Analysis I

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This document is intended for revision purposes. As a result, it does not contain any exposition. This is based off lectures given by Professor Gabriel Paternain in Lent 2021, but the order of content, as well as some of the proofs have been modified after the fact, primarily to provide simpler proofs for theorems. Note that this also contains theorems from examples sheets, as some are useful elsewhere.

Analysis I is on *Paper 1*.

Contents

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1 Limits and Convergence

Definition (Sequence)**.** A sequence (x_n), $(x_n)_{n=1}^{\infty}$ is a function $\mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$).

Definition (Convergence)**.** $x_n \to x$ as $n \to \infty$ if $\forall \varepsilon > 0, \exists N, \forall n \geq N, |x_n - x| < \varepsilon$.

Definition (Increasing Sequence)**.** (x_n) is increasing if $\forall n, x_n \leq x_{n+1}$

Definition (Strictly Increasing Sequence)**.** (x_n) is strictly increasing if $\forall n, x_n < x_{n+1}$

Definition (Monotone)**.** (x_n) is monotone if (x_n) is increasing or decreasing.

Definition (Supremum)**.** If a set $S \subseteq \mathbb{R}$ is nonempty and bounded above, $\sup S$ is the least upper bound of S. That is

- $\forall x \in S, x \leq \sup S$
- $\forall \varepsilon > 0, \exists x \in S, k - \varepsilon < x \leq k$

Theorem. *Every increasing sequence that is bounded above converges.*

Proof. Let $S = \{x_n : n \in \mathbb{N}\}$. Then S is bounded above and nonempty. We will show that $x_n \to \sup S$. By definition of the supremum, we know that given $\varepsilon > 0$, there exists N such that $k - \varepsilon < x_N \leq k$. As x_n is increasing, $\forall n \geq N$, $k - \varepsilon < x_N \leq x_n \leq k$. $\square$

Lemma. *(i) Limit is unique. If $x_n \to a$ and $x_n \to b$ as $n \to \infty$, then $a = b$.*

(ii) If $a_n \to a$ as $n \to \infty$, and $n_1 < n_2 < \dots$, then $a_{n_k} \to a$ as $k \to \infty$.

(iii) If $a_n = c$ for all n, then $a_n \to c$.

(iv) If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

(v) If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

(vi) If $a_n \to a$, $a_n \neq 0$ for all n, $a \neq 0$, then $1/a_n \to 1/a$.

(vii) If $a_n \leq A$ for all n, and $a_n \to a$, then $a \leq A$.

Proof. (i) Given $\varepsilon > 0$, we have $|a - b| \leq |a_n - a| + |a_n - b| < 2\varepsilon$. Setting $\varepsilon = |a - b|/3$ for $a \neq b$ we get a contradiction. So $a = b$.

(ii) Given $\varepsilon > 0$, exists N such that $\forall n \geq N$, $|a_n - a| < \varepsilon$. We note that $n_k \geq k$, so $\forall k \geq N$, $|a_{n_k} - a| < \varepsilon$.

(iii) Clearly $a_n - c = 0$, so N is arbitrary.

(iv) Clear from triangle inequality. $|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b|$.

(v) $|a_n b_n - ab| \leq |a_n b_n - a_n b| + |a_n b - ab| = |a_n||b_n - b| + |b||a_n - a|$. For n large enough, $|a_n| \leq |a| + 1$, $|a_n - a| < \varepsilon$, $|b_n - b| < \varepsilon$, so $|a_n b_n - ab| < \varepsilon(|a| + 1 + |b|)$.

(vi) For n large enough, $|a_n| \geq \frac{1}{2}|a|$. Then $\frac{1}{|a_n a|} < \frac{2}{|a|^2}$. Also, $|a_n - a| < \varepsilon$. With this, $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a_n - a}{a_n a}\right| < \frac{2}{|a|^2}\varepsilon$.

(vii) If $a > A$, then $\varepsilon = a - A > 0$, $|a_n - a| < a - A \implies a - a + A < a_n \implies A < a_n$. Contradiction. $\square$

Lemma. $\frac{1}{n} \to 0$

Proof. Clearly decreasing and bounded below so it converges to a. Note $\frac{1}{2n} \to \frac{1}{2}a$, but it is also a subsequence so $\frac{1}{2}a = a$. Hence $a = 0$. $\square$

