$. With this,

$$0 = (\gamma, v) = \sum_{i=1}^{m} c_i \underbrace{(\gamma, \alpha_i)}_{>0}$$

Hence $c_i = 0$ for $1 \leq i \leq m$. Similarly, the other c_i are zero as well. □

For (ii), see Humphreys §10.1. □

Corollary 12.9. We have a bijection

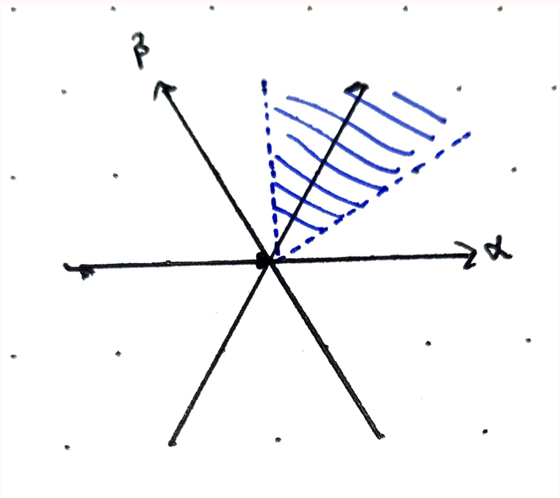
$$\{\text{Weyl chambers}\} \leftrightarrow \{\text{root bases}\}$$

Proof. Given a Weyl chamber C, we can choose $\gamma \in C$, and we have a root basis Δ_γ. Conversely given Δ, $\Delta = \Delta_\gamma$ for some γ, which in turn is in a Weyl chamber C. □

Notation 12.10. Write $C_\Delta = C_\gamma$ if $\Delta = \Delta_\gamma$, where C_γ is the Weyl chamber containing γ. We call it the *fundamental Weyl chamber relative to* Δ.

Example 12.11

For example, with the root system A_2, and the root basis $\{\alpha, \beta\}$, the fundamental Weyl chamber is



Definition 12.12 (height)

If $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a root basis, $\alpha \in \Phi$, say

$$\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$$

The *height* of α is

$$\sum_{i=1}^{\ell} c_i$$

Often it is useful when proving statements to induct on the height of a root.

Lemma 12.13. If $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a root basis, $\beta \in \Phi^+ \setminus \Delta$, then there exists i such that $\beta - \alpha_i \in \Phi$.

Proof. Given β, if $(\beta, \alpha_i) \leq 0$ for all i, then $\Delta \cup \{\beta\}$ is a linearly independent set. So there exists i such that

$$\left\langle \beta, \alpha_i^\vee \right\rangle > 0$$

Since $\langle \beta, \alpha_i^\vee \rangle \langle \alpha_i, \beta^\vee \rangle \in \{0, 1, 2, 3\}$, then $\langle \beta, \alpha_i^\vee \rangle = 1$ or $\langle \alpha_i, \beta^\vee \rangle = 1$. That is,

$$w_{\alpha_i}(\beta) = \beta - \alpha_i \text{ or } w_\beta(\alpha_i) = \alpha_i - \beta$$

In either case, $\beta - \alpha_i \in \Phi$. □

Corollary 12.14. If $\beta \in \Phi^+$, then β can be written as a sum

$$\beta = \sum_{j=1}^{n} \alpha_{i(j)}$$

where $\alpha_{i(j)}$ are not-necessarily distinct simple roots, and each partial sum is a root, i.e.

$$\sum_{j=1}^{k} \alpha_{i(j)} \in \Phi$$

Proof. Use the lemma and induction on the height of β. □

13 Facts about the Weyl group

Recall that W acts on the set of root bases, and so it preserves the Weyl chambers.

Lemma 13.1. If $w \in W$, $\lambda, \mu \in E$, then

$$\langle \lambda, \mu^\vee \rangle = \langle w(\lambda), w(\mu)^\vee \rangle$$

Using this, we can deduce that

Proposition 13.2. If Δ is a root basis, and $w \in W$, then

$$C_{w(\Delta)} = w(C_\Delta)$$

Lemma 13.3. For Φ a root system, Δ a root basis, and W the Weyl group, $\alpha \in \Delta$, then w_α permutes $\Phi^+ \setminus \{\alpha\}$.

Lecture 15

Proof. Take $\alpha_1 \in \Delta$, where $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, take $\beta \in \Phi^+$, $\beta \neq \alpha_1$. In particular, we can write

$$\beta = \sum_i c_i \alpha_i$$

with $c_i \in \mathbb{Z}_{\geq 0}$. In this case,

$$\begin{aligned} w_{\alpha_1}(\beta) &= \beta - \langle \beta, \alpha_1^\vee \rangle \alpha_1 \\ &= (c_i - \langle \beta, \alpha_1^\vee \rangle)\alpha_1 + \sum_{i=2}^{\ell} c_i \alpha_i \end{aligned}$$

Since β is a positive root and it is not α_1, $w_\alpha(\beta) \neq \pm\alpha_1$. Hence $c_i > 0$ for some $i \geq 2$, hence $w_{\alpha_1}(\beta)$ is a positive root. □

Theorem 13.4. (i) the Weyl group acts *simply transitively* (or *regular*, or *sharply transitively*) on the set of root bases (and on the set of Weyl chambers).

(ii) given a root basis Δ and $\alpha \in \Phi$, then there exists $w \in W$ such that $w(\alpha) \in \Delta$. This w is not necessarily unique.

(iii) if $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ is a root basis, then W is generated by $w_{\alpha_1}, \dots, w_{\alpha_\ell}$.

Proof. Omitted. See Humphreys §10.3 □

14 Classification of irreducible root systems

Throughout, let (Φ, E) be a root system, and a root basis $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Let W be the Weyl group of Φ.

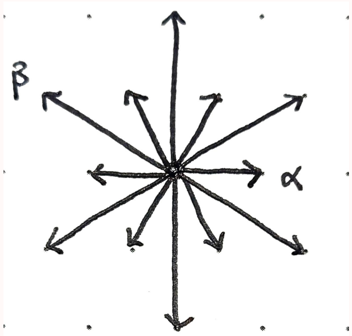
Definition 14.1 (Cartan matrix)
The *Cartan matrix* of Φ is the $\ell \times \ell$ matrix,

$$C = \left(\langle \alpha_i, \alpha_j^\vee \rangle\right)_{1 \le i,j \le \ell}$$

This is independent of the choice of root basis (up to permutation), since given Δ' another root basis, there exists $w \in W$ with $w(\Delta) = \Delta'$, and the action of W preserves $\langle \cdot, \cdot \rangle$.

Note $\det(C) \neq 0$. This follows from the fact that Δ is a basis of E.

Example 14.2
Recall the root system G_2, given by



Let $\alpha_1 = \alpha, \alpha_2 = \beta$. In this case, $\langle \alpha_1, \alpha_2^\vee \rangle = -1, \langle \alpha_2, \alpha_1^\vee \rangle = -3$. Hence the Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Example 14.3
Similarly, for $A_1 \times A_1$, we have

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

For A_2 we have

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and for B_2, we have

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

Proposition 14.4. Suppose (Φ', E') is another root system, with root basis $\{\alpha_1', \ldots, \alpha_\ell'\}$, with

$$\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_i', (\alpha_j')^\vee \rangle$$

Then the linear map $\alpha_i \mapsto \alpha_i'$ induces an isomorphism ϕ of root systems, with

$$\langle \phi(\alpha), \phi(\beta)^\vee \rangle = \langle \alpha, \beta^\vee \rangle$$

for all $\alpha, \beta \in \Phi$. Hence the Cartan matrix of Φ determines Φ up to isomorphism.

Proof. Since Δ is a basis for E, and Δ' a basis for E', we have a unique linear isomorphism $\phi : E \to E'$, with $\phi(\alpha_i) = \alpha_i'$. If $\alpha, \beta \in \Delta$, then

$$\begin{aligned} w_{\phi(\alpha)}(\phi(\beta)) &= w_{\alpha'}(\beta') = \beta' - \left\langle \beta', (\alpha')^\vee \right\rangle \alpha' \\ &= \phi(\beta) - \left\langle \beta, \alpha^\vee \right\rangle \phi(\alpha) \\ &= \phi(\beta - \left\langle \beta, \alpha^\vee \right\rangle \alpha) \\ &= \phi(w_\alpha(\beta)) \end{aligned}$$

That is, we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ {\scriptstyle w_\alpha}\downarrow & & \downarrow{\scriptstyle w_{\phi(\alpha)}} \\ E & \xrightarrow[\phi]{} & E' \end{array}$$

Now use theorem 13.4, the respective Weyl groups W, W' are generated by simple reflections, and so the map

$$w \mapsto \phi \circ w \circ \phi^{-1}$$

is an isomorphism $W \to W'$, sending w_α to $w_{\phi(\alpha)}$ for each $\alpha \in \Delta$. Each $\beta \in \Phi$ is conjugate under W to a simple root, say $\beta = w(\alpha)$ for some $\alpha \in \Delta$. This implies that

$$\phi(\beta) = \phi(w(\alpha)) = (\phi w \phi^{-1})\phi(\alpha) \in \Phi'$$

Hence ϕ maps Φ onto Φ'. Using the formula for reflections, ϕ preserves the Cartan integers, i.e.

$$\left\langle \alpha, \beta^\vee \right\rangle = \left\langle w(\phi(\alpha)), w(\phi(\beta))^\vee \right\rangle$$

□

Remark 14.5. The proposition suggests it is possible, in principle, to recover the root systems Φ from the Cartan integers. See Humphreys for a reference.

Definition 14.6 (Coxeter graph)

Recall if $\alpha \neq \pm\beta$, then

$$\left\langle \alpha, \beta^\vee \right\rangle \left\langle \beta, \alpha^\vee \right\rangle \in \{0, 1, 2, 3\}$$

The *Coxeter graph* of Φ is a graph with ℓ vertices, for $i \neq j$, we join the i-th vertex to the j-th with $\left\langle \alpha_i, \alpha_j^\vee \right\rangle \left\langle \alpha_j, \alpha_i^\vee \right\rangle$ edges.

