Modular Forms

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Michaelmas 2023*

Contents

Lecture 1

1 Introduction

Definition 1.1

Define the upper half plane

$$\mathfrak{H} = \{\tau \in \mathbb{C} \mid \text{im}(\tau) > 0\}$$

and the positive determinant group

$$\text{GL}_2(\mathbb{R})^+ = \{g \in \text{GL}_2(\mathbb{R}) \mid \det(g) > 0\}$$

and

$$\Gamma(1) = \text{SL}_2(\mathbb{Z})$$

Lemma 1.2. $\text{GL}_2(\mathbb{R})^+$ acts transitively on $\mathfrak{H}$ via Möbius transformations.

Proof. Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{R})^+$, and $\tau \in \mathfrak{H}$. Then

$$\text{Im}(g\tau) = \frac{1}{2i}\left(\frac{a\tau + b}{c\tau + d} - \frac{a\overline{\tau} + b}{c\overline{\tau} + d}\right) = \frac{1}{2i}\frac{(ad - bc)(\tau - \overline{\tau})}{|c\tau + d|^2} = \frac{\deg(g)\,\text{Im}(\tau)}{|c\tau + d|^2} > 0$$

For transitivity, for $x + iy \in \mathfrak{H}$,

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot i = x + iy$$

□

Definition 1.3

Let $g \in \text{GL}_2(\mathbb{R})^+$, $\tau \in \mathfrak{H}$, define $j(g, \tau) = c\tau + d$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This is called the *modular cocycle*.

If k is an integer, $f : \mathfrak{H} \to \mathbb{C}$ is any function, then

$$\begin{aligned} f|_k[g] &: \mathfrak{h} \to \mathbb{C} \\ &\tau \mapsto \det(g)^{k-1} f(g\tau) j(g, \tau) \end{aligned}$$

*Based on lectures by Jack Thorne. Last updated October 13, 2023.

is called the *weight k action of g on f*.

Lemma 1.4. This is a *right* action of $\mathrm{GL}_2(\mathbb{R})^+$. That is,

$$f|_k[gh] = (f|_k[g])\,|_k[h]$$

Proof.

$$\begin{aligned} f|_k[g]|_k[h](\tau) &= \det(h)^{k-1} f|_k[g](h\tau) j(h,\tau)^{-k} = \det(h)^{k-1}\det(g)^{k-1} f(gh\tau) j(g,h\tau)^{-k} j(h,\tau)^{-k} \\ &\stackrel{?}{=} \det(gh)^{k-1} f(gh\tau) j(gh,\tau)^{-k} \end{aligned}$$

Therefore, suffices to show that

$$j(gh,\tau) = j(g,h\tau)j(h,\tau)$$

Note that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $g\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau+b \\ c\tau+d \end{pmatrix} = j(g,\tau)\begin{pmatrix} g\tau \\ 1 \end{pmatrix}$. This means that,

$$j(gh,\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix} = gh\begin{pmatrix} \tau \\ 1 \end{pmatrix} = g\left(j(h,\tau)\begin{pmatrix} h\tau \\ 1 \end{pmatrix}\right) = j(g,h\tau)j(h,\tau)\begin{pmatrix} gh\tau \\ 1 \end{pmatrix}$$

□

Formulae:
For $g,h \in \mathrm{GL}_2(\mathbb{R})^+$, $\tau \in \mathfrak{h}$,

$$\mathrm{Im}(g\tau) = \frac{\det(g)\,\mathrm{Im}(\tau)}{|j(g,\tau)|^2}$$

and

$$j(gh,\tau) = j(g,h\tau)j(h,\tau)$$

Definition 1.5

Let $k \in \mathbb{Z}$, $\Gamma \leq \Gamma(1)$ a finite index subgroup. A *weakly modular function of weight k and level* Γ is a meromorphic function $f : \mathfrak{H} \to \mathbb{C}$, which is invariant under the weight k action of Γ. That is, such that for all $\gamma \in \Gamma$,

$$f|_k[\gamma] = f$$

We will define a modular form (when $\Gamma = \Gamma(1)$) next time, but they are weakly modular functions, which are holomorphic in $\mathfrak{H}$ and at ∞.

In fact, modular forms of fixed weight and level live in finite dimensional $\mathbb{C}$-vector spaces $M_k(\Gamma)$, which are the main objects of study in this course.

Why do we study modular forms?