1.1 Bolzano-Weierstrass

Theorem (Bolzano Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Proof. Suppose for all n, $x_n \in [a, b]$. At least one of $S_1 = \{n : x_n \in [a, \frac{a+b}{2}]\}$ and $S_2 = \{n : x_n \in [\frac{a+b}{2}, b]\}$ must be infinite. We can use this to generate new interval $[a_1, b_1]$ where $\{n : x_n \in [a_1, b_1]\}$ is infinite. Proceed inductively and we get two sequences a_n, b_n. Now note that $a_{n-1} \leq a_n \leq b_n \leq b_{n-1}$, and $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1})$. So $a_n \to a$, $b_n \to b$ and $a = b$. Now as $\{n : x_n \in [a_k, b_k]\}$ is infinite for all k, we may choose x_{n_k} such that $x_{n_k} \in [a_k, b_k]$ and $n_k > n_{k-1} > \cdots > n_1$. So $x_{n_k} \to a$. □

Proof. Alternatively, every bounded sequence has a monotonic subsequence. Consider the set $S = \{n : x_n \geq x_m \ \forall m \geq n\}$. If S is infinite, we can use this to construct our monotonic subsequence. If S is finite, then for all $n > \max(S)$, there exists $m \geq n$ such that $x_m > x_n$. We can use this to construct a monotonic subsequence. Clearly any monotonic bounded sequence converges. □

1.2 Cauchy Sequences

Definition (Cauchy)**.** (a_n) is Cauchy if given $\varepsilon > 0$, $\exists N$, $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$.

Lemma. *Every convergent sequence is Cauchy.*

Proof. $|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\varepsilon$ for n, m large enough. □

Theorem. *Every Cauchy sequence is convergent.*

Proof. First note that every Cauchy sequence is bounded, by setting $\varepsilon = 1$ in definition and taking the max of it and the first N terms. From Bolzano-Weierstrass, a_n has a convergent subsequence $a_{n_j} \to a$. Then $|a_n - a| \leq \left|a_n - a_{n_j}\right| + \left|a_{n_j} - a\right| < 2\varepsilon$ for n, j large enough. □

1.3 Series

Definition. For a sequence (a_n) (real or complex), $\sum\limits_{j=1}^{\infty} a_j$ converges to s if the partial sums $\sum\limits_{j=1}^{N} a_j$ converges to s as $N \to \infty$.

Lemma. *(i) If $\sum\limits_{j=1}^{\infty} a_j$ and $\sum\limits_{j=1}^{\infty} b_j$ converges, so does $\sum\limits_{j=1}^{\infty}(\lambda a_j + \mu b_j)$*

(ii) If for all $n \geq N$, $a_n = b_n$, then $\sum\limits_{j=1}^{\infty} a_j$ converges if and only if $\sum\limits_{j=1}^{\infty}$ converges.

Proof. (i) Write out partial sums and use distributivity.

(ii) Let $S_n = \sum\limits_{1}^{n} a_j = \sum\limits_{1}^{N-1} a_j + \sum\limits_{N}^{n} a_j$ and $D_n = \sum\limits_{1}^{n} b_j = \sum\limits_{1}^{N-1} b_j + \sum\limits_{N}^{n} b_j$. $S_n - d_n = \sum\limits_{1}^{N-1} a_j + \sum\limits_{1}^{N-1} b_j$ and is constant. So S_n converges if and only if D_n converges. □

Lemma. *If $\sum\limits_{1}^{\infty} a_j$ converges, then $a_j \to 0$.*

Proof. Let $S_n = \sum\limits_{1}^{n} a_j$. $a_n = S_n - S_{n-1} \to 0$. □

Proof. Let $S_n = \sum\limits_{1}^{n} 1/j$. $S_{2n} = S_n + 1/(n+1) + \ldots + 1/2n \geq S_n + 1/2$. So if $S_n \to a$, then $S_{2n} \to a$, and $a \geq a + 1/2$. Contradiction. □

1.4 Geometric Series

Definition (Geometric Series)**.** Given x, set $a_n = x^{n-1}$. $S_n = \sum_1^n a_j$.

Proposition. $S_n = \frac{1-x^n}{1-x}$.

Proof. Consider $(1-x)S_n$ and the terms cancel. □

Proposition. *S_n converges if and only if $|x| < 1$.*

Proof. If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$. To see this, wlog $0 < x < 1$, let $1/x = 1 + \delta$ where $\delta > 0$. Then $0 < x^n = 1/(1+\delta)^n \leq 1/(1+\delta n)$. By comparison with $1/n$, we can show $1/(1+\delta n) \to 0$.

If $|x| > 1$, again wlog $x > 1$, $x^n = (1+\delta)^n \geq 1 + \delta n \to \infty$.

