Topology of negatively curved manifolds

Shing Tak Lam

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1 Motivation

Recall from IB Geometry the Gauss-Bonnet theorem.

Theorem 1.1 (Gauss-Bonnet). Suppose S is a compact Riemannian surface without boundary. Then

$$\int_S \kappa \mathrm{d}A = 2\pi\chi(S) \tag{1}$$

where κ is the Gaussian curvature and χ is the Euler characteristic.

As a result, if suppose we wanted to find Riemannian surfaces with negative curvature everywhere. Then Gauss-Bonnet gives us a *topological* restriction on the surface.

Example 1.2 (Genus-g surface)

Let Σ_g be denote the g-holed torus. Then Σ_g has a Riemannian metric with negative curvature everywhere if and only if $g \geq 2$.

In this talk, we're going to see how this generalises to general manifolds. That is, we'll define a notion of "curvature" for general Riemannian manifolds, and we'll see how this gives restrictions on the fundamental group of the manifold[1].

Throughout, let (M, g) be a connected Riemannian manifold with dimension n, $\pi : \tilde{M} \to M$ be its universal cover, $G = G_D(\pi)$ be the deck group. Note $G \cong \pi_1(M, p)$. We equip $\tilde{M}$ with the pullback metric $\pi_*(g)$, so π is a local isometry.

Lemma 1.3 (Deck transformations are isometries)**.** $G_D(\pi) \leq \mathrm{Isom}(\tilde{M})$.

Proof. Let $f \in G_D(\pi)$, then f must be a local isometry, as $f \circ \pi = \pi$ and $\tilde{M}$ has the pullback metric.

Since the length of a curve near a point is preserved by local isometries, and by compactness, we can cover a curve by finitely many open neighbourhoods. So for any curve γ, $\mathrm{Length}(f \circ \gamma) = \mathrm{Length}(\gamma)$, so $d(f(p), f(q)) = d(p, q)$ and f must be a (global) isometry. $\square$

2 Exponential map and sectional curvature

First of all, a curve $\gamma : [a, b] \to M$ is called a *geodesic* if

$$\ddot{\gamma}^i(t) + \Gamma^i{}_{jk}\dot{\gamma}^j(t)\dot{\gamma}^k(t) = 0 \tag{2}$$

We don't really care about the exact equation here, or what $\Gamma^i{}_{jk}$ is. What we do care about is

Theorem 2.1. Let $p \in M$, $v \in T_pM$. Then there exists $\varepsilon > 0$, and a unique geodesic $c : [0, \varepsilon] \to M$ with $c(0) = p$ and $\dot{c}(0) = v$. In addition, c depends smoothly on p and v.

Proof. Equation (2) is a second order ODE, so the result follows from Picard-Lindelöf. $\square$

Now fix $p \in M$, and denote the geodesic from theorem 2.1 with by c_v. Then we have that

$$c_v(t) = c_{\lambda v}\left(\frac{t}{\lambda}\right) \qquad \text{for} \qquad \lambda > 0, t \in [0, \varepsilon]$$

[1] In fact, we'll be working with the deck group of the universal covering.

by uniqueness in theorem 2.1 and the chain rule. In particular, $c_{\lambda v}$ is defined on $[0, \varepsilon/\lambda]$. Since c_v and ε depends smoothly on v, and the unit sphere S in T_pM is compact, there exists $\varepsilon_0 > 0$ such that for all $v \in S$, c_v is defined on $[0, \varepsilon_0]$. Therefore, for any $w \in T_pM$ with $||w|| \leq \varepsilon_0$, c_w is defined on $[0, 1]$.

Definition 2.2 (Exponential map)
Let

$$V_p := \{v \in T_pM \mid c_v \text{ is defined on } [0, 1]\}$$

and define, $\exp_p : V_p \to M$, the exponential map of M at p by

$$\exp_p(v) = c_v(1)$$

Lemma 2.3. The exponential map $\exp_p$ maps a neighbourhood of $0 \in T_pM$ diffeomorphically onto a neighbourhood of $p \in M$.

Proof. Since T_pM is a vector space, we can identify $T_pM = T_0T_pM$. Then the derivative at 0, $d\exp_p(0)$ becomes a map $T_pM \to T_pM$. Computing, we find that

$$d\exp_p(v) = v$$

and the result follows by the inverse function theorem. □

With all of this in mind, we can now define the sectional curvature of M. Let σ be a 2-dimensional subspace of T_pM. Then $\sigma \cap V_p$ is an embedded surface in V_p, so by lemma 2.3, $S_\sigma = \exp_p(\sigma \cap V_p)$ is an embedded surface in M. S_σ is a Riemannian surface with the Riemannian metric given by restriction, so we can define the sectional curvature of σ, $\sec(\sigma)$ by the Gaussian curvature of S_σ at p.