1. They are related to the theory of elliptic functions. Let $E/\mathbb{C}$ be an elliptic curve. Let ω be a non-zero holomorphic 1-form. Then there exists a unique lattice $\Lambda \leq \mathbb{C}$, and an isomorphism of Riemann surfaces, $\phi : \mathbb{C}/\Lambda \to E$, such that $\phi^*(\omega) = \mathrm{d}z$. We can show that E is isomorphic to the elliptic curve $y^2 = 4x^3 - 60G_4(\Lambda)x - 140G_6(\Lambda)$, where for $k \in \mathbb{Z}$,

$$G_k(\Lambda) = \sum_{\lambda \in \Lambda \setminus 0} \lambda^{-k}$$

which converges for $\lambda > 2$. If $\tau \in \mathfrak{H}$, then we have an associated lattice $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}_\tau$, and the function

$$G_k(\tau) = G_k(\Lambda_\tau)$$

is a modular form of weight k and level $\Gamma(1)$. This is called the *Eisenstein series*. Moreover, $\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})$ can be identified (as a set) with the set of isomorphism classes of elliptic curves over $\mathbb{C}$.

2. Modular forms f have Fourier expansions
$$\sum_{n\in\mathbb{Z}} a_n q^n$$
where $a_n \in \mathbb{C}$, and often serve as generating functions for arithmetically interesting sequences a_n. One example is
$$\vartheta(q) = \sum_n e^{\pi i n^2 \tau}$$
If $k \in 2\mathbb{N}$, then ϑ^{2k} is a modular form, and on the other hand,
$$\vartheta^{2k} = \sum_{n\in\mathbb{Z}} r_k(n) e^{\pi i n \tau}$$
where $r_k(n)$ is the number of ways to write n as a sum of k squares. By expressing ϑ^{2k} in terms of ther modular forms, we can prove formulae such as
$$r_4(n) = 8 \sum_{d\mid n, 4\nmid d} d$$

3. The Riemann ζ function is an important object in number theory. Properties include
   (a) The Euler product
   $$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$
   (b) A meromorphic extension to $\mathbb{C}$,
   (c) A functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

   A *Dirichlet L-series* is a function of the form
$$\sum_{n\geq 1} a_n n^{-s}$$
which has similar properties is called an L-function. Modular forms can be used to construct interesting examples of L-functions. Take $M_k(\Gamma)$ and decompose them under the action of Hecke operators. In particular, if $\Gamma = \Gamma(1)$, we get a decomposition into lines, called Hecke eigenforms.

4. Connections to the Langland programme, which predicts (among other things) a relation between modular forms and other objects in arithmetic geometry. A special case of this is the *Modularity conjecture*, which says that there is a bijection between elliptic curves over $\mathbb{Q}$ (up to isogeny) and the set of Hecke eigenforms of weight 2. This implies Fermat's last theorem. The bijection is formulated in the language of Hecke operators and L-functions.

   Homework: Handout on Moodle called "Reminder of Complex Analysis".

Lecture 2

2 Modular forms on $\Gamma(1)$

Recall a *meromorphic function* on $U \subseteq \mathbb{C}$ is a closed subset $A \subseteq U$ and a holomorphic function $f : U \setminus A \to \mathbb{C}$, such that for every $a \in A$, there exists $\delta > 0$, such that
$$D_*(a, \delta) \subseteq U \setminus A$$
and there exists an integer $n \geq \mathbb{Z}$ such that
$$(z-a)^n f(z)$$
defines a holomorphic function on $D(a, \delta)$. Such an $a \in A$ is called a *pole* of f. f then has a *Laurent expansion*
$$\sum_{m\in\mathbb{Z}} a_m (z-a)^m$$
which is absolutely convergent on $D(a, \delta)$.

Lemma 2.1. Let f be a weakly modular function of weight k and level $\Gamma(1)$. Then there exists a unique meromorphic function $\tilde{f}$ on $D_*(0,1)$ such that

$$f(\tau)\tilde{f}(e^{2\pi i\tau})$$

Proof. By assumption, f is meromorphic on $\mathfrak{H}$, let $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma(1)$. Then

$$f|_k[\gamma](\tau) = f(\gamma\tau) = f(\tau)$$

as f is invariant under the weight k action of γ. But

$$f(\gamma\tau) = f(\tau+1)$$

Existence: Locally, let $a \in D_*(0,1)$, $\delta > 0$ be such that $D(a,\delta) \subseteq D_*(0,1)$. Define $\tilde{f}$ in the disc by

$$\tilde{f}(q) = f\left(\frac{1}{2\pi i}\log(q)\right)$$

where log is any branch of logarithm defined in $D(a,\delta)$. This is independent of the choice of branch of logarithm, since any two branches differ by $2\pi i$, and $f(\tau) = f(\tau+1)$. Therefore, this defines $\tilde{f}$ on $D_*(0,1)$.