For $x = 1$, $S_n = n$ and clearly diverges. For $x = -1$, S_n oscillates between 1 and -1. □

1.5 Convergence Tests

Theorem (Comparison Test)**.** *If $0 \leq b_n \leq a_n$ for all n, and $\sum_1^\infty a_n$ converges, then so does $\sum_1^\infty b_n$.*

Proof. Let $S_n = \sum_1^n a_j$ and $D_n = \sum_1^n b_j$, then $D_n \leq S_n \leq S$ for all n, and D_n is increasing, so it must converge. □

Theorem (Root Test)**.** *Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. If $a < 1$, then $\sum a_n$ converges. If $a > 1$, then $\sum a_n$ diverges.*

Proof. If $a < 1$, choose $a < r < 1$. Then for n large enough, $a_n^{1/n} < r$ and $a_n < r^n$. As $r < 1$, the geometric series converges. By comparison, so does a_n.

If $a > 1$, for n large enough, $a_n^{1/n} > 1$, so $a_n > 1$. Then $\sum a_n$ can't converge as the terms don't tend to zero. □

Theorem (Ratio Test)**.** *Suppose $a_n \geq 0$, $\frac{a_{n+1}}{a_n} \to l$ as $n \to \infty$. If $l < 1$, then $\sum a_n$ converges, if $l > 1$, then $\sum a_n$ diverges.*

Proof. If $l < 1$, choose $l < r < 1$. For $n \geq N$, we have $0 < \frac{a_{n+1}}{a_n} < r$. Then $a_n = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_n < a_N r^{n-N}$. By comparison to geometric, $\sum a_n$ converges.

If $l > 1$, then $a_n > a_N r^{n-N}$ by a similar method. By comparison to geometric, $\sum a_n$ diverges. □

Theorem (Cauchy's Condensation Test)**.** *Let a_n be a sequence of decreasing positive terms. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges.*

Proof. First suppose if $\sum a_n = A$. $2^{n-1}a_{2^n} \leq \sum_{m=2^{n-1}+1}^{2^n} a_m$, so $\sum_{n=1}^N 2^{n-1}a_{2^n} \leq \sum_{n=1}^N \sum_{m=2^{n-1}+1}^{2^n} a_m = \sum_{m=2}^{2^N} a_n$. Hence $\sum_{n=1}^N 2^n a_{2^n} \leq 2\sum_{m=2}^N a_m \leq 2(A - a_1)$. So $\sum_{n=1}^N 2^n a_{2^n}$ is increasing and bounded above, thus it converges.

For the reverse implication, $\sum_{m=2^{n-1}+1}^{2^n} a_m \leq 2^{n-1}a_{2^{n-1}}$, so $\sum_{m=2}^{2^N} = \sum_{n=1}^N \sum_{m=2^{n-1}-1}^{2^n} a_m \leq \sum_{n=1}^N 2^{n-1}a_{2^{n-1}}$. Thus $\sum a_n$ is bounded and increasing, so it converges. □

Proposition. *$\sum \frac{1}{n^k}$ converges if and only if $k > 1$.*

Proof. As $\frac{n}{n+1} < 1$, $(\frac{n}{n+1})^k < 1$, so $0 < \frac{1}{n^k} < \frac{1}{(n+1)^k}$. $2^n a_{2^n} = 2^n(1/(2^n))^k = (2^{1-k})^n$. So $\sum 2^n a_{2^n}$ is a geometric series. This converges if and only if $2^{1-k} < 1$, ie $k > 1$. □

Theorem (Alternating Series Test)**.** *If a_n is decreasing, and $a_n \to 0$, then $\sum(-1)^{n+1}a_n$ converges.*

Proof. Let $S_n = \sum_{j=1}^n (-1)^{j+1}a_j$. Then $S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) = S_{2n-2} + (a_{2n-1} - a_{2n})$. As a_n is decreasing, $S_{2n} \geq S_{2n-2}$. Rearranging, $S_{2n} = a_1 - (a_2 - a_3) - \cdots - a_{2n} \leq a_1$. So S_{2n} is bounded above and increasing, thus it converges. Say $S_{2n} \to S$. $S_{2n+1} = S_{2n} + a_{2n+1} \to S$ as well. Hence $S_n \to S$ by choosing N to be the max of the N from S_{2n} and S_{2n+1}. □

1.6 Absolute Convergence

Definition (Absolute Convergence)**.** Let $a_n \in \mathbb{C}$, then $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

Theorem. *Absolute convergence implies convergence.*

Proof. First suppose $a_n \in \mathbb{R}$. Let $v_n = \frac{|a_n|+a_n}{2}$, $w_n = \frac{|a_n|-a_n}{2}$. Then $v_n, w_n \geq 0$, $a_n = v_n - w_n$, $|a_n| = v_n + w_n \geq v_n, w_n$. if $\sum |a_n|$ converges, by comparison $\sum v_n$ and $\sum w_n$ converges. So $\sum a_n = \sum v_n - w_n = \sum v_n - \sum w_n$ converges.