3 Cartan's theorem

First of all, we will need a few things which requires a bit more machinery to prove, so we will black box it.

Suppose M has nonpositive sectional curvature. Then

Theorem 3.1 (Cartan-Hadamard)**.** $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$, and $\exp_p : T_pM \to M$ is a covering map.

We omit the proof here, since it uses some machinery which we don't have yet.

Lemma 3.2 (Geodesics are unique)**.** Let $p, q \in \tilde{M}$ be distinct points. Then there exists a unique geodesic γ from p to q.

Proof. Since $\exp_p : T_p\tilde{M} \to \tilde{M}$ is a diffeomorphism, write $q = \exp_p(v)$ for some $v \in T_p\tilde{M}$. Now let γ be a geodesic from p to q. By reparametrisation, without loss of generality $\gamma : [0, 1] \to \tilde{M}$, with $\gamma(0) = p$ and $\gamma(1) = q$. From theorem 2.1, we have that γ is uniquely determined by $\dot{\gamma}(0) = w$. Therefore, we must have that $\gamma = c_w$, so $q = \exp_p(w)$. But $\exp_p$ is a diffeomorphism (hence a bijection), so $v = w$ and γ is unique. $\square$

Define $f_p(x) = \frac{1}{2}d(x, p)^2$ for $x, p \in \tilde{M}$. Then $f_p : \tilde{M} \to \mathbb{R}$ is smooth, since the distance is a smooth function of $\exp_p$ and in $\mathbb{R}^n$, $x \mapsto ||x||^2$ is smooth.

Lemma 3.3. Let c be a geodesic on $\tilde{M}$. Then $f_p \circ c$ is strictly convex.

Proof. Since $g(x) = x^2$ is strictly convex. $\square$

We can extend the definition of convexity from $\mathbb{R}^n$ to any Riemannian manifold, by saying that a function $f : M \to \mathbb{R}$ is (strictly) convex if $f \circ c$ is (strictly) convex for any geodesic c.

Theorem 3.4 (Cartan)**.** Let $F : \tilde{M} \to \tilde{M}$ be an isometry with finite order (so $F^k = \mathrm{id}$ for some k). Then $\tilde{F}$ has a fixed point.

Example 3.5

Let us first consider the simplest case where $\tilde{M} = \mathbb{R}^n$ with the Euclidean norm. Let R be a rotation, $x \in M$. Say $R^k = \mathrm{id}$. Then consider $x, Rx, \ldots, R^{k-1}x$. The average of these points must in fact be in the axis of the rotation. Similarly if T is a reflection, then $(x + Tx)/2$ must be in the plane of reflection.

We can generalise this idea of considering the average of the orbits to the general case. First, we define the centre of mass of points $p_1, \ldots, p_k \in M$ to be

$$\mathrm{cm}\{p_1, \ldots, p_k\} = \arg\min_x(\max\{f_{p_1}(x), \ldots, f_{p_k}(x)\})$$

This is well defined since a stictly convex function has a unique minimum. With this, we can now prove Cartan's theorem.

Proof. Let k minimal be such that $F^k = \mathrm{id}$, $p \in \tilde{M}$ be arbitrary. Define $q = \mathrm{cm}\{p, F(p), \ldots, F^{k-1}(p)\}$. We'll show $F(q) = q$.

Let $f(x) = \max\{f_p(x), f_{F(p)}(x), \ldots, f_{F^{k-1}(p)}(x)\}$. Then

$$\begin{aligned}
f(F(q)) &= \max\{f_p(F(q)), \ldots, f_{F^{k-1}(p)}(F(q))\} \\
&= \frac{1}{2}\max\{d(F(q), p), \ldots, d(F(q), F^{k-1}(p))\}^2 \\
&= \frac{1}{2}\max\{d(F(q), F^k(p)), \ldots, d(F(q), F^{k-1}(p))\}^2 \\
&= \frac{1}{2}\max d(q, F^{k-1}(p)), \ldots, d(q, F^{k-2}(p))^2 \\
&= f(q)
\end{aligned}$$

But f is strictly convex, so it has a unique minimum. □

With this result, we can now prove our first result about $\pi_1(M)$.

Corollary 3.6. Let M be a complete Riemannian manifold with nonpositive curvature. Then $\pi_1(M, p)$ is torsion free.