Uniqueness: Since the map $\tau \mapsto e^{2\pi i\tau}$ is surjective, $\tilde{f}$ is unique. □

Suppose $\tilde{f}$ extends to a meromorphic function on $D(0,1)$, then there exists $\delta > 0$ such that $\tilde{f}$ has a Laurent expansion

$$\tilde{f}(q) = \sum_{n\in\mathbb{Z}} a_n q^n$$

which is absolutely convergent on $D_*(0,\delta)$. In particular, in the region where $\mathrm{Im}(\tau) > \frac{1}{2\pi}\log(\delta)$, we have

$$f(\tau) = \sum_n a_n q^n$$

where $q = e^{2\pi i\tau}$. We call this the *q-expansion of a weakly modular function f*.

Definition 2.2

Let f be a weakly modular function of weight k and level $\Gamma(1)$. We say that f is *meromorphic at* ∞ if $\tilde{f}$ extends to a meromorphic function on $D(0,1)$.

We say that f is *holomorphic at* ∞ if $\tilde{f}$ extends, and is holomorphic at $q=0$. In this case, we define

$$f(\infty) = \tilde{f}(0) = \lim_{\mathrm{Im}(\tau)\to\infty} f(\tau)$$

We say that f *vanishes at* ∞ if f is holomorphic at ∞, and $f(\infty) = 0$.

Definition 2.3 (modular form)

A *modular function* (of weight k and level $\Gamma(1)$) is a weakly modular function, which is meromorphic at ∞. A *modular form* is a weakly modular function which is holomorphic in $\mathfrak{H}$, and holomorphic at ∞.

A *cuspidal modular form* is a modular form which vanishes at ∞.

Remark 2.4. We let

$$M_k(\Gamma(1))$$

be the set of *modular forms of weight k and level* $\Gamma(1)$, and

$$S_k(\Gamma(1)) \subseteq M_k(\Gamma(1))$$

be the set of *cuspidal modular forms of weight k and level* $\Gamma(1)$. These are $\mathbb{C}$-vector spaces. If k is odd, then they are zero. To see this, consider the matrix

$$\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1)$$

and the weight k-action is

$$f|_k[\gamma](\tau) = (-1)^k f(\tau) = f(\tau)$$

We will now consider even weights only.

If $k \in \mathbb{Z}$ is even, let

$$G_k(\tau) = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k} = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (m\tau + n)^{-k}$$

where $\Lambda_\tau = \mathbb{Z}_\tau \oplus \mathbb{Z} \leq \mathbb{C}$.

If $\gamma \in \Gamma(1)$, then we formally have

$$G_k|_k[\gamma](\tau) = G_k(\gamma\tau) j(\gamma, \tau)^{-k} = \sum_{\lambda \in \Lambda_{\gamma\tau} \setminus 0} \lambda^{-k} j(\gamma, \tau)^{-k}$$

But if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\Lambda_{\gamma\tau} = \mathbb{Z}\frac{a\tau + b}{c\tau + d} \oplus \mathbb{Z} = (c\tau + d)^{-1} \left(\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}(c\tau + d)\right) = (c\tau + d)^{-1} \Lambda_\tau$$

Therefore,

$$G_k|_k[\gamma](\tau) = \sum_{\lambda \in (c\tau + d)^{-1}\Lambda_\tau \setminus 0} \lambda^{-k} (c\tau + d)^{-k} = \sum_{\lambda \in \Lambda_\tau \setminus 0} \lambda^{-k} = G_k(\tau)$$

This is justified only when the series defining $G_k(\tau)$ converges absolutely.

Proposition 2.5. Let $k > 2$ be an even integer. Then $G_k(\tau)$ converges absolutely, and defines a modular form of weight k and level $\Gamma(1)$, with

$$G_k(\infty) = 2\zeta(k)$$

where ζ is the Riemann ζ-function. We call G_k the *weight k Eisenstein series*.

Remark 2.6. We will see later that $M_2(\Gamma(1)) = 0$, so this is optimal.

Proof. We want to show absolute and local uniform convergence in $\mathfrak{H}$, since this shows that G_k is holomorphic.