If $a_n \in \mathbb{C}$, then let $a_n = x_n + iy_n$. $|x_n|, |y_n| \leq |a_n|$, so if $\sum |a_n|$ converges, by comparison so do $\sum |x_n|$ and $\sum |y_n|$. Thus $\sum x_n$ and $\sum y_n$ converges. Then $\sum a_n = \sum x_n + iy_n = \sum x_n + i \sum y_n$ converges. □

Definition (Conditional Convergence)**.** A sequence is conditionally convergent if it is convergent and not absolutely convergent.

Definition (Rearrangement)**.** Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection, then $a'_n = a_{\sigma(n)}$ is a rearrangement of a_n.

Theorem. *If $\sum a_n$ is absolutely convergent, then every rearrangement will sum to the same value.*

Proof. We shall first show this for $a_n \in \mathbb{R}$. Let a'_n be a rearrangement of a_n. Let $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=1}^{n} a'_i$. For now, let's consider $a_n \geq 0$. Given n, we can find $q = \max\{\sigma(i) : 0 \leq i \leq n\}$ such that every term in s_n is in t_q. Then $s_n \leq t_q \leq t$ for all n, and as s_n is increasing and bounded, $s_n \to s \leq t$. Similarly $t \leq s$, so $s = t$.

New define $v_n = \frac{|a_n|+a_n}{2}$, $w_n = \frac{|a_n|-a_n}{2}$ and v'_n, w'_n similarly. As $\sum |a_n|$ converges, so do $\sum v_n$, $\sum v'_n$, $\sum w_n$ and $\sum w'_n$. Then v'_n is a rearrangement of v_n, and w'_n is a rearrangement of w_n. So $\sum v_n = \sum v'_n$ and $\sum w_n = \sum w'_n$. Result follows.

For complex a_n, let $a_n = x_n + iy_n$. Then $\sum |x_n|$ and $\sum |y_n|$ converges. Then use the above. □

Theorem (Riemann Rearrangement Theorem)**.** *If $\sum a_n$ is conditionally convergent, then given $x \in \mathbb{R}$, there is a rearrangement a'_n such that $\sum a'_n = x$.*

Proof. Let b_n be the positive terms in a_n, c_n be the negative terms, d_n be the zeroes. Then $\sum b_n$ and $\sum c_n$ must both diverge. Now add on terms from b_i until we go over x, then add on c_i until we go below x, and so on. This converges to x as $|x - x_n|$ is bounded by $|b_n|$ and $|c_n|$. Intersperse the d_n as appropriate. □

2 Continuity

Let $E \subseteq \mathbb{C}$, E nonempty, $f : E \to \mathbb{C}$, $a \in E$.

Definition (Continuity (1))**.** f is continuous at $a \in E$ if for every sequence $z_n \in E$, $z_n \to a$, we have that $f(z_n) \to f(a)$

Definition (Continuity (2))**.** f is continuous at $a \in E$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in E$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

Proposition. *The two definitions of continuity are equivalent.*

Proof. (2) $\implies$ (1). Given $\varepsilon > 0$, we require $|z_n - a| < \delta$, and this is true for all $n \geq N$.

(1) $\implies$ (2). Suppose $f(z_n) \to f(a)$ whenever $z_n \to a$. Also suppose that (according to 2), f is not continuous at a. That is, $\exists \varepsilon > 0, \forall \delta > 0, \exists x, |x - a| < \delta \wedge |f(x) - f(a)| \geq \varepsilon$. From this we get z_n such that $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \varepsilon$. Contradiction, as we must have that $f(z_n) \to f(a)$ as $z_n \to a$. □

Proposition. *Let $f, g : E \to \mathbb{C}$ be continuous at a. Then so is $f + g$, fg, λf for $\lambda \in \mathbb{C}$. If $f(z) \neq 0$ for all $z \in E$, then $1/f$ is also continuous.*

Proof. Using sequence definition of continuity, these are properties of sequences which we have already shown. □

Definition (Continuous Function)**.** $f : E \to \mathbb{C}$ is continuous if it is continuous at every $x \in E$.

Theorem. *Let $f : A \to \mathbb{C}$, $g : B \to \mathbb{C}$, $f(A) \subseteq B$, f continuous at a, g continuous at $f(a)$. Then $g \circ f : A \to \mathbb{C}$ is continous at a.*

Proof. Take any sequence $z_n \to a$. Set $w_n = f(z_n)$. Then $g(w_n) \to g(f(a))$ by continuity. □

2.1 Limits of Functions

Definition (Limit Point)**.** Let $E \subseteq \mathbb{C}$, $a \in E$ is a limit point of E if for all $\delta > 0$, exists $z \in E$ such that $0 < |z - a| < \delta$.

