Local fields

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Contents

Lecture 1

Makeup lecture Thursday Week 9. Four examples sheets/examples classes.

1 Basic theory

For example, we have Diophantine equations

$$f(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$$

and we would like to find solutions to

$$f(x_1, \dots, x_n) = 0$$

in $\mathbb{Z}^n$. Instead of trying to solve this equation, we can try to solve

$$f(x_1, \dots, x_n) \equiv 0 \pmod{p}$$

and also modulo higher powers of p.

Local fields packages all of this information together.

1.1 Absolute value

Definition 1.1 (absolute value)
Let K be a field. An *absolute value* on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$, satisfying:

(i) $|x| = 0$ if and only if $x = 0$,

(ii) $|xy| = |x||y|$ for all $x, y \in K$,

(iii) the triangle inequality $|x + y| \leq |x| + |y|$ for all $x, y \in K$.

We call the pair $(K, |\cdot|)$ a *value field*.

Example 1.2
If $K = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, then the usual absolute value is an absolute value (in the above sense). We denote this as the $|x|_\infty$.

Example 1.3 (trivial absolute value)

*Based on lectures by Rong Zhou. Last updated October 7, 2023.

For any field K, define

$$|x| = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}$$

This defines a absolute valie. In this course, we will ignore this.

Example 1.4 (p-adic absolute value)
Let p be a prime number, and let $K = \mathbb{Q}$. For a nonzero $x \in \mathbb{Q}$, we can write write

$$x = p^n \frac{a}{b}$$

where $(a, p) = (b, p) = 1$. Then we can define the *p-adic absolute value* as follows:

$$|x|_p = \begin{cases} 0 & x = 0 \\ p^{-n} & x = p^n \frac{a}{b} \text{ as above} \end{cases}$$

For the axioms, (i) is clear. For the other two, write

$$y = p^m \frac{c}{d}$$

(ii) is

$$|xy|_p = \left| p^{n+m} \frac{ac}{bd} \right|_p = p^{-(n+m)} = |x|_p |y|_p$$

as the numerator and denominator are coprime to p. For (iii), without loss of generality, we can assume $m \geq n$. Then

$$|x + y|_p = \left| p^n \left(\frac{ad + p^{m-n} bc}{bd} \right) \right| \leq p^{-n} = |x|_p = \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p$$

The inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}$$

is called the *ultrametric inequality*.

An absolute value on K induces a metric on K, defined by

$$d(x, y) = |x - y|$$

This then induces a topology on K.

Definition 1.5 (equivalence of absolute values)
Suppose $|\cdot|$ and $|\cdot|'$ are absolute values on a field K. We say that these absolute values are *equivalent* if they induce the same topology. An equivalence class of absolute values is called a *place*.

Proposition 1.6. Let $|\cdot|$ and $|\cdot|'$ are (non-trivial) absolute values on K. Then the following are equivalent:

(i) $|\cdot|$ and $|\cdot|'$ are equivalent,

(ii) for all $x \in K$, $|x| < 1$ if and only if $|x|' < 1$,

(iii) there exists $c > 0$ such that

$$|x|^c = |x|'$$

for all $x \in K$.

Proof. **(i) $\Longrightarrow$ (ii)** Note that $|x| < 1$ if and only if $x^n \to 0$ with respect to the topology. Since the topologies are equivalent, this is true if and only if $|x|' < 1$.

(ii) $\implies$ (iii) Note $|x|^c = |x|'$ is equivalent to saying

$$c \log |x| = \log |x|'$$

Let $a \in K^\times$, such that $|a| > 1$. This exists since we assumed the absolute value is non-trivial. We need to show that for all $x \in K^\times$,

$$\frac{\log |x|}{\log |a|} = \frac{\log |x|'}{\log |a|'}$$

Suppose not. Then without loss of generality, for some $x \in K^\times$,

$$\frac{\log |x|}{\log |a|} < \frac{\log |x|'}{\log |a|'}$$

Choose $m, n \in \mathbb{Z}$ such that

$$\frac{\log |x|}{\log |a|} < \frac{m}{n} < \frac{\log |x|'}{\log |a|'}$$

Rearranging,

$$n \log |x| < m \log |a|$$
$$n \log |x|' > m \log |a|'$$

Exponentiating, we find that

$$\left|\frac{x^n}{a^m}\right| < 1 \quad \text{and} \quad \left|\frac{x^n}{a^m}\right|' > 1$$

Contradiction.