Let $A \geq 2$, and define

$$\Omega_A = \left\{ \tau \in \mathfrak{H} \;\middle|\; \mathrm{Im}(\tau) \geq \frac{1}{A} \text{ and } \mathrm{Re}(\tau) \in [-A, A] \right\}$$

We will show uniform convergence in Ω_A. If $\tau \in \Omega_A$, $x \in \mathbb{R}$, then

$$|\tau + x| \geq \begin{cases} \frac{1}{A} & |x| \leq 2A \\ \frac{|x|}{2} & |x| \geq 2A \end{cases}$$

Therefore,

$$\tau + x \geq \max\left\{\frac{1}{A}, \frac{|x|}{2A^2}\right\} \geq \max\left\{\frac{1}{A^2}, \frac{|x|}{2A^2}\right\} \geq \frac{1}{2A^2} \max\{1, |x|\}$$

If $(m, n) \in \mathbb{Z}^2$, then we get that

$$|m\tau + n| \geq \frac{1}{2A^2} \max\{|m|, |n|\}$$

If $\tau \in \Omega_A$, then

$$\begin{aligned}
\sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} |m\tau + n|^{-k} &\leq (2A^2)^k \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \max\{|m|, |n|\}^{-k} \\
&= (2A^2)^k \sum_d d^{-k} \cdot \#\{(m, n) \in \mathbb{Z}^2 \setminus 0 \mid \max\{|m|, |n|\} = d\} \\
&= (2A^2)^{-k} \sum_d d^{1-k} \\
&= 8(2A^2)^k \zeta(k - 1) < \infty
\end{aligned}$$

as $k > 2$. This shows uniform convergence in Ω_A by the Weierstrass M-test.

We now know that G_k is holomorphic in $\mathfrak{H}$, and invariant under the weight k action of $\Gamma(1)$. It remains to show to show G_k is holomorphic at ∞, with $G_k(\infty) = 2\zeta(k)$. It suffices to show that

$$\lim_{\operatorname{Im}(\tau)\to\infty} G_k(\tau) = 2\zeta(k)$$

By uniform convergence, we can exchange the limit and sum. But

$$\lim_{\operatorname{Im}(\tau)\to\infty} (m\tau + n)^{-k} = \begin{cases} 0 & m \neq 0 \\ n^{-k} & m = 0 \end{cases}$$

□

Lecture 3

If we consider the *q-expansion*

$$f(\tau) = \sum_{n\geq 0} a_n q^n$$

where $q = e^{2\pi i \tau}$, then we have that

$$f(\infty) = \tilde{f}(0) = a_0$$

We define

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 + \sum_{n\geq 1} a_n q^n$$

and we will see that $a_n \in \mathbb{Q}$ for all $n \geq 1$. We can construct more modular forms from these: if

$$f \in M_k(\Gamma(1)) \quad \text{and} \quad g \in M_\ell(\Gamma(1))$$

then

$$fg \in M_{k+\ell}(\Gamma(1))$$

Exercise: Check. For fg holomorphic at ∞, we can use that the q-expansions multiply.

For example,

$$E_4^3, E_6^2 \in M_{12}(\Gamma(1))$$

and in fact,

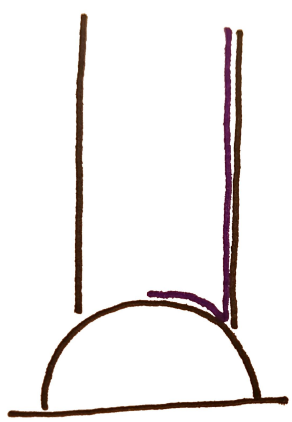
$$\Delta = E_4^2 - E_6^2 \in S_{12}(\Gamma(1))$$

and Δ is called the *Ramanujan Δ function*.

Next, we want to show $M_k(\Gamma(1))$ is finite dimensional. We will study the space $\Gamma(1)\backslash\mathfrak{H}$. To do this, we will introduce a *fundamental set* $\mathcal{F}' \subseteq \mathfrak{H}$ for the $\Gamma(1)$-action, which contains exactly one element from each $\Gamma(1)$-orbit.

$$\mathcal{F} = \left\{ \tau \in \mathcal{H} \;\middle|\; |\operatorname{Re}(\tau)| \leq \frac{1}{2}, |\tau| \geq 1 \right\}$$

$$\mathcal{F}' = \left\{ \tau \in \mathcal{F} \;\middle|\; \operatorname{Re}(\tau) < \frac{1}{2} \text{ and if } |\tau| = 1 \text{ then } -\frac{1}{2} \leq \operatorname{Re}(\tau) \leq 0 \right\}$$



Define

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then every element of $\mathcal{F}$ is conjugate under S, T to an element of $\mathcal{F}'$. In particular, $T(\tau) = \tau + 1$, $S(\tau) = -\overline{\tau}$.