Proposition. *a is a limit point if and only if there exists $z_n \in E$ such that $z_n \to a$ and $z_n \neq a$ for all n.*

Proof. Reverse implication is clear. For forward implication use $\delta = 1/n$ to construct z_n. □

Definition (Limit of Function)**.** For $f : E \to \mathbb{C}$, a a limit point of E, then $\lim_{z \to a} f(z) = l$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $|z - a| < \delta$, $|f(z) - l| < \varepsilon$.

Or equivalently, $f(z_n) \to l$ for every sequence $z_n \in E$, $z_n \neq a$, $z_n \to a$.

Proposition. *$\lim_{z \to a} f(z) = f(a)$ if and only if f is continuous at a.*

Proof. Clear from definition. □

Proposition. *If $a \in E$ is not a limit point, then f is continuous at a.*

Proof. Choose δ small enough such that a is the only point left. □

Lemma. *(i) Limit of functions is unique*

(ii) $f(z) \to A$ and $g(z) \to B$ as $z \to a$ implies $f(z) + g(z) \to A + B$ as $z \to a$.

(iii) $f(z) \to A$ and $g(z) \to B$ as $z \to a$ implies $f(z)g(z) \to AB$ as $z \to a$.

(iv) If in addition, $B \neq 0$, $g(z) \neq 0$ then $f(z)/g(z) \to A/B$.

Proof. Clear from sequence definition of limits and properties of the limits of sequences. □

2.2 Intermediate Value Theorem

Theorem (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous, wlog $f(a) < f(b)$, then for any $f(a) < \eta < f(b)$, there exists $a < c < b$ such that $f(c) = \eta$.*

Proof. Let $S = \{x \in [a, b] : f(x) < \eta\}$. Clearly S is bounded above by b and $a \in S$. Set $c = \sup S$. From definition of the supremum, there exists $c - 1/n < x_n < c$ for all n. Then $x_n \to c$, $f(x_n) \to f(c)$ by continuity, and $f(x_n) < \eta$. So $f(c) \leq \eta$.

For n large enough, $c + 1/n < b$, and $c + 1/n \to c$, so $f(c + 1/n) \to f(c)$. As $c + 1/n > c$, $f(c + 1/n) \geq \eta$, so $f(c) \geq \eta$ and we are done. □

2.3 Bounds of a Continuous Function

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists k such that $|f(x)| \leq k$ for all $x \in [a, b]$.*

Proof. Suppose not. Then for all $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. As x_n is bounded, from Bolzano-Weierstrass there exists a convergent subsequence $x_{n_j} \to x$. So $f(x_{n_j}) \to f(x)$. But $\left|f(x_{n_j})\right| > n_j$, so $\left|f(x_{n_j})\right| \to \infty$. Contradiction. □

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.*

Proof. Let $A = \{f(x) : x \in [a, b]\}$. From theorem above, A is bounded above and clearly nonempty. Set $M = \sup A$. Then for all n, there exists $x_n \in [a, b]$ such that $M - 1/n < f(x_n) < M$. From Bolzano-Weierstrass, let $x_{n_j} \to x$ be a convergent subsequence. Then by continuity, $f(x_{n_j}) \to f(x)$. But we also have that $f(x_{n_j}) \to M$. So $f(x) = M$. x_1 can be found using the inf. □

Proof. Let A and M be as above. Suppose if for all x, $f(x) < M$. Then $g(x) = 1/(M - f(x))$ is well defined, positive and continuous. So there exists k such that $|g(x)| < k$. Thus $f(x) \leq M - 1/k$ for all x. Contradiction as M is least upper bound. □

2.4 Inverse Functions

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be a continuous strictly increasing function. Let $c = f(a)$ and $d = f(b)$. Then $f : [a, b] \to [c, d]$ is a bijection, and $g = f^{-1} : [c, d] \to [a, b]$ is strictly increasing and continuous.*

Proof. Surjective comes from the IVT, Injective comes from that it is strictly increasing. So f is a bijection. Let $g = f^{-1}$. If $x_1 < x_2$, then we must have $g(x_1) < g(x_2)$. Otherwise $x_1 = f(g(x_1)) \geq f(g(x_2)) = x_2$.

For $k \in (c, d)$, given $\varepsilon > 0$, choose $\delta = \min(f(k) - f(k - \varepsilon), f(k + \varepsilon) - f(k))$. Then result follows. Proof for $k = c$ and $k = d$ are similar. □

3 Differentiability

Let E be a subset of $\mathbb{C}$.