Proof. If there is any deck transformation f with finite order, then it must have a fixed point by theorem 3.4 (Cartan's theorem). But this means that $F = \mathrm{id}$. □

4 Preissmann's theorem

Theorem 4.1 (Preissmann)**.** Let M be a compact manifold with negative curvature. Then every nontrivial abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

To prove this, we will need a few preliminary results. Let $f : M \to M$ be an isometry. Then an *axis* for f is a geodesic $c : \mathbb{R} \to M$ such that $f \circ c$ is a reparametrisation of c. Since f maps geodesics to geodesics, and geodesics are parametrised by arc length, we have that

$$f \circ c(t) = c(\pm t + a)$$

If we have $-$, then f fixes $c(a/2)$. When we have $f(c(t)) = c(t + a)$, we call a the *period* of F with respect to c.

Given an isometry $f : M \to M$, the displacement function of f is $\delta_F : M \to \mathbb{R}$, defined by

$$\delta_f(x) = d(f(x), x)$$

Lemma 4.2. If δ_f has a positive minimum, then f has an axis.

Proof. Suppose δ_f attains it's minimum at $p \in M$, $c : [0, 1] \to M$ be a segment from p to $f(p)$. Then $f \circ c$ is a segment from $f(p)$ to $f(f(p))$ with the same speed.

We claim that these two meet at an angle π at $f(p)$, so we can join them to get a geodesic $c : [0, 2] \to M$. Fix $t \in [0, 1]$, then

$$\begin{aligned}
\delta_f(p) &\leq \delta_f(c(t)) \\
&= d(c(t), f(c(t))) \\
&\leq d(c(t), c(1)) + d(c(1), f(c(t))) \\
&= d(c(t), c(1)) + d(f(c(0)), f(c(t))) \\
&= d(c(t), c(1)) + d(c(0), c(t)) \\
&= d(c(1), c(0)) \\
&= \delta_f(p)
\end{aligned}$$

So equality holds in all of these, and the segment given by $c|_{[t,1]}$ followed by $f \circ c|_{[0,t]}$ is a geodesic[2]. But by theorem 3.1 (Cartan-Hadamard), $\exp_p$ is defined on all of T_pM, so we can extend c to a geodesic $c : \mathbb{R} \to M$. These two must agree, so $(f \circ c)(t) = c(1 + t)$. Repeating this argument we see that c is an axis of F with period 1. $\square$

Lemma 4.3. If $f : \tilde{M} \to \tilde{M}$ a non-trivial deck transformation of the universal cover $\tilde{M}$ of M, then δ_f has a positive minimum. The axis corresponding to this minimum is mapped to a closed geodesic in M whose length is minimal in it's homotopy[a] class. Moreover, we have the lower bound

$$\delta_f(x) \geq C_M > 0$$

for some constant C_M[b] depending only on M.

[a] unbased homotopy

[b] $C_M = 2\,\mathrm{inj}(M)$ is positive since M is compact,and $C_M = \infty$ if M is simply connected.

Proof. Omitted. $\square$

Lemma 4.4 (Triangles in negative curvature). Let T be a triangle in M with sides a, b, c and angles α, β, γ. Then

[2] Since geodesics are distance minimising.

(i) $\alpha + \beta + \gamma \leq \pi$, with equality if and only if T is degenerate, i.e. all three vertices lie on a line.

(ii) $a^2 + b^2 - 2ab\cos(\gamma) \leq c^2$.

Lemma 4.5. Let $A \subseteq \mathbb{R}$ be a subgroup, $a = \inf\{x \in A \mid x > 0\}$. If $a = 0$ then A is dense, and if $a > 0$ then $A = a\mathbb{Z}$.

With these, we are now ready to prove Preissmann's theorem.

Proof of Preissmann's theorem. As in the corollary of Cartan's theorem, we will be working with the deck group instead. By the lemmas above, we know that any non-trivial deck transformation F has an axis. First we will show that this axis must be unique.

Suppose c_1, c_2 are axes for F.

First we consider the case where c_1 and c_2 intersect at some $p \in M$. Then since they are axes and f has no fixed points, $f(p)$ must also be on both of them. So c_1 and c_2 intersect at at least two points. By theorem 3.1 (Cartan-Hadamard), we have a unique geodesic between p and $f(p)$. So in fact $c_1 = c_2$.

Now suppose c_1 and c_2 do not intersect. Fix $p_1 \in c_1$ and $p_2 \in c_2$. Let σ be the segement joining p_1 to p_2. Then $f \circ \sigma$ is the segment joining $f(p_1)$ to $f(p_2)$. Since f is an isometry that preserves c_1 and c_2, the angle between the axes and σ must be the same as the angle between $f \circ \sigma$ and the axes. So $p_1, p_2, F(p_2), F(p_1)$ define a quadrilateral with angle sum 2π. Hence by lemma 4.4 (angle sum), it is degenerate, so all points lie on one geodesic, and $c_1 = c_2$. Contradiction.