Proposition 2.7. Let $G = \Gamma(1)/\{\pm I\}$. Then

(i) For every $\tau \in \mathfrak{H}$, τ is $\Gamma(1)$-conjugate to an element of $\mathcal{F}'$,

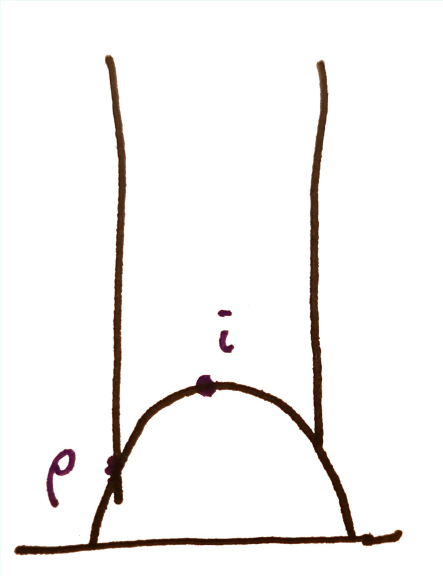
(ii) if $\tau, \tau' \in \mathcal{F}'$ are $\Gamma(1)$-conjugate, then $\tau = \tau'$,

(iii) if $\tau \in \mathcal{F}'$, then $\mathrm{Stab}_G(\tau) = 1$, except for

$$\mathrm{Stab}_G(i) = \langle S \rangle \quad \text{and} \quad \mathrm{Stab}_G(\rho) = \langle ST \rangle$$

where $\rho = e^{2\pi i/3}$.

(iv) $\Gamma(1)$ is generated by S, T.



Proof. Define $H = \langle S, T \rangle \leq G$.

Claim 2.8. Every $\tau \in \mathfrak{H}$ is H-conjugate to an element of $\mathcal{F}'$.

Proof of claim. By an easy observation, and as $S, T \in \mathfrak{H}$, it suffices to show every $\tau \in \mathfrak{H}$ is H-conjugate to an element of $\mathcal{F}$.

Let $\tau \in \mathfrak{H}$, and recall that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$$

In particular, for all $R > 0$, the intersection $H\tau \cap \{\mathrm{Im}(\tau') > R\}$ is finite, since $\mathrm{Im}(\tau) > R$ if and only if

$$|c\tau + d|^2 < \frac{\mathrm{Im}(\tau)}{R}$$

But $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$ is a lattice, and so its intersection with any compact subset is finite.

In particular, there exists $h \in H$ such that $\mathrm{Im}(h\tau) \geq \mathrm{Im}(h'\tau)$ for all $h' \in H$. That is, the maximum value of the imaginary part is attained. By replacing τ with $h\tau$, we can assume

$$\mathrm{Im}(\tau) \geq \mathrm{Im}(h\tau)$$

for all $h \in H$. Moreover, $\text{Im}(T\tau) = \text{Im}(\tau)$, and so we can assume that

$$-\frac{1}{2} \leq \text{Re}(\tau) \leq \frac{1}{2}$$

Setting $h = S$, we have that

$$\text{Im}(\tau) \geq \text{Im}(S\tau) = \frac{\text{Im}(\tau)}{|\tau|^2}$$

Hence $|\tau| \geq 1$. □

Note that this also proves (i). Now given $\tau, \tau' \in \mathcal{F}'$, and suppose $\gamma\tau = \tau'$ for some

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$$

We want to show that $\gamma = \pm I$ or $\tau = i$ or $\tau = \rho$. Without loss of generality, $\text{Im}(\tau') = \text{Im}(\gamma\tau) \geq \text{Im}(\tau)$. This means that

$$\text{Im}(\gamma(\tau)) = \frac{\text{Im}(\tau)}{|c\tau + d|^2} \geq \text{Im}(\tau)$$

and so $|c\tau + d| \leq 1$.

If $\tau \in \mathcal{F}'$ then $\text{Im}(\tau) \geq \sqrt{3}2$, with equality if and only if $\tau = \rho$. Hence

$$|c\tau + d| \geq |c|\, \text{im}(\tau) \geq |c|\frac{\sqrt{3}}{2}$$

Hence $|c| = 0$ or 1. Therefore, $c = 0, 1$ or -1.