Definition (Differentiable)**.** $f : E \to \mathbb{C}$ is differentiable at x if the limit

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

exists, and we define $f'(x)$ to be its value.

Remark. Let $\varepsilon(h) = f(x + h) - f(x) - hf'(x)$. Then

$$\lim_{h\to 0} \frac{\varepsilon(h)}{h} = 0$$

Definition (Alternative definition for Differentiability)**.** $f : E \to \mathbb{C}$ is differentiable at x if there exists constant $A \in \mathbb{C}$ and $\varepsilon : \mathbb{C} \to \mathbb{C}$ such that

$$f(x + h) = f(x) + hA + \varepsilon(h)$$

where $\lim_{h\to 0} \dfrac{\varepsilon(h)}{h} = 0$. If such an A exists, then it is unique. We denote it by $f'(x)$.

Proposition. *Differentiable implies continuous.*

Proposition. *(i) If $f(x) = c$ for all $x \in E$, then f is differentiable, and $f'(x) = 0$.*

(ii) If f and g are differentiable, then so is $f + g$. $(f + g)'(x) = f'(x) + g'(x)$.

(iii) If f and g are differentiable, then so is fg, $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

(iv) If f is differentiable at x, and $f(x) \neq 0$ for all $x \in E$, then $1/f$ is differentiable at x, and $(1/f)'(x) = -f'(x)/(f(x))^2$.

Proof. (i) Clear from definition.

(ii) From properties of limits, we have that $\lim f + g = \lim f + \lim g$.

(iii) Write $f(x + h)g(x + h) - f(x)g(x)$ as $f(x + h)(g(x + h) - g(x)) + g(x)(f(x + h) - f(x))$.

(iv) $1/f(x + h) - 1/f(x) = -(f(x + h) - f(x))/(f(x)f(x + h))$. □

Proposition (Quotient Rule)**.** $(f/g)' = \dfrac{f'g - fg'}{g^2}$.

Theorem (Chain Rule). *If $f : U \to \mathbb{C}$ is differentiable at $a \in U$, and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, with $f(U) \subseteq V$, then $g \circ f$ is differentiable at a, with derivative*
$(g \circ f)'(a) = f'(a)g'(f(a))$.

Proof. Let $f(x) = f(a) + (x-a)f'(a) + (x-a)\varepsilon_f(x)$ and $g(x) = g(b) + (x-b)g'(b) + (x-b)\varepsilon_g(x)$, where $\lim_{x\to a} \varepsilon_f(x) = \lim_{x\to b} \varepsilon_g(x) = 0$. Now

$$g(f(x)) = g(b) + (f(x) - b)g'(b) + (f(x) - b)\varepsilon_g(f(x))$$

Setting $b = f(a)$, we get that

$$\begin{aligned}
g(f(x)) &= g(f(a)) + (f(x) - f(a))g'(f(a)) + (f(x) - f(a))\varepsilon_g(f(x)) \\
&= g(f(a)) + (g'(f(a)) + \varepsilon_g(f(x)))(f(x) - f(a)) \\
&= g(f(a)) + (g'(f(a)) + \varepsilon_g(f(x)))(x-a)(f'(a) + \varepsilon_f(x)) \\
&= g(f(a)) + (x-a)g'(f(a))f'(a) + (x-a)(\varepsilon_g(f(x))f'(a) + g'(f(a))\varepsilon_f(x) + \varepsilon_f(x)\varepsilon_g(f(x)))
\end{aligned}$$

□

Then suffices to show that

$$\lim_{x\to a} \varepsilon_g(f(x))f'(a) + g'(f(a))\varepsilon_f(x) + \varepsilon_f(x)\varepsilon_g(f(x)) = 0$$

Which follows from the definitions of ε_f and ε_g.

3.1 Mean Value Theorem

Theorem (Rolle's Theorem). *Let $f : [a,b] \to \mathbb{R}$, f continuous on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. As f is continuous, there exists $m, M \in [a,b]$ such that for all $x \in [a,b]$, $f(m) \leq f(x) \leq f(M)$. If $f(m) = f(a) = f(M)$, then f is constant. Otherwise, we may assume that $f(m) < f(a)$ or $f(a) < f(M)$.

If $f(a) < f(M)$, then we must have that $f'(M) = 0$. Suppose if $f'(M) > 0$. Then by the definition of the derivative, $\dfrac{f(M+h) - f(M)}{h}$ tends to a positive limit. Let h be positive and small enough such that $M + h < b$ and $\dfrac{f(M+h) - f(M)}{h} > 0$. Then $f(M+h) > f(M)$. Contradiction. Proceed similarly for $f'(M) < 0$. Thus we must have that $f'(M) = 0$ and $M \in (a,b)$.