The Coxeter graph determines $\left\langle \alpha_i, \alpha_j^\vee \right\rangle$ in the case where all the roots have the same length (i.e. it is simply-laced). In the case where more than one root length occurs (e.g. for B_2, G_2), the graph fails to tell us which of a pair of vertices should correspond to a short simple root, and which corresponds to a long simple root (in the case where the vertices are joined by two or three edges).

Remark 14.7. The Coxeter graph determines W as it determines the order of products of the generators of W.

Definition 14.8

The *Dynkin diagram* of Φ is the Coxeter graph, but if a multiple edge between vertices occurs, we add an arrow to point to the shorter root.

Example 14.9

For ranks 1 and 2, we have the Dynkin diagrams

1. A_1:
2. $A_1 \times A_1$:
3. A_2:
4. B_2:
5. G_2: [a]

[a] in lectures G_2 was drawn the other way around, i.e.

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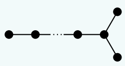
Remark 14.10. The maximum number of edges between two vertices in a Dynkin diagram is 3, and a root system is simply laced if and only if its Dynkin diagram has no multiple edges.

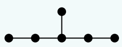
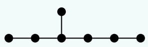
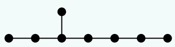
Exercise: Φ is irreducible if and only if its Dynkin diagram is (simply) connected.

Theorem 14.11. Let Φ be an irreducible root system, then its Dynkin diagram is one of the following:

(I) Classical root systems (with rank ℓ):

- A_ℓ ($\ell \geq 1$):
- B_ℓ ($\ell \geq 2$):
- C_ℓ ($\ell \geq 3$):
- D_ℓ ($\ell \geq 4$): 

(II) Exceptional root systems:

- E_6: 
- E_7: 
- E_8: 
- F_4:
- G_2:

Remark 14.12. The restriction on ℓ is included so that we don't have repetitions. For example, $B_2 = C_2$ and so on.

Proof. See Humphreys. Alternatively, do five pages of Euclidean geometry. □

Theorem 14.13. For every Dynkin diagram $\mathcal{D}$ listed above, there exists a simple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{t}$, roots Φ corresponding to $\mathfrak{t}$, such that the Dynkin diagram corresponding to Φ is $\mathcal{D}$.

Proof sketch. For A_ℓ, let $e_1, \ldots, e_{\ell+1}$ be the standard basis of $\mathbb{R}^{\ell+1}$. Let

$$\Phi = \left\{ e_i - e_j \mid i \neq j \right\} \subseteq \mathbb{R}^{\ell+1}$$

We can see that Φ spans an ℓ-dimensional subspace E of $\mathbb{R}^{\ell+1}$. Then Φ is a root system in E, and it has a root basis given by

$$\alpha_i = e_i - e_{i+1}$$

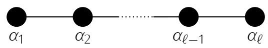
Note for $i < j$,

$$e_i - e_j = (e_i - e_{i+1}) + (e_[i+1] - e_{i+2}) + \cdots + (e_{j-1} - e_j)$$

Also,

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} -1 & |i-j| = 1 \\ 2 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Hence the Dynkin diagram in this case is A_ℓ:



Now note w_{α_i} flips the i-th and $i+1$-th coordinates, so $W = S_{\ell+1}$. The corresponding Lie algebra is $\mathfrak{sl}_{\ell+1}(\mathbb{C})$, with Cartan subalgebra $\mathfrak{t}$ of diagonal matrices, and

$$\alpha_i \left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{\ell+1} \end{pmatrix} \right) = t_i - t_{i+1}$$

For the other classical root systems, with e_i a basis of $\mathbb{R}^\ell$ and $\mathfrak{t}$ the Cartan subalgebra of diagonal matrices.

Type	$\Phi \subseteq \mathbb{R}^\ell$	$\Delta \subseteq \Phi$	W	$\mathfrak{g}$	Dimension
B_ℓ	$\{\pm e_i, \pm e_i \pm e_j \mid i \neq j\}$	$\{e_i - e_{i+1}\} \cup \{e_\ell\}$	$S_\ell \ltimes C_2^\ell$	$\mathfrak{so}_{2\ell+1}$	$2\ell^2 + \ell$
C_ℓ	$\{\pm 2e_i, \pm e_i \pm e_j \mid i \neq j\}$	$\{e_i - e_{i+1}\} \cup \{2e_\ell\}$	$S_\ell \ltimes C_2^\ell$	$\mathfrak{sp}_{2\ell}$	$2\ell^2 + \ell$
D_ℓ	$\{\pm e_i \pm e_j \mid i \neq j\}$	$\{e_i - e_{i+1}\} \cup \{e_{\ell-1} + e_\ell\}$	$S_\ell \ltimes C_2^{\ell-1}$	$\mathfrak{so}_{2\ell}$	$2\ell^2 - \ell$

For the Weyl group for B_ℓ, S_ℓ acts on the coordinates, and C_2 acts as a sign change on each coordinate. For the exceptional types, see Humphreys (or Erdmann and Wildon). We summarise some of the results:

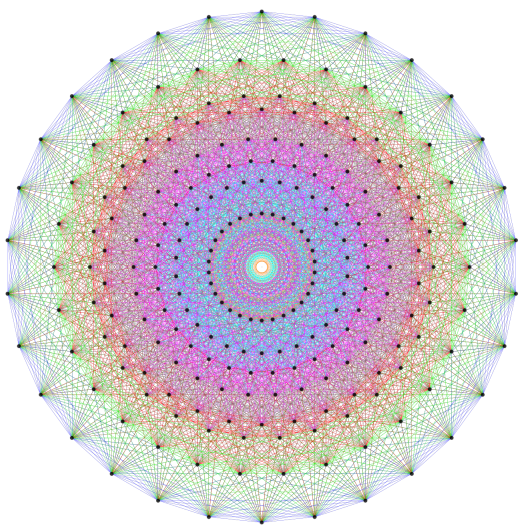
- G_2, there are 12 roots, $E = \langle e_1 + e_2 + e_3 \rangle^\perp \leq \mathbb{R}^3$
- F_4, $E = \mathbb{R}^4$, $|\Phi| = 48$, $|W| = 1152$,
- E_6, E_7, E_8: First do E_8, and find root systems of types E_7, E_6 as subsets. For E_8: $E = \mathbb{R}^8$, $|\Phi| = 240$, $|W| = 2^14 \cdot 3^5 \cdot 5^2 \cdot 7$.

□

For E_8, let $\alpha_1, \dots, \alpha_8$ be its root basis, and we have a *Coxeter element*

$$w_c = \prod_{i=1}^{8} w_{\alpha_i}$$

which has order 30. There is a plane of $\mathbb{R}^8$, on which w_c acts as a rotation. Image stolen from Wikipedia:



Remark 14.14. To look up root systems, see the Spherical explorer.

Remark 14.15. To do computations, it is useful to compute things in terms of a root basis.

Remark 14.16. We have Lie algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ corresponding to the exceptional Lie algebras.

In particular, $\mathfrak{g}_2$ is the algebra of "derivations of octonions $\mathbb{O}$", where a *derivation* is a linear map δ such that

$$\delta(ab) = \delta(a)b + a\delta(b)$$

$\mathbb{O}$ is an 8-dimensional normed division algebra over $\mathbb{R}$, and it has a one-dimensional centre span$\{1\}$, on which $\mathfrak{g}_2$ acts trivially. There is a representation $\mathfrak{g}_2 \to \mathfrak{so}_7$, which is the lowest dimensional non-trivial representation. See Humphreys §19.3. Others can be constructed, see Fulton-Harris §22.4.

Remark 14.17. Given Φ, there is a natural construction of a Lie algebra with Φ as its root system.

Lecture 17

So far, we have found correspondences

$$\{\text{simple Lie algebra } \mathfrak{g} \text{ with CSA } \mathfrak{t}\} \twoheadrightarrow \{\text{irred. root systems } \Phi\} \leftrightarrow \{\text{connected Dynkin diagrams}\}$$

We will now show that

- the root system corresponding to $\mathfrak{g}$ is independent of the choice of Cartan subalgebra $\mathfrak{t}$,
- two Lie algebras with the same root system are isomorphic.

9 Brief introduction to inner automorphisms

An *automorphism of* $\mathfrak{g}$ is an isomorphism $\mathfrak{g} \to \mathfrak{g}$. The group of all such is called Aut($\mathfrak{g}$). For example, if $\mathfrak{g} = \mathfrak{gl}(V)$ or $\mathfrak{sl}(V)$, $A \in \mathrm{GL}(V)$, then

$$x \mapsto AxA^{-1}$$

is an automorphism of g.

Let V be a finite dimensional, choose $x \in \mathfrak{g}$ such that ad(x) is nilpotent, say $(\mathrm{ad}(x))^m = 0$. Then

$$\exp(\mathrm{ad}(x)) = 1 + \mathrm{ad}(x) + \frac{(\mathrm{ad}(x))^2}{2} + \cdots + \frac{(\mathrm{ad}(x))^{m-1}}{(m-1)!}$$

It is easy to see that $\exp(\mathrm{ad}(x)) \in \mathrm{Aut}(\mathfrak{g}) = \mathrm{GL}(\mathfrak{g})$, and an automorphism of this form is called *inner*. The subgroup of Aut($\mathfrak{g}$) generated by these is called Inn($\mathfrak{g}$), and this is a normal subgroup of Aut($\mathfrak{g}$). This is because if $\phi \in \mathrm{Aut}(\mathfrak{g})$, $x \in \mathfrak{g}$, then $\phi\,\mathrm{ad}(x)\phi^{-1} = \mathrm{ad}(\phi(x))$, and so

$$\phi \exp(\mathrm{ad}(x))\phi^{-1} = \exp(\mathrm{ad}(\phi(x)))$$

Lemma 9.1. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be a complex Lie algebra, and $x \in \mathfrak{g}$ nilpotent. Then ad(x) is nilpotent, and

$$\exp(x)y\exp(x)^{-1} = \exp(\mathrm{ad}(x))y$$

for all $y \in \mathfrak{g}$.