Finally, let g be any deck transformation which commutes commutes with f. Suppose c is an axis for f with period 1, then

$$g(c(t+1)) = g(f(c(t))) = f(g(c(t)))$$

This implies that $g \circ c$ is an axis for f, so by uniqueness must be c itself. Now let $H = \langle f, g \rangle$ be the subgroup of the deck group generated by f and g. Any element of H has c as an axis, so we get a map $\Phi : H \to \mathbb{R}$ which sends an isometry to its period. This is a group homomorphism, with trivial kernel since any deck transformation with a fixed point must be the identity.

Since any positive period is greater than $\frac{C_M}{\|\dot{c}\|}$, H must be cyclic. □

Example 4.6 (Torus)

Recall from IB Geometry the Gauss-Bonnet theorem, that

$$\int_S \kappa \mathrm{d}x = 2\pi\chi(S)$$

where $\chi(S)$ is the Euler characteristic of S. From this, we can show that there is no metric on $T^2 = S^1 \times S^1$ with negative curvature everywhere, since $\chi(T^2) = 0$.

By Preissmann's theorem, this generalises to all T^n, $n \geq 2$. To see this, note $\pi_1(T^n) = Z^{\oplus n}$, so there are non-trivial, non-cylic abelian subgroups of $\pi_1(T^n)$ for $n \geq 2$.

In fact, even more is true.

Theorem 4.7. If M is a compact Riemannian manifold with negative sectional curvature. Then $\pi_1(M)$ is non-abelian.

Proof. Suppose $\pi_1(M)$ is abelian. Then by Preissmann's theorem, $\pi_1(M)$ is infinite cyclic, generated by f. Let $\tilde{c}$ be the axis for f, and fix a point $\tilde{p}$ on c. Let $\tilde{b}$ be a geodesic orthogonal to $\tilde{c}$ at $\tilde{p}$, with $\tilde{b}(0) = p$ and $t > 0$ fixed. Let $b = \pi \circ \tilde{b}$ and $c = \pi \circ \tilde{c}$, $p = \pi(\tilde{p})$. Let α_t be a geodesic from $b(t)$ to p. We will show $\text{Length}(\alpha_t) = t$.

Let $\tilde{\alpha}_t$ be the lift of α_t starting at $b(t)$. Since c is the axis for f, we must have that the end point of $\tilde{\alpha}(t)$ lies on c. So by lemma 4.4 part (ii), we must have that

$$\text{Length}(\tilde{\alpha}_t) \geq \text{Length}(\tilde{b})$$

But since the covering map π is a local isometry, we must also have that

$$\text{Length}(\tilde{\alpha}_t) = \text{Length}(\alpha_t) \leq \text{Length}(b) = \text{Length}(\tilde{b}) = t$$

So $\text{Length}(\alpha_t) = t$ is unbounded. Contradiction as M is compact. □

In fact, *even more is true.*

Theorem 4.8 (Byers)**.** If M is a compact Riemannian manifold with negative sectional curevature, then any nontrivial solvable subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$. Furthermore, $\pi_1(M)$ does not have a (infinite) cyclic subgroup of finite index.

5 Growth of the fundamental group

One easy corollary of Cartan's theorem is that $\pi_1(M)$ must be infinite. Preissmann's theorem says that the group is "very non-abelian". Milnor proved a theorem about one way of measuring how non-abelian a group is. First, we need some group theory.

Let G be a finitely generated group, $S = \{g_1, \ldots, g_p\}$ a set of generators, S closed under inverse. Then given $x \in G$, it's *word norm* is defined to be

$$||x|| = \inf\{\ell \mid x = g_{i_1} \cdots g_{i_\ell}\}$$

and define it's *growth function*

$$\gamma(s) = |\{x \in G \mid ||x|| \leq s\}|$$

We have an easy upper bound of $\gamma(s) \leq p^s$. Intuitively, if a group is "somewhat abelian", then we would get some cancellation between the elements, so it can't grow that quickly. Conversely, if a group is "highly non-abelian", then not much cancellation will occur and we'll get exponential growth.

Theorem 5.1 (Milnor)**.** If M is a compact Riemannian manifold with negative curvature, then the growth function of $\pi_1(M)$ is at least exponential. That is,

$$\gamma(s) \geq a^s$$

for some constant $a > 1$.