The case for $f(m) < f(a)$ is similar. □

Theorem (Mean Value Theorem). *Let $f : [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). Then there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$*

Proof. Let $\varphi(x) = f(x) - kx$, where $k = \dfrac{f(b) - f(a)}{b-a}$. Then $\varphi(a) = \varphi(b)$. Using Rolle's Theorem, there exists c such that $\varphi'(c) = 0$ and $\varphi'(x) = f'(c) - k$, so $\dfrac{f(b) - f(a)}{b-a} = f'(c)$, and the result follows. □

Corollary. *Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Then*

- *If $\forall x \in (a,b), f'(x) > 0$, then f is strictly increasing on $[a,b]$.*
- *If $\forall x \in (a,b), f'(x) \geq 0$, then f is increasing on $[a,b]$.*
- *If $\forall x \in (a,b), f'(x) = 0$, then f is constant on $[a,b]$.*

Theorem (Cauchy's Mean Value Theorem). *Let $f, g : [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). Then there exists $t \in (a,b)$ such that*

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a))$$

Proof. Let $\varphi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}$. By expanding the determinant, we find that φ is continuous on $[a, b]$ and differentiable on (a, b). By properties of the determinant, we have that $\varphi(a) = \varphi(b) = 0$. By Rolle, there must be a $t \in (a, b)$ such that $\varphi'(t) = 0$. Expand the determinant and we get that $\varphi'(t) = (g(b) - g(a))f'(t) - (f(b) - f(a))g'(t)$, and result follows. □

3.2 Inverse Function Theorem

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b) with $f'(x) > 0$. Then let $f(a) = c$, $f(b) = d$. $f : [a, b] \to [c, d]$ is a bijection. In addition, f^{-1} is differentiable, with derivative*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. As $f'(x) > 0$, f is strictly increasing. Thus f is a bijection and has a continuous strictly increasing inverse as shown previously. Let g be the inverse to f and $y = f(x)$.

Given $k \neq 0$, let $h = g(y + k) - g(y)$. Then $f(x + h) = y + k = f(x) + k$. Consequently,

$$\frac{g(y + k) - g(y)}{k} = \frac{h}{f(x + h) - f(x)} \to \frac{1}{f'(x)} = \frac{1}{f'(g(y))}$$

by continuity, as when $k \to 0$, $h \to 0$ by the continuity of g. □

3.3 Taylor's Theorem

Theorem (Taylor's Theorem with Lagrange's Remainder)**.** *Suppose f and it's derivatives up to order $n - 1$ are continuous in $[a, a + h]$. The n-th derivative exists for $x \in (a, a + h)$. Then*

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a + \theta h)$$

where $\theta = \theta(h) \in (0, 1)$.

Proof. For $0 \leq t \leq h$, define

$$\varphi(t) = f(a + t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{t^n}{n!}B$$

where B is chosen such that $\varphi(h) = 0$. In addition, we see that $\varphi(0) = \varphi'(0) = \cdots = \varphi^{(n-1)}(0) = 0$. As $\varphi(h) = 0$, we have some $h_1 \in (0, h)$ such that $\varphi'(h_1) = 0$. Then using Rolle again we have $h_2 \in (0, h_2)$ such that $\varphi''(h_2) = 0$. Repeat this until we get that $\varphi^{(n-1)}(0) = \varphi^{(n-1)}(h_{n-1}) = 0$. Then finally we have $h_n \in (0, h_{n-1})$ such that $\varphi^{(n)}(h_n) = 0$. Then $h_n \in (0, 1)$, so we can let $h_n = \theta h$.

Finally, $\varphi^{(n)}(t) = f^{(n)}(a + t) - B$, so $B = f^{(n)}(a + \theta h)$. Setting $t = h$ gives us the desired result. □

Theorem (Taylor's Theorem with Cauchy's Remainder)**.** *Suppose f and it's derivatives up to order $n - 1$ are continuous in $[0, h]$. The n-th derivative exists for $x \in (0, h)$. Then*

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

where $R_n = \dfrac{(1 - \theta)^{n-1}f^{(n)}(\theta h)h^n}{(n-1)!}$, $\theta \in (0, 1)$.