Proof. Humphreys §2.3. □

Example 9.2 1. $\mathrm{Inn}(\mathfrak{sl}_n(\mathbb{C})) = \mathrm{GL}_n(\mathbb{C})/Z$,

2. $\mathrm{Inn}(\mathfrak{so}_n(\mathbb{C})) = \mathrm{SO}_n(\mathbb{C})/Z$,

3. $\mathrm{Inn}(\mathfrak{sp}_n(\mathbb{C})) = \mathrm{Sp}_n(\mathbb{C})/Z$,

Let G be a matrix Lie group, and $\mathfrak{g} = T_e G$ its Lie algebra. We have the *exponential map* $\exp : \mathfrak{g} \to G$. For $g \in G$, define

$$\begin{aligned} C_g : G &\to G \\ x &\mapsto gxg^{-1} \end{aligned}$$

for the conjugation map. Differentiating this, at $e \in G$, we get

$$\begin{aligned} \mathrm{Ad}_g : \mathfrak{g} &\to \mathfrak{g} \\ x &\mapsto gxg^{-1} \end{aligned}$$

In particular, $\mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g})$, and so $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ defines a representation. This map also happens to be smooth, and so we can differentiate it again, to get

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

which is the matrix commutator. What lemma 9.1 says is then

$$\mathrm{Ad}_{\exp(x)} = \exp(\mathrm{ad}(x))$$

In fact, this is true for general Lie groups, we have that if $\varphi : G \to H$ is a homomorphism, then

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & H \\ \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow{\ \mathrm{d}\varphi\ } & \mathfrak{h} \end{array}$$

commutes. When $\varphi = \mathrm{Ad}$, $\mathrm{d}\varphi = \mathrm{ad}$, we get that $\exp(\mathrm{ad}(x)) = \mathrm{Ad}_{\exp(x)}$.

15 Conjugacy results

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{t}$ a Cartan subalgebra, Φ the root system corresponding to $\mathfrak{t}$, and so we have a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

and $\Delta \subseteq \Phi$ a root basis.

Lemma 15.1. If $\mathfrak{t}'$ is another Cartan subalgebra of $\mathfrak{g}$, then there exists an (inner) automorphism $\Psi \in \mathrm{Inn}(\mathfrak{g})$, with $\psi(\mathfrak{t}) = \mathfrak{t}'$.

Proof. Humphreys §16.4, Carter page 34. Read at your peril. □

Definition 15.2 (rank)

The *rank* of a Lie algebra $\mathfrak{g}$ is the dimension of a Cartan subalgebra, which is independent of the choice of Cartan subalgebra. If $\mathfrak{g}$ is semisimple, then

$$\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\Phi)$$

where Φ is the root system of $\mathfrak{g}$ corresponding to a Cartan subalgebra $\mathfrak{t}$.

Lemma 15.3. If $\mathfrak{t}'$ is another Cartan subalgebra of $\mathfrak{g}$, with root system Φ', then Φ and Φ' are isomorphic.

Proof. Let $\psi \in \mathrm{Inn}(\mathfrak{g})$, be as in lemma 15.1. Take $t \in \mathfrak{t}, \alpha \in \Phi, e_\alpha \in \mathfrak{g}_\alpha$. Then

$$[\psi(t), \psi(e_\alpha)] = \psi([t, e_\alpha]) = \psi(\alpha(t)e_\alpha) = \alpha(t)\psi(e_\alpha)$$

As $\psi(e_\alpha)$ spans the root space for $\mathfrak{t}'$,

$$\Phi' = \{\alpha \circ \psi^{-1} \mid \alpha \in \Phi\} = (\psi^{-1})^*(\Phi)$$

□

Theorem 15.4. If $\mathfrak{g}'$ is a semisimple Lie algebra, with root system Φ (the same as $\mathfrak{g}$), then $\mathfrak{g} \cong \mathfrak{g}'$.

Proof. See Carter Ch 7, using the theory of (finite) structure constants. Choose a basis h_α of $\mathfrak{t}$, and e_α in each root space $\mathfrak{g}_\alpha$, so that

$$[e_\alpha, e_{-\alpha}] = h_\alpha$$

This gives a basis of $\mathfrak{g}$ (consistent with $\mathfrak{sl}_2$ theory), with

$$[h_\alpha, h_\beta] = 0 \text{ for } \alpha \neq \beta$$
$$[h_\alpha, e_\beta] = \beta(h_\alpha)e_\beta$$
$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta}e_{\alpha+\beta} & \alpha + \beta \in \Phi \\ h_\alpha & \beta = -\alpha \\ 0 & \alpha + \beta \notin \Phi \cup \{0\} \end{cases}$$

□

16 Weights

Example 16.1

Let $\mathfrak{g} = \mathfrak{so}_5$, we have simple roots α, β for the root system Φ of $\mathfrak{g}$, which is of type B_2. Recall

$$\mathfrak{m}_\alpha = \mathfrak{g}_\alpha \oplus \langle[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]\rangle \oplus \mathfrak{g}_{-\alpha} \cong \mathfrak{sl}_2$$

We can decompose the adjoint representation of $\mathfrak{g}$ under the action of $\mathfrak{m}_\alpha$.

If $e_\alpha \in \mathfrak{g}_\alpha$, then

$$e_\alpha \cdot \mathfrak{g}_\gamma = \mathfrak{g}_{\alpha+\gamma}$$

for all $\gamma \in \Phi$. That is, each α-root string corresponds to an irreducible subrepresentation $\mathfrak{m}_\alpha$ of $\mathfrak{g}$. In fact,

$$\mathfrak{g}|_{\mathfrak{m}_\alpha} = V(0) \oplus V(2) \oplus V(2) \oplus V(2)$$

Similarly,

$$\mathfrak{g}|_{\mathfrak{m}_\beta} = V(0) \oplus V(0) \oplus V(0) \oplus V(1) \oplus V(1) \oplus V(2)$$

Let (Φ, E) be a root system, and fix a base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of simple roots.

Definition 16.2 (root lattice, weight lattice)

The *root lattice* is

$$\mathbb{Z}\Phi = \left\{\sum_{\alpha\in\Phi} c_\alpha \alpha \mid c_\alpha \in \mathbb{Z}\right\} \subseteq E$$

and the *weight lattice* is

$$X = \{\lambda \in E \mid \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$$

In the case of a semisimple Lie algebra $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{t}$, we have

$$\{\mathfrak{t} \in \mathfrak{t}^* \mid \beta(h_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$$

The elements of X are called *weights*.

Note

- $\mathbb{Z}\Phi \subseteq X$,
- if $\lambda \in X$, with $w(\lambda) \in X$ for all $w \in W$, since $\langle \lambda, \alpha^\vee \rangle = \langle w(\lambda), w(\alpha)^\vee \rangle$.
- the root lattice is a lattice in E, since it is the $\mathbb{Z}$-span of an $\mathbb{R}$-basis.

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Lemma 16.3. $\lambda \in X$ if and only if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in \Delta$.

Proof. Examples sheet 3. □

Definition 16.4 (fundamental weights)
For each $1 \le i \le \ell$, define $\omega_i \in E$ by

$$\left\langle \omega_i, \alpha_j^\vee \right\rangle = \delta_{ij}$$

We call the $\{\omega_i\}$ the *fundamental weights with respect to* Δ.

By lemma 16.3,

$$X = \left\{ \sum_i c_i \omega_i \,\middle|\, c_i \in \mathbb{Z} \right\}$$

Moreover, $X/\mathbb{Z}\Phi$ is a finite group, called the *fundamental group*. Moreover,

$$\left| \frac{X}{\mathbb{Z}\Phi} \right| = \det(C)$$

where C is the Cartan matrix of $\mathfrak{g}$. The number $|X/\mathbb{Z}\Phi|$ is sometimes called the *index of connection*.

Example 16.5
For $\mathfrak{g} = \mathfrak{sl}_2$, $\Phi = \{\pm\alpha\}$, $\mathbb{Z}\Phi = \mathbb{Z}\alpha$, and $\langle \alpha, \alpha^\vee \rangle = 2$, and so $X = \mathbb{Z}(\alpha/2)$. In this case, $|X/\mathbb{Z}\Phi| = 2 = \det\big((2)\big)$. More generally for type A_ℓ, $|X/\mathbb{Z}\Phi| = \ell + 1$.

Definition 16.6 (dominant)
$\lambda \in X$ is called *dominant* if $\langle \lambda, \alpha^\vee \rangle \ge 0$ for all $\alpha \in \Phi^+$. If the inequality is strict for all α, it is called *strongly dominant*. The set of dominant weights is denoted X^+.

This is equivalent to:

- λ lies in the closure of the fundamental Weyl chamber with respect to Δ,
- $\lambda = \sum_i c_i \omega_i$, where each $c_i \ge 0$.

Now assume $\mathfrak{g}$ is a semisimple Lie algebra, with a Cartan subalgebra $\mathfrak{t}$ and root system Φ. Choose $e_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Phi$, with

$$[e_\alpha, e_{-\alpha}] = h_\alpha$$

and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ a finite dimensional representation over $\mathbb{C}$.

Lemma 16.7.

$$V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda$$

where

$$V_\lambda = \{v \in V \mid tv = \lambda(t)v \text{ for all } t \in \mathfrak{t}\}$$

Proof. Clear from lemma 10.4, where the commuting semisimple elements are the basis elements of $\mathfrak{t}$. □

Recall for $\lambda, \mu \in \mathfrak{t}^*$, we write

$$\mu \leq \lambda \iff \lambda - \mu = \sum_i k_i \alpha_i$$

where each $k_i \geq 0$. If V is a representation of $\mathfrak{g}$, we say

- The *weight* of a non-zero $v \in V$ is λ if $v \in V_\lambda$,
- $\lambda \in \mathfrak{t}^*$ is a *highest weight* if $V_\lambda \neq 0$ and if $V_\mu \neq 0$, then $\mu \leq \lambda$.

Proposition 16.8. (i) if $v \in V_\lambda$, then $e_\alpha V_\lambda = V_{\alpha+\lambda}$,

(ii) if V_λ is non-zero, then $\lambda \in X$. That is, $\lambda(h_\alpha) \in \mathbb{Z}$ for all α,

(iii) $\dim(V_\lambda) = \dim(V_{w(\lambda)})$ for all $w \in W$.