If $c = 0$, then

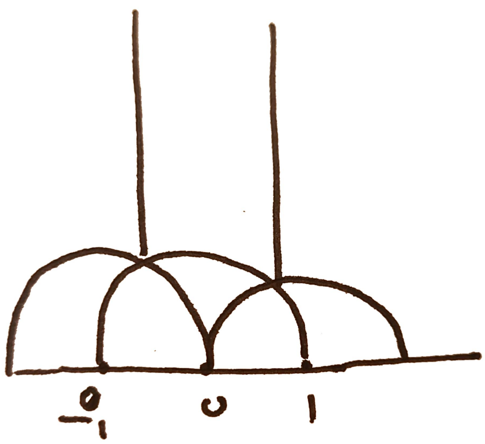
$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

In this case, $ad = 1$, and so $\gamma = \pm T^n$ for some T. But as the real part of τ and τ' lie in $[-1/2, 1/2)$, we must have $n = 0$, and so $\gamma = \pm I$.

If $c = 1$, then

$$\gamma = \begin{pmatrix} a & b \\ 1 & d \end{pmatrix}$$

and $|\tau + d| \leq 1$. The only circles centered at integers, and with radius 1 which intersects $\mathcal{F}'$ are centred at $-d = -1$ and $d = 0$.



The only possibilities are $d = 0$, and so $|\tau| = 1$, or $d = 1$, and so $\tau = \rho$.

If $c = 1, d = 0, |\tau| = 1$, then

$$\gamma = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$$

and $\gamma(\tau) = a - \tau^{-1} = a - \overline{\tau}$, and

$$\text{Re}(\gamma\tau) = a - \text{Re}(\tau) \in \text{Re}(\mathcal{F}' \cap \{|\tau| = 1\}) = [-1/2, 0]$$

But we also have that $\mathrm{Re}(\gamma(\tau)) \in a - [-1/2, 0] = a + [0, 1/2]$. But the intersection between

$$[-1/2, 0] \cap (a + [0, 1/2])$$

is non-empty only if $a = 0$, and the unique point in the intersection is 0, or $a = -1$, and the unique point of intersection is $\mathrm{Re}(\tau) = \mathrm{Re}(\gamma\tau) = -1/2$. The first case is $\gamma = i$ and the second is $\gamma = \rho$.

If $a = 0$, then

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -S$$

and if $a = -1$, then

$$\gamma = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

Computing,

$$ST = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

and

$$(ST)^2 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$$

and

$$(ST)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Now if $c = 1, d = 1, \tau = \rho$, then

$$\gamma = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix}$$

Then

$$\gamma\rho = \frac{a\rho + b}{\rho + 1} = \rho$$

We have that $\rho^2 + \rho + 1 = 0$. In particular, $\rho + 1 = -\rho^2$. Hence

$$a\rho + b = \rho^2 + \rho = -1$$

But since $a, b \in \mathbb{Z}$, and $\rho \in \mathbb{C}$, which is linearly independent of $\mathbb{R}$, and so $a\rho + b = -1$ implies $a = 0$ and $b = -1$. Therefore,

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = -ST$$

The case $c = -1$ can be reduced to the case $c = 1$ by replacing $\gamma = -\overline{\gamma}$.

We have now shown (ii) and (iii). That is, $\Gamma(1)$ is generated by S, T. As $S^2 = -I$, it suffices to show $G = H$. Choose $\tau \in \mathrm{Int}(\mathcal{F})$, then $\mathrm{Stab}_G(\tau) = \{I\}$. Let $g \in G$, there exists $h \in H$ such that $hg\tau \in \mathcal{F}'$. But then by (ii) and (iii), we must therefore have $hg\tau = \tau$. This then implies $hg \in \mathrm{Stab}_G(\tau) = 1$, and so $g = h^{-1} \in H$. □

Lecture 4

> **Notation 2.9.** For $\tau \in \mathfrak{H}$, write $e_\tau = |\mathrm{Stab}_G(\tau)|$.

Let f be a non-zero modular function of weight k and level $\Gamma(1)$. If $\tau \in \mathfrak{H}$, write $v_\tau(f)$ for the order of the function f at τ. That is, if $v_\tau(f) = n$, then

$$f(z) = (z - \tau)^n g(z)$$

in a neighbourhood of τ, g holomorphic and non-vanishing.

Define the *order of f at ∞*:

$$v_\infty(f) = v_0(\tilde{f})$$

where $\tilde{f}$ is the meromorphic function on the unit disc, with

$$f(\tau) = \tilde{f}(e^{2\pi i\tau})$$

Note that these are well defined as $f \neq 0$. With this,

Proposition 2.10. Let f be a non-zero modular function of weight k and level $\Gamma(1)$. Then

$$\sum_{\tau\in\Gamma(1)\backslash\mathfrak{H}} \frac{1}{e_\tau} v_\tau(f) + v_\infty(f) = \frac{k}{12}$$

Proof. First we need to check that the sum is well defined. If $\tau \in \mathfrak{H}$, then e_τ and $v_\tau(f)$ only depends on the $\Gamma(1)$ orbit of τ. Moreover, it only has finitely many non-zero terms.