(iii) $\implies$ (ii) is clear. □

Remark 1.7. $|\cdot|_\infty^2$ is *not* an absolute value, since it does not satisfy the triangle inequality. Some authors replace the triangle inequality by

$$|x + y|^\beta \le |x|^\beta + |y|^\beta$$

for some $\beta > 0$.

In this course, we are mainly interested in the following:

Definition 1.8 (non-archimedean absolute value)
An absolute value $|\cdot|$ on K is *non-archimedean* if it satisfies the *ultrametric inequality*

$$|x + y| \le \max\{|x|, |y|\}$$

If $|\cdot|$ is not non-archimedean, then it is *archimedean*.

Example 1.9
- $|\cdot|_\infty$ on $\mathbb{R}$ is archimedean.
- $|\cdot|_p$ on $\mathbb{Q}$ is non-archimedean, see example 1.4.

Lemma 1.10. Let $(K, |\cdot|)$ be non-archimedean, $x, y \in K$. If $|x| < |y|$, then $|x - y| = |y|$.

Proof.

$$|x - y| \le \max\{|x|, |y|\} = |y|$$

where we use the fact that $|-y| = |y|$ (follows from the axioms of an absolute value). Conversely,

$$|y| \le \max\{|x|, |x - y|\} \le |x - y|$$

since we can't have $|y| \le |x|$. □

Convergence is easier for non-archimedean absolute values.

Proposition 1.11. Let $(K, |\cdot|)$ be a non-archimedean valued field, and (x_n) be a sequence in the field K. Suppose $|x_n - x_{n+1}| \to 0$. Then (x_n) is Cauchy. In particular, if K is complete, then (x_n) converges.

Proof. For $\varepsilon > 0$, choose N such that for all $n \geq N$, $|x_n - x_{n+1}| < \varepsilon$. Then for $n, m \geq N$,

$$|x_n - x_m| = |(x_n - x_{n+1}) + \cdots + (x_{m-1} - x_m)| < \varepsilon$$

by the ultrametric inequality (and induction). □

Example 1.12

Consider the 5-adic metric, we will construct a sequence (x_n) in $\mathbb{Q}$ such that

(i) $x_n^2 + 1 \equiv 0 \pmod{5^n}$,

(ii) $x_n \equiv x_{n+1} \pmod{5^n}$

Take $x_1 = 2$, and suppose we have defined x_n. Let $x_n^2 + 2 = a5^n$, and set $x_{n+1} = x_n + b5^n$. Then

$$x_{n+1}^2 + 1 = x_n^2 + 2bx_n5^n + b^25^{2n} + 1 = a5^n + 2bx_n5^n + \underbrace{b^25^{2n}}_{\equiv 0 \pmod{5^{n+1}}}$$

Therefore, all we need to do is choose b such that $a + 2bx_n \equiv 0 \pmod 5$. Then $x_{n+1}^2 + 1 \equiv 0 \pmod{5^{n+1}}$.

By the proposiiton, (ii) tells us that (x_n) is Cauchy. On the other hand, suppose $x_n \to \ell \in \mathbb{Q}$. By the usual properties of limits, $x_n^2 \to \ell^2$. But (i) tells us that $x_n^2 \to -1$. Hence $\ell^2 = -1$, contradiction.

Using this, $(\mathbb{Q}, |\cdot|_5)$ is not compete.

Definition 1.13 (p-adic numbers)

The *p-adic numbers* $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

Analogy: Compare this with the construction of $\mathbb{R}$ as the completion of $\mathbb{Q}$ with respect to the usual absolute value.