Proof. For (i), fix $t \in \mathfrak{t}$. Then

$$t(e_\alpha v) = ([t, e_\alpha] + e_\alpha t)v = \alpha(t)e_\alpha v + e_\alpha \lambda(t)v = (\alpha + \lambda)(t)e_\alpha v$$

For (ii), consider $V|_{\mathfrak{m}_\alpha}$. Then h_α acts by integer weights, so $\lambda(h_\alpha) \in \mathbb{Z}$.
For (iii), first of all, it is enough to assume $w = w_\alpha$. Now

$$V|_{\mathfrak{m}_\alpha} = \bigoplus_j V^{(j)}$$

where $V^{(j)}$ are $\mathfrak{m}_\alpha$ irreducible representations. The h_α weight spaces of $V^{(j)}$ are 1-dimensional, and so we can choose a basis $v_1, \ldots, v_n$ for V_λ, with each v_i being in a distinct $V^{(j)}$. Now it suffices to show that given $v_i \in V^{(j)}$, there exists $x \in \mathfrak{m}_\alpha$ such that $xv_i \in V_{w_\alpha(\lambda)}$. But we know that $w_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, and that the set

$$\{e_\alpha^k v_i, e_{-\alpha}^k v_i \mid \alpha \in \mathbb{Z}\}$$

spans $V^{(j)}$.

Define $M = \max\{k \mid e_\alpha^k v_i \neq 0\}$, $m = \max\{k \mid e_{-\alpha}^k v_i \neq 0\}$. We need to prove that

$$-m \leq -\langle \lambda, \alpha^\vee \rangle \leq M$$

However

$$(\lambda + M\alpha)(h_\alpha) = -(\lambda - m\alpha)(h_\alpha)$$

and so $\lambda(h_\alpha) = m - M$. But $\lambda(h_\alpha) = \langle \lambda, \alpha^\vee \rangle$ and so we are done. □

Definition 16.9 (highest weight vector)
$v \in V$ is a *highest weight vector* if

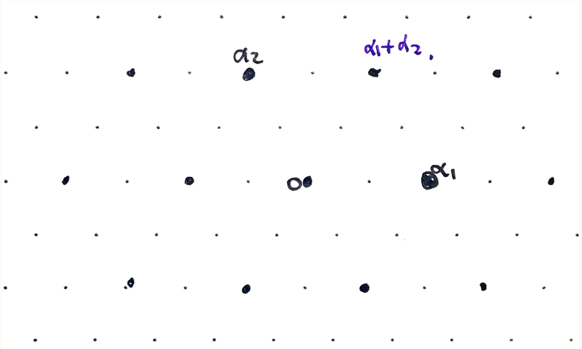
- $v \neq 0$,
- $v \in V_\lambda$ for some λ,
- $e_\alpha v = 0$ for all $\alpha \in \Phi^+$.

In the examples sheet, we show that there is a root α_0 of maximal height with respect to the basis Δ called the *highest root*. Any non-zero element of $\mathfrak{g}_{\alpha_0}$ is a highest weight vector with respect to the adjoint action.

17 The PBW theorem

Example 17.1

Let $\mathfrak{g} = \mathfrak{sl}_3$, the root lattice is



and we have a unique highest weight $\alpha_1 + \alpha_2$ for the adjoint representation.

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Example 17.2

Let $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{t}$ be the Cartan subalgebra of diagonal matrices, with basis

$$h_{\alpha_1} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \qquad h_{\alpha_2} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

Let V be the defining representation of $\mathfrak{g}$, with the standard basis elements $\{e_1, e_2, e_3\}$. We look for $|\lambda_i \in \mathfrak{t}^*$ such that

$$V = \bigoplus_{\lambda} V_\lambda$$

We know that

$$\begin{aligned} h_{\alpha_1} e_1 &= e_1 \\ h_{\alpha_1} e_2 &- -e_2 \\ h_{\alpha_1} e_3 &= 0 \\ h_{\alpha_2} e_1 &= 0 \\ h_{\alpha_2} e_2 &= e_2 \\ h_{\alpha_2} e_3 &= -e_3 \end{aligned}$$

Take

$$\begin{aligned} \lambda_1(h_{\alpha_1}) &= 1 & \lambda_1(h_{\alpha_2}) &= 0 \implies V_{\lambda_1} = \langle e_1 \rangle \\ \lambda_2(h_{\alpha_1}) &= -1 & \lambda_2(h_{\alpha_2}) &= 1 \implies V_{\lambda_1} = \langle e_2 \rangle \\ \lambda_3(h_{\alpha_1}) &= 0 & \lambda_3(h_{\alpha_2}) &= -1 \implies V_{\lambda_1} = \langle e_3 \rangle \end{aligned}$$

Note $\lambda_1 = \omega_1$, $\lambda_2 = -\omega_1 + \omega_2$ and $\lambda_3 = -\omega_2$, see page 81 in notes. In this case, e_1 is a highest weight vector.

See Fulton-Harris Lectures 12 and 13 for more examples.

Lemma 17.3. (i) V has a highest weight vector,

(ii) if $v \in V_\lambda$ is a highest weight vector, then λ is dominant.

Proof. For (i), choose any non-zero element $v_0 \in V_\lambda$ for some λ. If v_0 is a highest weight vector then we are done. Otherwise, choose $\alpha \in \Phi^+$ such that $e_\alpha(v_0) \neq 0$. Let

$$k' = \max\{k \mid e_\alpha^k v_0 \neq 0\}$$

and

$$v_1 = e_\alpha^{k'} v_0 = V_{\lambda + k'\alpha}$$

Repeat this argument, it must terminate as V is finite dimensional, and each v_i lives in a distinct weight space, as we always add on a positive root.

For (ii), let $\alpha \in \Phi^+$, we need to show that $\langle \lambda, \alpha^\vee \rangle > 0$. Consider $\mathfrak{m}_\alpha = \langle e_\alpha, f_\alpha = e_{-\alpha}, h_\alpha \rangle$ acting on V. Then $e_\alpha v = 0$ and $h_\alpha v = \lambda(h_\alpha)v$. So v is a highest weight vector for any $\mathfrak{m}_\alpha \cong \mathfrak{sl}_2$ acting on V, hence $\lambda(h_\alpha) > 0$ by $\mathfrak{sl}_2$ theory. □

Next, we will show that there is a correspondence

$$\{\text{f.d. irred. reps of } \mathfrak{g}\} \leftrightarrow \{\text{dominant weights}\}$$

17.1 Universal enveloping algebra

For now, let k be any field. We will associate to each Lie algebra $\mathfrak{g}$ over k an associative unital algebra (which in general is infinite dimensional over k), which is generated "as freely as possible" by the Lie algebra $\mathfrak{g}$ subject to the commutation relations in $\mathfrak{g}$.

Definition 17.4 (tensor algebra)
Let V be a vector space over k, defined the *tensor algebra* of V as

$$\mathcal{T}(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

where (by convention) $V^{\otimes 0} = k$. On $\mathcal{T}(V)$, we have an associative product defined on homogeneous generators by

$$(v_1 \otimes \cdots \otimes v_m) \otimes (u_1 \otimes \cdots \otimes u_n) = v_1 \otimes \cdots \otimes v_m \otimes u_1 \otimes \cdots \otimes u_n \in V^{\otimes(m+n)}$$

Definition 17.5 (symmetric algebra)
The *symmetric algebra* on V is

$$\mathcal{S}(V) = \mathrm{Sym}(V) = \frac{\mathcal{T}(V)}{I}$$

where I is the (two-sided) ideal generated by

$$\{x \otimes y - y \otimes x \mid x, y \in V\}$$

Notice

$$\mathcal{S}(V) = \bigoplus_{n \geq 0} S^n(V)$$

and we can identify

$$\mathcal{S}(V) = k[V]$$

for the algebra of polynomials on V.

Note both $\mathcal{T}(V)$ and $\mathcal{S}(V)$ are graded algebras.

Definition 17.6 (universal enveloping algebra)
Given an arbitrary Lie algebra $\mathfrak{g}$ over k (could be infinite dimensional), then the *univeral enveloping*

algebra $\mathcal{U}(\mathfrak{g})$ is the associative k-algebra

$$\mathcal{U}(\mathfrak{g}) = \frac{\mathcal{T}(\mathfrak{g})}{J}$$

where J is the (two-sided) ideal generated by

$$\{x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g}\}$$

Some facts/exercises:

- we often write $x \otimes y$ as xy,
- if V is a representation of $\mathfrak{g}$, then V is a $\mathcal{U}(\mathfrak{g})$-module, with

  $$(x_1 \otimes \cdots \otimes x_n)v = x_1 \cdots x_n v$$

  This is well defined as $(x \otimes y - y \otimes x)(v) = xyv - yxv = [x, y]v$,
- if V is a finite dimensional representation of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we defined the Casimir element $\Omega = ef + fe + \frac{1}{2}h^2 \in \mathfrak{gl}(V)$. Ω is naturally an element of $\mathcal{U}(\mathfrak{g})$, independent of V. In general, if $\mathfrak{g}$ is a semisimple Lie algebra complex Lie algebra, with basis $\{x_1, \ldots, x_n\}$, with dual basis $\{y_1, \ldots, y_n\}$ with respect to the Killing form. Then we define the *Casimir element*

  $$\Omega = \sum_{i=1}^{n} x_i y_i \in \mathcal{U}(\mathfrak{g})$$

  Moreover, $\Omega \in Z(\mathcal{U}(\mathfrak{g}))$
- $\mathcal{U}(\mathfrak{g})$ is not graded, since the generators of J are not homogeneous. For example, $\mathfrak{g} \otimes \mathfrak{g}$ is not closed under addition. But it does have a filtration. Let U_n be the image of

  $$\bigoplus_{i=0}^{n} \mathfrak{g}^{\otimes i}$$

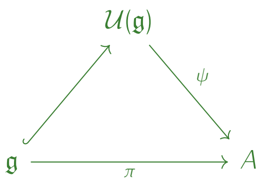
  in $\mathcal{U}(\mathfrak{g})$, then $U_n U_m \subseteq U_{m+n}$.

  Exercise (utterly horendous): If $x \in U_n, y \in U_m$, then $xy - yx \in U_{m+n-1}$.

The universal property of the universal enveloping algebra is: If A is an (associative unital, not necessarily commutative) algebra over k, $\pi : \mathfrak{g} \to A$ a k-linear map, such that

$$\pi([x, y]) = \pi(x)\pi(y) - \pi(y)\pi(x)$$

Then there exists a unique k-algebra homomorphism ψ, making



commute.