If $\gamma \in \Gamma(1)$, $\tau \in \mathfrak{H}$, then

$$\mathrm{Stab}_{\Gamma(1)}(\tau) \quad \text{and} \quad \mathrm{Stab}_{\Gamma(1)}(\gamma\tau)$$

are conjugate subgroups of $\Gamma(1)$. In particular, they are isomorphic, and thus $e_\tau = e_{\gamma\tau}$. On the other hand,

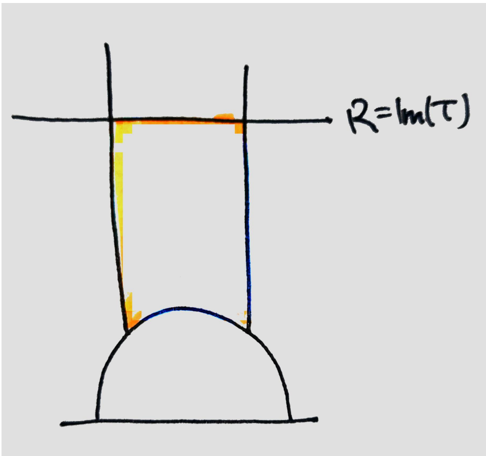
$$f(\gamma\tau) = f(\tau)j(\gamma,\tau)^{-k}$$

But j as a function on $\mathfrak{H}$ is a non-vanishing holomorphic function. Hence $v_{\gamma\tau}(f) = v_\tau(f)$.

Since f is a modular function, $\tilde{f}$ is a meromorphic function on the unit disc $D(0,1)$. Hence there exists $\delta > 0$, such that $\tilde{f}$ is holomorphic and non-vanishing in $D_*(0,\delta)$. Hence there exists $R > 0$, such that f is holomorphic, and non-vanishing on

$$\{\tau \in \mathfrak{H} \mid \mathrm{Im}(\tau) > R\}$$

Since each orbit intersects $\mathcal{F}$, to show that the sum is finite, it suffices to show f only has finitely many zeroes and poles in $\mathcal{F}$ with $\mathrm{Im}(\tau) \le R$. But this is true as the set of zeroes and poles of a meromorphic function are discrete, and $\mathcal{F} \cap \{\mathrm{Im}(\tau) \le R\}$ is compact.



To prove the identity, we will use contour integration.

Pullback formula: If $u : U \to V$ is a holomorphic map between open subsets of $\mathbb{C}$, and a holomorphic function $g : V \to \mathbb{C}$, and a path γ in U, then

$$\int_{u\circ\gamma} g(z)\mathrm{d}z = \int_\gamma u^*g(z)\mathrm{d}z = \int_\gamma g(u(z))u'(z)\mathrm{d}z$$

A nice case of this is when $g(z) = \frac{h'(z)}{h(z)}$ for a holomorphic non-vanishing function h. In this case,

$$g(z)\mathrm{d}z = \mathrm{d}\log(h)$$

and

$$\int_{u\circ\gamma} \mathrm{d}\log(h) = \int_\gamma u^*(\mathrm{d}\log(h)) = \int_\gamma \mathrm{d}(\log(h)\circ u) = \int_\gamma \frac{(h\circ u)'(z)}{(h\circ u)(z)}\mathrm{d}z$$

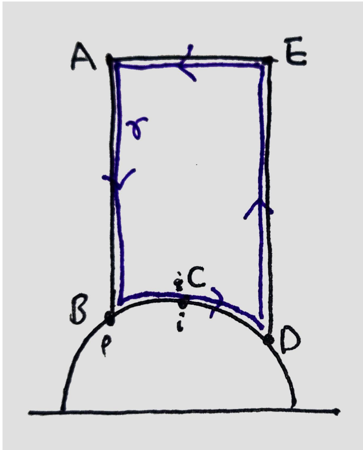
Cauchy's argument principle: If $U \subseteq \mathbb{C}$ a simply connected open subset, $\gamma \subseteq U$ a simple, positively oriented closed path. Let g be meromorphic in U, with no zeroes or poles on γ. Then

$$\frac{1}{2\pi i}\oint_\gamma \mathrm{d}\log(g) = \frac{1}{2\pi i}\oint_\gamma \frac{g'(z)}{g(z)}\mathrm{d}z = \sum_{a\in\mathrm{Int}(\gamma)} v_a(g)$$