Another motivation for the universal enveloping algebra: If G is a simple compact Lie group, with Lie algebra $\mathfrak{g}$, then $\mathcal{U}(\mathfrak{g})$ is the algebra of left invariant differential operators on G. We can also use this to motivate the Casimir element. G has a natural bi-invariant metric. Then we can consider the Laplace-Beltrami operator

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d}$$

This is central in $\mathcal{U}(\mathfrak{g})$, and the corresponding element in $\mathfrak{g}$ is the Casimir element.

Definition 17.7 (associated graded algebra)
Given any filtration $F_0 \subseteq F_1 \subseteq F_2$, we call

$$\mathrm{gr}(F) = \bigoplus_i F_i / F_{i-1}$$

the *associated graded algebra*.

In our case,

$$\mathrm{gr}(\mathcal{U}(\mathfrak{g})) = U_0 \oplus \left(\bigoplus_{n \geq 1} U_n / U_{n-1} \right)$$

See also the Commutative Algebra course.

Lecture 20

17.2 PBW theorem

Since $\mathfrak{gl}(V)$ is an associative unital algebra, a representation $\mathfrak{g} \to \mathfrak{gl}(V)$ is equivalent to a k-algebra homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathfrak{gl}(V)$. Therefore, it would be useful to understand the structure of $\mathcal{U}(\mathfrak{g})$, and the PBW theorem is one part of this.

Theorem 17.8 (Poincaré-Birkhoff-Witt)**.** There exists an isomorphism of algebras

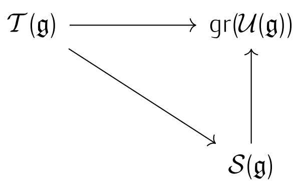
$$\mathcal{S}(\mathfrak{g}) \cong \mathrm{gr}(\mathcal{U}(\mathfrak{g}))$$

Equivalently, if $\{x_1, \dots, x_n\}$ is a basis for $\mathfrak{g}$, then

$$x_1^{k_1} \cdot x_n^{k_n}$$

is a basis for $\mathcal{U}(\mathfrak{g})$, and so $\mathfrak{g}$ embeds into $\mathcal{U}(\mathfrak{g})$.

Proof.* Omitted. For the first part, see Humphreys §17.4. We have a map $\mathfrak{g} \to \mathcal{U}_n$ by inclusion, then consider the composition of this with the quotient map. Hence we get a map from the tensor algebra to the associated graded algebra, which by the exercise at the end of the last lecture, factors through the symmetric algebra, $\mathcal{S}(\mathfrak{g})$.



It's not too hard to show the map is surjective, but it is hard to show that it is injective.

For the basis, a basis for $\mathcal{S}(\mathfrak{g})$ gives an associated basis of $\mathrm{gr}(\mathcal{U}(\mathfrak{g}))$, which in turn gives a basis for $\mathcal{U}(\mathfrak{g})$. See Humphreys §17.3 Corollary C. □

Lemma 17.9. Suppose V is a representation of $\mathfrak{g}$, and $v \in V$. Then the minimal subrepresentation of V which contains v is

$$\mathcal{U}(\mathfrak{g})v = \{uv \mid u \in \mathcal{U}(\mathfrak{g})\}$$

Proof. It is clear that $\mathcal{U}(\mathfrak{g})v$ contains everything we want, as it contains all elements of the form $x_1 \cdots x_n v$ for all $x_i \in \mathfrak{g}$. It also contains all scalar multiples, and all of the sums of the above. □

Example 17.10

Let V be a infinite dimensional $\mathbb{C}$-vector space, with basis $v_0, v_1, \ldots$. Define an $\mathfrak{sl}_2$-action on V by

$$\begin{aligned} ev_0 &= 0 \\ hv_0 &= 0 \\ fv_i &= v_{i+1} \end{aligned}$$

We claim that v_0 and v_1 are highest weight vectors for the $\mathfrak{sl}_2$-action. We need that $ev_0 = ev_1 = 0$. For $j = 1$,

$$ev_1 = efv_0 = ([e, f] + fe)v_0 = hv_0 = 0$$

We also require that $\langle v_0 \rangle$ and $\langle v_1 \rangle$ to contain their images under h. $hv_0 = 0$ so this is clear, and for v_1,

$$hv_1 = hfv_0 = ([h, f] + fh)v_0 = [h, f]v_0 = -2fv_0 = -2v_1$$

So $v_1 \in V_{-2}$ is a highest weight vector.

Also note that

$$W = \operatorname{span}\{v_1, \ldots, v_n\}$$

is a subrepresentation of V, and V/W is one dimensional, and so $V/W \cong V(0)$.

More generally, if $V^{(n)}$ is a $\mathbb{C}$-vector space, with basis $v_0, \ldots, v_m$, and with $\mathfrak{sl}_2$ action given by

$$\begin{aligned} ev_0 &= 0 \\ hv_0 &= nv_0 \\ fv_i &= v_{i+1} \end{aligned}$$

Then v_{n+1} is a highest weight vector. If we let

$$W^{(n)} = \operatorname{span}\{v_{n+1}, \ldots, v_m\}$$

and we have that

$$V^{(n)}/W^{(n)} \cong V(n)$$

18 Highest weight modules and Verma modules

As usual, let $\mathfrak{g}$ be a semisimple Lie algebra, with a Cartan subalgebra $\mathfrak{t}$ and roots Φ, and a base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of simple roots. If V is a representation, then we have weight spaces

$$V_\lambda = \{v \in V \mid tv = \lambda(t)v \text{ for all } t \in \mathfrak{t}\}$$

Remark 18.1.

- V_λ makes sense even if V is infinite dimensional.
- the definition of a highest weight vector also makes sense if V is infinite dimensional.
- if $e_\alpha \in \mathfrak{g}_\alpha$ is non-zero, then
  $$e_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$$
  which still makes sense if V is infinite dimensional.

Definition 18.2 (highest weight module)

A representation V of $\mathfrak{g}$ is a *highest weight module* if V contains a highest weight vector v, such that

$$V = \mathcal{U}(\mathfrak{g})v$$

Note that Humphreys calls this a *standard cyclic module*, but the modern terminology is highest weight module.

Example 18.3

Any finite dimensional irreducible representation v of $\mathfrak{g}$ is a highest weight module. This follows as v has to contain a highest weight vector v, and we saw that $\mathcal{U}(\mathfrak{g})v$ is a subrepresentation of V containing v. Thus equality holds as V is irreducible.

Example 18.4

In example 17.10, v_0 is a highest weight vector, and $v_i = f^i v_0$, and so

$$V = \mathcal{U}(\mathfrak{g})v_0$$

is a highest weight module.

Remark 18.5.

- Not every highest weight module is irreducible,
- If V is an infinite dimensional weight module, $v \in V_\lambda$ a highest weight vector, then λ does not have to be dominant.

Notation 18.6. Define

$$\eta^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

$$\eta^- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$$

With this, we have that

$$\mathfrak{g} = \eta^+ \oplus \mathfrak{t} \oplus \eta^-$$

Lemma 18.7. Suppose V is a highest weight module, with a highest weight vector v such that $V = \mathcal{U}(\mathfrak{g})v$. Then in fact

$$V = \mathcal{U}(\eta^-)v$$

Proof. Choose a basis $x_1, \dots, x_n$ of η^-, a basis $t_1, \dots, t_\ell$ of $\mathfrak{t}$, and a basis $y_1, \dots, y_n$ of η^+. Then

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\eta^-) \otimes \mathcal{U}(\mathfrak{t}) \otimes \mathcal{U}(\eta^+)$$

and by PBW,

$$\mathcal{U}(\mathfrak{g})v = \operatorname{span}\{x_1^{k_1} \cdots x_n^{k_n} t_1^{m_1} \cdots t_\ell^{m_\ell} y_1^{r_1} \cdots y_n^{r_n} v\}$$

But $y_i v = 0$ for all i, and $t_i v \in \operatorname{span}\{v\}$, and so

$$\mathcal{U}(\mathfrak{g})v = \mathcal{U}(\eta^-)v$$

□

Intuitively, since v is a highest weight vector, it is in the kernel of all of the $y_i \in \eta^+$. So the weight can only decrease.

Proposition 18.8. Let V be a highest weight module, with highest weight vector $v_\lambda \in V_\lambda$, with $V = \mathcal{U}(\mathfrak{g})v_\lambda$. Then

(i)

$$V = \bigoplus_{\mu \in D(\lambda)} V_\mu$$

where

$$D(\lambda) = \{\lambda - \sum_{i=1}^{\ell} k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geq 0}\}$$

is the *descent set of* λ.

(ii) Any submodule of V is a direct sum of weight spaces V_μ,

(iii) $\dim(V_\lambda) = 1$ and any other V_μ is finite dimensional,

(iv) V is irreducible if and only if every highest weight vector lies in V_λ,

(v) V contains a maximal (proper) subrepresentation.

Lecture 21

Proof. Recall that $V = \mathcal{U}(\eta^-)v_\lambda$, and so by considering expressions of the form

$$e_{-\beta_1} \cdots e_{-\beta_r} v_\lambda$$

where the β_i are positive roots, $r \geq 0$, $e_{-\beta}$ a non-zero vector in $\mathfrak{g}_{-\beta}$. These expressions span V.

In this case, the weight of such an expression is $\lambda - (\beta_1 + \cdots + \beta_r)$, and so the generators live in $V_{\lambda - \sum \beta_i}$, where $\beta_i \in \Phi^+$. This shows (i). Moreover, this also shows (iii), since given μ there exists only a finite number of ways to write

$$\mu = \nu - \sum_i \beta_i$$

where the β_i are positive roots.

For (ii), let $W \leq V$ be a submodule. Write $w \in W$ as a sum

$$w = \sum_{k=1}^n v_k$$

where $v_k \in V_{\lambda_k}$, the λ_k are distinct. We need to show that all the v_k are in W. If not, then we can choose a w with n minimal, $n > 1$.

In particular, none of the v_k is in W. Find $t \in \mathfrak{t}$, for which $\mu_1(t) \neq \mu_2(t)$. Then

$$tw = \sum_i \mu_i(t) v_i \in W$$

as does

$$(t - \mu_1(t)\,\mathrm{id})w = (\mu_2(t) - \mu_1(t))v_2 + \cdots + (\mu_n(t) - \mu_1(t))v_n$$

The right hand side is non-zero. But since n is minimal, this forces $v_2 \in W$.

For (iv), suppose V has a highest weight vector $v_\mu \in V_\mu$, where $\mu \neq \lambda$. Then $\mathcal{U}(\mathfrak{g})v_\mu$ is a subrepresentation, and it does not contain v_λ. To see this, the weights for $\mathcal{U}(\mathfrak{g})v_\mu$ are of the form

$$\mu - \sum_i k_i \alpha_i$$

Hence $\mathcal{U}(\mathfrak{g})v_\mu$ is a non-trivial proper subrepresentation. Conversely, suppose $U \subsetneq V$ is a non-trivial proper subrepresentation. We can write U as a direct sum of V_μ. Choose μ such that $v_\mu \in U$, and if we write

$$\mu = \lambda - \sum_i k_i \alpha_i$$

have $\sum_i k_i$ minimal. Let $v_\mu \in V_\mu$ be non-zero, $\alpha \in \Phi^+$, $e_\alpha \in \mathfrak{g}_\alpha$. Then

$$e_\alpha v_\mu \in V_{\mu+\alpha} \cap U = 0$$

Hence v_μ is a highest weight vector.

(v) has been left as an exercise. Let W_1, W_2 be submodules of V. If $v_\lambda \in W_i$, then $W_i = V$. So we may assume that $v_\lambda \notin W_i$. We claim that $W_1 + W_2$ is a proper subrepresentation. But this follows from the fact shown in (ii) that $W_1 + W_2$ decomposes into weight spaces, and by (iii), the λ weight space is one dimensional.

Therefore, the sum of all proper subrepresentations must also be a subrepresentation, since if it contains v_λ, a finite sum of proper subrepresentations must contain v_λ. With this, maximality is clear. □

Other easy exercises (Humphreys):

- show that V is indecomposable as a $\mathfrak{g}$-module,
- show that every non-zero homomorphic image of V is also a highest weight module

18.1 Verma modules

Definition 18.9 (highest weight)
If V is a highest weight module with $v \in V_\lambda$ a highest weight vector, $V = \mathcal{U}(\mathfrak{g})v$, then we say that V is of *highest weight* λ.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$, with corresponding root system Φ and a root basis $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Choose a basis $h_1, \dots, h_\ell$ of $\mathfrak{t}$, with $h_i = h_{\alpha_i}$. For $\lambda \in \mathfrak{t}^*$, let $J(\lambda)$ be the (left) ideal of $\mathcal{U}(\mathfrak{g})$ generated by

- e_α for $\alpha \in \Phi^+$,
- $h_i - \lambda(h_i)1$ for each i

That is, $J(\lambda)$ comprises elements

$$\sum_{\alpha \in \Phi^+} u_\alpha e_\alpha + \sum y_i \left(h_i - \lambda(h_i)1\right)$$

where $u_\alpha, y_i \in \mathcal{U}(\mathfrak{g})$. $J(\lambda)$ is a left module for $\mathcal{U}(\mathfrak{g})$.

Definition 18.10 (Verma module)
Let $M(\lambda)$ be the quotient space

$$M(\lambda) = \frac{\mathcal{U}(\mathfrak{g})}{J(\lambda)}$$

This is a $\mathcal{U}(\mathfrak{g})$-module, with action

$$u(v + J(\lambda)) = uv + J(\lambda)$$

and we say that $M(\lambda)$ is the *Verma module associated to* λ.

Proposition 18.11. $M(\lambda)$ is a highest weight module, with highest weight λ.
Moreover, $M(\lambda)$ is universal. That is, for $m_\lambda \in M(\lambda)_\lambda$ a highest weight vector, V any other highest weight module with highest weight λ, and highest weight vector v_λ, then there exists a unique $\mathfrak{g}$-equivariant linear map $M(\lambda) \to V$, sending m_λ to v_λ.

Proof. Let $m_\lambda = 1 + J(\lambda) \in M(\lambda)$. This is a generator for $M(\lambda)$ as a $\mathcal{U}(\mathfrak{g})$-module. Then

$$h_i m_\lambda - h_i + J(\lambda) = \lambda(h_i)1 + J(\lambda)$$

and if $\alpha \in \Phi^+$, $e_\alpha \in \mathfrak{g}_\alpha$, then

$$e_\alpha m_\lambda = e_\alpha + J(\lambda) = J(\lambda) = 0$$

In this case, m_λ is a highest weight vector, with highest weight λ, and $M(\lambda) = \mathcal{U}(\mathfrak{g})m_\lambda$. So any other highest weight vector is a scalar multiple of this one.
By the PBW theorem, if $\Phi^+ = \{\beta_1, \dots, \beta_r\}$, then

$$e_{-\beta_1} \cdots e_{-\beta_r} m_\lambda$$

is a basis for $M(\lambda)$. Define

$$\varphi : M(\lambda) \to V$$
$$e_{-\beta_1} \cdots e_{-\beta_r} m_\lambda \mapsto e_{-\beta_1} \cdots e_{-\beta_r} v_\lambda$$

□

Lecture 22

Remark 18.12. Humphreys calls $M(\lambda)$ the universal *standard cyclic modules*.

Lemma 18.13. Given a weight $\lambda \in \mathfrak{t}^*$, there is a unique irreducible highest weight module with highest weight λ, called $V(\lambda)$.

Proof. We know that $M(\lambda)$ has a unique maximal (proper) submodule J, by proposition 18.8. Then $M(\lambda)/J$ is irreducible. Uniqueness follows from the universal property of the Verma module. □

Example 18.14

In example 17.10, we had $V = M(0)$, $J = \langle v_1, \ldots \rangle$, and $M(0)/J \cong V(0)$, which is the trivial representation of $\mathfrak{sl}_2$.

Example 18.15

See Erdmann-Wildon Example 15.12, they give an example of an irreducible Verma module, for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. This shows that $\mathfrak{sl}_2(\mathbb{C})$ has infinite dimensional irreducible representations.

Remark 18.16. Verma modules are building blocks for the 'category $\mathcal{O}$'. Although each $M(\lambda)$ is infinite dimensional, when we viewed as a $\mathcal{U}(\mathfrak{g})$-modules, it has finite length. That is, there exists submodules

$$0 = M_0 \leq M_1 \leq \cdots \leq M_r = M(\lambda)$$

such that M_{i+1}/M_i are simple for all i.

See Humphreys 'category $\mathcal{O}$' book.

Remark 18.17. In 1985, Drinfeld and Jimbo independently defined *quantum groups*, by 'deforming' the universal enveloping algebras of Lie algebras. These have numerous applications in theoretical physics, knot theory, and representation theory of algebraic groups.

Theorem 18.18. $V = V(\lambda)$ is a finite dimensional irreducible $\mathfrak{g}$-module if and only if λ is dominant.

Proof. If V is finite dimensional, then for each simple root α_i, let $\mathfrak{m}_{\alpha_i}$ be the corresponding copy of $\mathfrak{sl}_2$. Then V is also a finite dimensional module for $\mathfrak{m}_i$, and a highest weight vector for $\mathfrak{g}$ is a highest weight vector for $\mathfrak{m}_i$. Since, there exists a highest weight vector of weight λ, then the weight for the Cartan subalgebra $\mathfrak{t}_i \subseteq \mathfrak{m}_{\alpha_i}$ is determined by the $\lambda(h_i)$, since $h_i(v) = \lambda(h_i)v = \langle \lambda, \alpha_i^\vee \rangle v$. This forces $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$.

For the converse, $V(\lambda)$ is a direct sum of finite dimensional weight spaces, as it is the quotient of the Verma module. The idea is to show that the set of weights

$$\Pi(\lambda) = \{\mu \mid V(\lambda)_\mu \neq 0\}$$

is finite. Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ be a root base, and for each i, let $\{x_i, y_i, h_i\}$ be a $\mathfrak{m}_{\alpha_i}$-triple.

We need that in $\mathcal{U}(\mathfrak{g})$,

$$[x_j, y_i^{k+1}] = 0 \text{ for } i \neq j \tag{i}$$

$$[x_i, y_i^{k+1}] = -(k+1)y_i^k(k \cdot 1 - h_i) \tag{ii}$$

for $k \geq 0$. See Humphreys §21.2.

Claim 18.19. $V(\lambda)$ contains a non-zero finite dimensional $\mathfrak{m}_{\alpha_i}$-module, for each i.

Proof. Let $v \in V(\lambda)$ be a highest weight vector. As λ is dominant,

$$n_i = \left\langle \lambda, \alpha_i^\vee \right\rangle \in \mathbb{Z}_{\geq 0}$$

Let $u = y_i^{n_i+1}v$. We will show that $u = 0$. By (i), for $i \neq j$,

$$x_j u = y_i^{n_i+1}(x_j v) + [x_j, y_i^{n_i+1}]v = 0$$

Next, for (ii),

$$x_i u = y_i^{n_i+1}(x_i v) + [x_i, y_i^{n_i+1}]v = -(n_i+1)y_i^{n_i}(n_i - h_i)v = 0$$

Suppose if $u \neq 0$, then from the above, u would be a highest weight vector of weight $\lambda - (n_i+1)\alpha_i < \lambda$. Contradiction, the highest weight is unique. So

$$W = \left\langle v, y_i v, \dots, y_i^{n_i+1} v \right\rangle$$

is a non-zero finite dimensional $\mathfrak{m}_{\alpha_i}$ subrepresentation of $V(\lambda)$. To see that $x_i W \subseteq W$, use (ii). □

Claim 18.20. For each i, $V(\lambda)$ is the sum of all finite dimensional $\mathfrak{m}_{\alpha_i}$ subrepresentations contained in it.

Proof. Let W be the sum of all finite dimensional $\mathfrak{m}_{\alpha_i}$-modules contained in $V(\lambda)$. We will show that W is a $\mathfrak{g}$-submodule of $V(\lambda)$. But $V(\lambda)$ is irreducible, and by claim 18.19, $W \neq 0$, and so $W = V(\lambda)$.