With this, we can apply this to our modular function f. Choose $R > 0$ such that f has no zeroes or poles τ with $\mathrm{Im}(\tau) \geq R$. Consider the contour integral

$$\frac{1}{2\pi i} \oint_\gamma \mathrm{d}(\log(f))$$

where γ is the contour



By our choice of R, there are no poles on the line segment AE. We first consider the case where f has no zeroes or poles on γ. In this case, by the argument principle,

$$\frac{1}{2\pi i} \oint_\gamma \mathrm{d}\log(f) = \frac{1}{2\pi i} \left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA} \right) \mathrm{d}\log(f) = \sum_{\tau \in \Gamma(1)\backslash \mathfrak{H}} \frac{1}{e_\tau} v_\tau(f)$$

as $v_\tau(f) \neq 0$, $e_\tau(f) = 1$ under the assumptions.

Applying the pullback formula with $u(\tau) = \tau + 1$.

$$u(AB) = u(ED) \qquad f \circ u = f$$

and this gives us that

$$-\int_{DE} \mathrm{d}\log f = \int_{ED} \mathrm{d}\log g = \int_{u(AB)} \mathrm{d}\log f = \int_{AB} \mathrm{d}\log(f \circ u) = \int_{AB} \mathrm{d}\log f$$

Hence

$$\left(\int_{AB} + \int_{DE} \right) \mathrm{d}\log f = 0$$

Now let $q = e^{2\pi i \tau}$. Then $f = \tilde{f} \circ q$, and $q(AE)$ is a positively oriented circle around 0 in $D(0, 1)$. So

$$\frac{1}{2\pi i} \int_{AE} \mathrm{d}\log f = \frac{1}{2\pi i} \int_{AE} \mathrm{d}\log\left(\tilde{f} \circ q\right) = \frac{1}{2\pi i} \int_{q(AE)} \mathrm{d}\log\left(\tilde{f}\right) = v_\infty(f)$$

Now let $v(\tau) = S(\tau) = -\frac{1}{\tau}$. Then $v(BC) = DC$, and

$$f|_k[S](\tau) = f(-1/\tau)\tau^{-k} = f(\tau)$$

and so, $f \circ v(\tau) = f(\tau)\tau^k$. Hence

$$\begin{aligned}
\int_{DC} \mathrm{d}\log f &= \int_{v(BC)} \mathrm{d}\log(f) \\
&= \int_{BC} \mathrm{d}\log(f \circ v) \\
&= \int_{BC} \mathrm{d}\log\left(f(\tau)\tau^k\right) \\
&= \int_{BC} \mathrm{d}\log(f) + k \int_{BC} \mathrm{d}\log(\tau) \\
&= \int_{BC} \mathrm{d}\log(f) + k\log(C) - \log(B)
\end{aligned}$$

where log is any branch of logarithm defined on BC.

We can compute this, as $B = \rho, C = i$. Hence

$$\int_{CD} \mathrm{d}\log f = -\int_{BC} \mathrm{d}\log(f) + k\left(\frac{2\pi i}{3} - \frac{2\pi i}{4}\right)$$

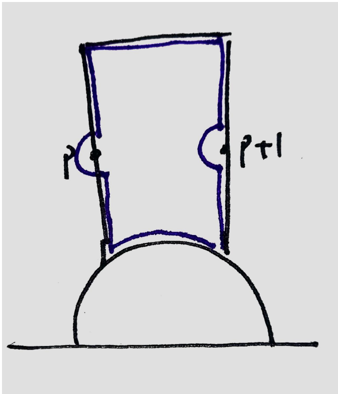
Therefore,

$$\left(\int_{BC} + \int_{CD}\right) \mathrm{d}\log f = \frac{2\pi i k}{12}$$

Hence

$$\sum_{\tau \in \Gamma(1)\backslash\mathfrak{H}} \frac{1}{e_\tau} v_\tau(f) = \frac{1}{2\pi i}\left(\int_{AB} + \int_{BC} + \int_{CD} + \int_{DE} + \int_{EA}\right) \mathrm{d}\log(f) = \frac{1}{2\pi i}\left(\frac{2\pi i k}{12} - 2\pi i v_\infty(f)\right)$$

Rearranging we get the result in this case. If there are poles on γ, then we will need to modify the contour. For example, if there was a pole at the point $P \in AB$, we can consider the modified contour:



For poles at B or C, we will leave to the examples sheet. □

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