For $x \in \mathfrak{g}, w \in W$, we need to show that $xw \in W$. But then $w \in W'$ for some finite dimensional $\mathfrak{m}_{\alpha_i}$-module W'. Let

$$x = \sum_{\beta \in \Phi \cup \{0\}} x_\beta$$

where $x_\beta \in \mathfrak{g}_\beta$. Then $x_\beta w \in \mathfrak{g}_\beta W' = W''$. Now consider

$$W'' = \operatorname{span}_\beta\{\mathfrak{g}_\beta W'\}$$

Then W'' is finite dimensional, and it is clearly $\mathfrak{m}_{\alpha_i}$-invariant, as

$$x_i W'' \subseteq \operatorname{span}_\beta\{x_i x_\beta W'\}$$

But

$$x_i x_\beta W' = x_\beta(x_i W') + [x_i, x_\beta]W' \subseteq x_\beta W' + \mathfrak{g}_{\alpha_i+\beta} W' \subseteq W''$$

Similarly, repeat for y_i, h_i. Thus, $xw \in W'' \subseteq W$. □

Claim 18.21. The Weyl group acts on $\Pi(\lambda)$ by permutations.

Assuming the claim, $\Pi(\lambda)$ decomposes as a disjoint union of orbits, this will mean that it is now enough to show there are only finitely many orbits, as W is finite. First, we will show

Lecture 23

Claim 18.22. If $\mu \in \Pi(\lambda)$, then $w_i(\mu) \in \Pi(\lambda)$, where $w_i = w_{\alpha_i}$. Also,

$$\dim(V(\lambda)_\mu) = \dim(V(\lambda)_{w_i(\mu)})$$

Proof. Since $V(\lambda)_\mu$ is finite dimensional, there exists a finite-dimensional $\mathfrak{m}_{\alpha_i}$-module U containing $V(\lambda)_\mu$. Pick an element $0 \neq w \in V(\lambda)_\mu$. We have that

$$h_i w = \mu(h_i) w$$

Hence $w \in U$ is a weight vector for h_i. Thus by $\mathfrak{sl}_2$ theory, $\mu(h_i) = \mu(\alpha_i^\vee) = m \in \mathbb{Z}$. Hence all weights are in $\mathbb{Z}$, as this holds for all i. Since m appears as a weight of U, so does $-m$. Moreover, we have that $\dim(U_m) = \dim(U_{-m})$.

If $m \geq 0$, then $y_i^m w \neq 0$, and $y^i m \in U_{-m}$. But

$$y_i^m w \in V(\lambda)_{\mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i} = V(\lambda)_{w_i(\mu)}$$

as weights add. If $m < 0$, the same argument with x_i^{-m} works.

To get the equality of dimensions, if $w_1, \dots, w_r$ is a basis of $V(\lambda)_\mu$, then $w_1, \dots, w_r$ is linearly independent in U_m. Applying y_i^m (or x_i^{-m}), the results are still linearly independent, which live in $V(\lambda)_{w_i(\mu)}$. With this,

$$\dim(V(\lambda)_\mu) \leq \dim(V(\lambda)_{w_i(\mu)})$$

But we can just swap μ and $w_i(\mu)$, as $w_i^2 = \mathrm{id}$. □

Claim 18.23. For $\mu \in \Pi(\lambda)$, its Weyl orbit $W\mu$ contains a dominant weight.

Proof. The orbit $W\mu$ is finite, so there exists $\eta \in W_\mu$ which is maximal with respect to $\leq$. Then we know that η is dominant, since if not, then

$$\langle \eta, \alpha_i^\vee \rangle < 0$$

for some i, and so $w_i(\eta) \in W\mu$, with $w_i(\eta) = \eta - \langle \eta, \alpha_i^\vee \rangle$ with $w_i(\eta) \geq \eta$. Contradiction. □

Claim 18.24.

$$S = \{\eta \mid \eta \text{ dominant}, \eta \leq \lambda\}$$

is finite.

Proof. If $\eta \in S$, then $\lambda - \eta$ is a sum of positive roots, with nonnegative coefficients. hence η lies in a discrete set.

Moreover, $\lambda + \eta$ is dominant, and so

$$\langle \lambda + \eta, \alpha_i^\vee \rangle \geq 0$$

for all i. In particular,

$$(\lambda + \eta, \lambda - \eta) \geq 0 \implies (\lambda, \lambda) \geq (\eta, \eta)$$

and so S is a subset of a compact set. Thus S is compact and discrete, and so finite. □

From claim 18.23, any W-orbit of $\Pi(\lambda)$ contains a dominant weight, i.e. an element of S. But S is finite and so there are only finitely many orbits. □

We've just shown that there exists a bijection

$$\{\text{dominant weights } \lambda\} \leftrightarrow \{\text{finite dimensional irreducible representations } V(\lambda) \text{ of } \mathfrak{g}\}$$

19 The Weyl character formula

Let $\mathfrak{g}$ be a semisimple Lie algebra, with Cartan subalgebra $\mathfrak{t}$, root basis $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, weight lattice X, Weyl group W.

Example 19.1

Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

Let us compute $\langle \rho, \alpha_i^\vee \rangle$ for A_1, A_2, B_2.

For A_2, B_2, $\Delta = \{\alpha_1, \alpha_2\}$, we claim that

$$\rho = \omega_1 + \omega_2$$

In particular,

$$\langle \rho, \alpha_i^\vee \rangle = 1$$

For A_1, $\Delta = \{\alpha_1\}$, $\rho = \alpha_1/2$, and

$$\langle \rho, \alpha_1^\vee \rangle = 1$$

We claim that

$$\rho = \sum_{i=1}^{\ell} \omega_i$$

where the ω_i are the fundamental dominant weights[3]. It suffices to show that $\langle \rho, \alpha_j^\vee \rangle = 1$ for all $\alpha_j \in \Delta$. In this case,

$$w_{\alpha_j}(\rho) = \rho - \langle \rho, \alpha_j^\vee \rangle \alpha_j$$

[3] Recall that this means $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. By construction they are dominant.

But we also have that

$$w_{\alpha_j}(\rho) = w_{\alpha_j}\left(\frac{1}{2}\sum_{\alpha\in\Phi^+\setminus\{\alpha_j\}}\alpha + \frac{1}{2}\alpha_j\right)$$

We know that w_{α_j} permutes $\Phi^+ \setminus \{\alpha_j\}$, and so

$$w_{\alpha_j}(\rho) = \frac{1}{2}\sum_{\alpha\in\Phi^+\setminus\{\alpha_j\}}\alpha - \frac{1}{2}\alpha_j = \rho - \alpha_j$$

Recall from theorem 18.18 that

$$\Pi(\lambda) = \{\mu \mid V(\lambda)_\mu \neq 0\}$$

This leads to the questions:

- what is $\Pi(\lambda)$?
- what is $\dim(V(\lambda)_\mu)$?

Definition 19.2

We have a partial ordering $\preceq$ on X, defined by

$$\mu \preceq \lambda \text{ iff } \lambda - \mu = \sum_{i=1}^{\ell} k_i\alpha_i$$

where $k_i \in \mathbb{Z}_{\geq 0}$ for all i.

Now note that

$$\Pi(\lambda) = \{\mu \mid \mu \preceq \lambda\}$$

and to determine $\Pi(\lambda)$, we only need to find the dominant weights in it. This is because (Humphreys Lemma 13.2A) each weight is conjugate under the Weyl group to a unique dominant weight. See claim 18.23 for existence. For our purposes, we won't need uniqueness.

Proposition 19.3. Suppose λ, μ are dominant weights. Then $\mu \in \Pi(\lambda)$ if and only if $\mu \preceq \lambda$.

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Proof. Suppose $\mu \preceq \lambda$. Then

$$\lambda - \mu = \sum_{\alpha\in\Phi^+} k_\alpha\alpha$$

where $k_\alpha \in \mathbb{Z}_{\geq 0}$. We will induct on $\sum k_\alpha$. We've already done the case where $\sum k_\alpha = 0$. Now suppose

$$\mu = \lambda - \alpha$$

for some $\alpha \in \Phi^+$. Then

$$\left\langle \mu, \alpha^\vee \right\rangle = \left\langle \lambda, \alpha^\vee \right\rangle - 2 \geq 0$$

by dominance, and so $\langle \alpha, \alpha^\vee \rangle \geq 2$. Let $v_\lambda \in V_\lambda$ be non-zero. Since

$$h_\alpha v_\lambda = n v_\lambda$$

fo some $n \geq 2$, and then we know that

$$e_{-\alpha}v_\lambda \neq 0$$

By the usual $\mathfrak{sl}_2$ theory, $e_{-\lambda}v_\lambda \in V(\lambda)_{\lambda-\alpha} = V(\lambda)_\mu$, and so μ is a weight.

Now suppose we know the claim is true for $\sum k_\alpha = n-1$. We assume now that $\sum k_\alpha = n$, so that

$$\mu = \lambda - \beta_1 - \cdots - \beta_n$$

We have two cases:

Case 1: For some i, j distinct, $\langle \beta_i, \beta_j^\vee \rangle < 0$. Without loss of generality, $i < j$. In this case, $\beta_i + \beta_j$ is a positive root. Thus, we have that

$$\sum_{i=1}^{n} \beta_i = \sum_{k=1}^{i-1} \beta_k + \sum_{k=i+1}^{j-i} \beta_k + \sum_{k=j+1}^{n} \beta_k + (\beta_i + \beta_j)$$

which is a sum of $n-1$ positive roots, and so we are done by induction.

Case 2: For all i, j distinct, $\langle \beta_i, \beta_j^\vee \rangle \geq 0$. For this,

Claim 19.4.

$$\lambda - \sum_{i=1}^{r} \beta_r \in \Pi(\lambda)$$

for all $1 \leq r \leq n$.

Proof. By induction on r. See notes. □

□

Example 19.5

For G_2 we have the root basis $\{\alpha_1, \alpha_2\}$, with α_1 short, and we would like to compute $\Pi(2\omega_1)$.

add diagram

Now ω_1 is such that $\langle \omega_1, \alpha_1^\vee \rangle = 1$, and $\langle \omega_1, \alpha_2^\vee \rangle = 0$. The dominant weights of $\Pi(2\omega_1)$ are the dominant μ with $\mu \preceq 2\omega_1$ by the proposition.

First, $2\omega_2 = 4\alpha_1 + 2\alpha_2$, and $\omega_2 = 3\alpha_1 + 2\alpha_2$. Then $\omega_2 \preceq 2\omega_2$. But $\omega_1 + \omega_2 \not\preceq 2\omega_1$. Hence the dominant weights in $\Pi(2\omega_1)$ are

$$\omega_1, 2\omega_1, \omega_2, 0$$

The Weyl conjugates of ω_1 are the short roots, and the Weyl conjugates of ω_2 are all the long roots. So

$$\begin{aligned}\Pi(2\omega_1) &= \{\text{short root}\} \cup \{2(\text{short root})\} \cup \{\text{long root}\} \\ &= \Phi \cup \{\pm 2\omega_1, \pm 2\alpha_1, \pm 2(\alpha_1 + \alpha_2), 0\}\end{aligned}$$

See Humpreys page 68, 69, or Fulton-Harris pages 339-359. This is a very typical exam question.

Definition 19.6 ((formal) character)

Let $\mathbb{Z}[X]$ be the free $\mathbb{Z}$-module with basis

$$\{e^\mu \mid \mu \in X\}$$

with multiplication

$$e^\mu e^\lambda = e^{\mu+\lambda}$$

This makes $\mathbb{Z}[X]$ into a commutative ring, with $1 = e^0$.

Let V be a finite dimensional representation of $\mathfrak{g}$, then the *(formal) character* of V is

$$\mathrm{ch}(V) = \sum_{\mu \in X} \dim(V_\mu) e^\mu \in \mathbb{Z}[X]$$

$\mathbb{Z}[X]$ is the group ring generated by the lattice X.

Example 19.7

For $\mathfrak{g} = \mathfrak{sl}_2$, then

$$X = \mathbb{Z} \cdot \frac{\alpha}{2}$$

Let $z = e^{\alpha/2}$, then

$$\mathrm{ch}(V(n)) = z^n + \cdots + z^{-n}$$

Exercise: Find the character of the adjoint represenation of $\mathfrak{sl}_3$.
Recall from examples sheet 3 that for $w \in W$, $\ell(w)$ is the minimal n such that

$$w = w_{\alpha_1} \cdots w_{\alpha_n}$$

where $\alpha_i \in \Delta$. The *sign* of w is

$$\mathrm{sign}(w) = (-1)^{\ell(w)}$$

Example 19.8
If $\mathfrak{g} = \mathfrak{sl}_n$, then for $w \in W$, the sign of w as above, is the same as the sign of w in S_n.

Theorem 19.9 (Weyl character formula)**.** Let λ be a dominant weight, and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_j \omega_j$$

Then

$$\mathrm{ch}(V(\lambda)) = \frac{\sum_{w \in W} \mathrm{sign}(w) e^{w(\lambda+\rho)}}{e^\rho \prod_{\alpha \in \Phi^+}(1 - e^{-\alpha})}$$

Proof. Fulton-Harris Chapter 24, Grojnowski §7, Humphreys 24.3. □

Corollary 19.10 (Weyl denominator formula)**.**

$$e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{w \in W} \mathrm{sign}(w) e^{w(\rho)}$$

Proof. Plug $\lambda = 0$ into the Weyl character formula. Note $\mathrm{ch}(V(\lambda)) = 1$. □

Corollary 19.11 (Weyl dimension formula)**.** If λ is dominant, then

$$\dim(V(\lambda)) = \frac{\prod_{\alpha \in \Phi^+} \langle \lambda + \rho, \alpha^\vee \rangle}{\prod_{\alpha \in \Phi^+} \langle \rho, \alpha^\vee \rangle} = \prod_{\alpha \in \Phi^+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}$$

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**Proof*.* By definition,

$$\mathrm{ch}(V(\lambda)) = \sum_{\mu \in X^+} \dim(V_\mu) e^\mu$$

We'd like to substitute $e^\mu = 1$, for any μ, but we would get 0/0.
Indeed, for $\mu \in X$, $p \in \mathbb{Z}[X]$, define

$$F_\mu(p) : \mathbb{R}_{\geq 0} \to \mathbb{R}$$
$$F_\mu(e^\lambda)(q) = q^{-(\mu,\lambda)}$$

and extend linearly over p. Note F_μ is multiplicative and $F_\mu(p)$ is C^1 on $\mathbb{R}_{>0}$. Clearly $F_0(e^\lambda) = 1$, and

$$F_0(\mathrm{ch}(V(\lambda))) = \dim(V(\lambda))$$

First apply F_μ to the Weyl denominator formula, we get

$$q^{-(\rho,\mu)} \prod_{\alpha\in\Phi^+} (1-q^{(\alpha,\mu)}) = \sum_{w\in W} \operatorname{sign}(w) q^{-(w\rho,\mu)} = \sum_{w\in W} \operatorname{sign}(w) q^{-(\rho,w\mu)} \tag{$*$}$$

Since $\operatorname{sign}(w) = \operatorname{sign}(w^{-1})$ and $(wx, y) = \left(x, w^{-1}y\right)$.

Now apply F_ρ to the Weyl character formula, we get

$$F_\rho(\operatorname{ch}(V(\lambda)))(q) = \frac{\sum_{w\in W} q^{-(\rho,w(\lambda+\rho))}}{q^{-(\rho,\rho)}\prod_{\alpha\in\Phi^+}(1-q^{-(\rho,\alpha)})}$$

Note we need $(\rho, \alpha) \neq 0$ for all α. But recall that $(\rho, \alpha_i) = 1 > 0$ for all i, and so $(\rho, \alpha) > 0$ for all $\alpha \in \Phi^+$.

Using $(*)$ with $\mu = \lambda + \rho$,

$$F_\rho(\operatorname{ch}(V(\lambda))) = \frac{q^{-(\rho,\lambda+\rho)}\prod_{\alpha\in\Phi^+}(1-q^{(\alpha,\lambda+\mu)})}{q^{-(\rho,\rho)}\prod_{\alpha\in\Phi^+}(1-q^{(\rho,\alpha)})}$$

where we applied $(*)$ to the numerator. Finally, note that

$$F_\rho(\operatorname{ch}(V(\lambda)))(q) = \sum \dim(V(\lambda)_\mu) q^{-(\rho,\mu)}$$

Taking the limit $q \to 1$, and using L'Hôpital's rule,

$$\dim(V(\lambda)) = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)}$$

as required. ☐

Example 19.12

For $\mathfrak{g} = \mathfrak{sl}_2$, $\omega_1 = \frac{1}{2}\alpha = \rho$, and $X^+ = \{m\omega_1\}$, and so

$$\dim(V(\lambda)) = \frac{(m+1)(\alpha,\alpha)}{(\alpha,\alpha)} = m+1$$

Example 19.13

For $\mathfrak{g} = \mathfrak{sl}_3$, $\Phi^+ = \{\alpha, \beta, \alpha+\beta\}$. Let

$$\lambda = m_1\omega_1 + m_2\omega_2$$

and

$$\rho = \alpha + \beta = \omega_1 + \omega_2$$

Computing:

$$\begin{aligned}
(\lambda+\rho,\alpha) &= m_1+1\\
(\lambda+\rho,\beta) &= m_2+1\\
(\lambda+\rho,\alpha+\beta) &= m_1+m_2+2\\
(\rho,\alpha) &= 1\\
(\rho,\beta) &= 1\\
(\rho,\alpha+\beta) &= 2
\end{aligned}$$

and so

$$\dim(V(\lambda)) = \frac{(m_1+1)(m_2+1)(m_1+m_2+2)}{2}$$

Exercise: Compute the dimensions of the finite dimensional irreducible representations of B_2 and G_2. See Humphreys page 140.

Example 19.14 (A very common tripos question)
For $\mathfrak{g} = \mathfrak{sp}_4(\mathbb{C})$, which is of type B_2. Let α_1 be a short root. Suppose $\lambda = a\omega_1 + b\omega_2$ is a dominant weight. In this case,

$$\prod_{\alpha \in \Phi^+} \langle \rho, \alpha^\vee \rangle = 6$$

Next, $\lambda + \rho = (a+1)\omega_1 + (b+1)\omega_2$. Hence

$$\prod_{\alpha \in \Phi^+} \langle \lambda + \rho, \alpha^\vee \rangle = (a+1)(b+1)(a+2b+3)(a+b+2)$$

Let V be the defining representation. For its highest weight,

$$\dim(V(\omega_1)) = 4$$

and so if W is a non-trivial representation of $\mathfrak{sp}_4$, and not isomorphic to $V(\omega_1)$, then by the dimension formula, we need $\dim(W) > 4$. Hence $V \cong V(\omega_1)$.

Finally, to decompose $V \otimes V$ into irreducible subrepresentations, we need to find $\lambda_1, \dots, \lambda_r$ such that

$$V \otimes V = V(\lambda_1) \oplus \cdots \oplus V(\lambda_r)$$

Let $v \in V(\omega_1)$ be a highest weight vector. Then $v \otimes v$ is a highest weight vector, with weight $2\omega_1$. That is, $V(2\omega_1)$ is a subrepresentation of $V \otimes V$. But $2\omega_1 = 2\alpha_1 + \alpha_2$, and so $V(2\omega_1)$ is the adjoint representation.

In particular,

$$\dim(V(2\omega_1)_{\omega_1}) = 1$$

Finally, take a basis $\{v_\gamma\}$ where $\gamma \in \Pi(\omega_1)$ be a basis of weight vectors of V. Then

$$\{v_{\gamma_1} \otimes v_{\gamma_2} \mid \gamma_1, \gamma_2 \in \Pi(\omega_1)\}$$

is a basis of weight vectors for $V \otimes V$. Using this,

$$\dim((V \otimes V)_{\omega_1}) = 1$$
$$\dim((V \otimes V)_{\omega_2}) = 2$$
$$\dim((V \otimes V)_0) = 4$$

All other weight spaces correspond to non-dominant weights. Next,

$$\dim(V(2\omega_1)_{\omega_2}) = 1$$
$$\dim(V(\omega_2)) = 5$$

and $V(\omega_2)$ has to be a subrepresentation of $V \otimes V$. By counting dimensions, we get

$$V \otimes V = V(2\omega_1) \oplus V(\omega_2) \oplus V(0)$$

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