Proof. Define for $t \in [0, h]$

$$F(t) = f(h) - f(t) - (h - t)f'(t) - \cdots - \frac{(h - t)^{n-1}}{(n-1)!}f^{(n-1)}(t)$$

Then $F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \cdots - \dfrac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t) = -\dfrac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$

Set $\varphi(t) = F(t) - \left(\dfrac{h-t}{h}\right)^p F(0)$. Then $\varphi(0) = \varphi(h) = 0$. By Rolle, there exists $\theta \in (0,1)$ such that $\varphi'(\theta h) = 0$. Thus $F'(\theta h) + \dfrac{p(1-\theta)^{p-1}}{h}F(0) = 0$. So

$$-\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h}\left(f(h) - f(0) - h'f(0) - \cdots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)\right) = 0$$

and

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!p(1-\theta)^{p-1}}f^{(n)}(\theta h)$$

Letting $p = n$ we get Lagrange's Remainder, letting $p = 1$ we get Cauchy's Remainder. □

4 Power Series

Definition (Power Series). Power series are series of the form $\sum_n a_n z^n$, where $a_n, z \in \mathbb{C}$.

Proposition (Binomial Series). *For* $|x| < 1$, $r \in \mathbb{R}$,

$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n$$

where $\binom{r}{n} = \dfrac{r(r-1)\dots(r+n-1)}{n!}$ *are the generalised binomial coefficients.*

Proof. Clearly $f^{(n)}(x) = r(r-1)\dots(r+n-1)(1+x)^{r-n}$. From Lagrange's Form of the Remainder, we have that $R_n = \dfrac{x^n}{n!}f^{(n)}(\theta x) = \dfrac{r(r-1)\dots(r+n-1)}{n!}x^n(1+\theta x)^{r-n} = \binom{r}{n}x^n(1+\theta x)^{r-n}$. We first assume that $0 < x < 1$. Then $(1+\theta x)^{r-n} \leq 1$ for $n > r$.

Now consider the series $\sum_n \binom{r}{n}x^n$. Using the ratio test with $a_n = \binom{r}{n}x^n$, then $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(r-n)x}{n+1}\right| \to |x|$ as $n \to \infty$. Consequently, the original series is absolutely convergent, and as a result, $a_n \to 0$. Thus for $0 < x < 1$ and $n > r$, we have that

$$|R_n| \leq \left|\binom{r}{n}x^n\right| = |a_n| \to 0$$

To prove this for $-1 < x < 0$, we will need to use Cauchy's form of the remainder.

$$R_n = r\binom{r-1}{n-1}x^n(1+\theta x)^{r-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

.

For $x \in (-1,1)$, we have that $\left|\dfrac{1-\theta}{1+\theta x}\right| < 1$, so

$$|R_n| \leq \left|r\binom{r-1}{n-1}x^n\right|(1+\theta x)^{r-1}$$

By considering the sign of $r-1$, we have that $(1+\theta x)^{r-1} < \max(1,(1+x)^{r-1})$. Let $k_r = |r|\max(1,(1+x)^{r-1})$. Then

$$|R_n| \leq k_r|r - 1n - 1x^n| \to 0$$

So $|R_n| \to 0$ for $|x| < 1$. □

4.1 Radius of Convergence

Lemma. *If* $\sum_{n=0}^{\infty} a_n z_1^n$ *converges, and* $|z| < |z_1|$*, then* $\sum_{n=0}^{\infty} a_n z^n$ *converges absolutely.*

Proof. Since $\sum a_n z_1^n$ converges, we must have that $a_n z_1^n \to 0$. So there exists k such that $\left|a_n z_1^n\right| \leq k$ for all n. Then $|a_n z^n| \leq k \left|\frac{z}{z_1}\right|^n$. Since the geometric series $\sum \left|\frac{z}{z_1}\right|^n$ converges, result follows by comparison. □

Theorem. *A power seris either*

(i) Converges absolutely for all $z \in \mathbb{C}$

(ii) Converges absolutely for all $z \in \mathbb{C}$*,* $|z| < R$ *and diverges for all* $|z| > R$

(iii) Converges for $z = 0$ *only.*

Proof. Let $S = \{X \in \mathbb{R} : x \geq 0, \sum a_n x^n \text{ converges}\}$. Clearly 0 is in S, and by the lemma above, if $x_1 \in S$, then $[0, x_1] \subseteq S$. If S is not bounded above, then $S = [0, \infty)$ and we have case (i).

On the other hand, if S is bounded above, let $R = \sup S$. For $z \in \mathbb{C}$, if $|z| < R$, then there exists $R_0 \in S$ such that $|z| < R_0 < R$. Then by the lemma above $\sum a_n z^n$ converges.

Now suppose $|z| > R$. If $\sum a_n z^n$ converges, then this contradicts the fact that R is a supremum. □

Lemma. *If* $\left|\frac{a_{n+1}}{a_n}\right| \to l$ *then* $R = \frac{1}{l}$.

Proof. By ratio test. □

4.2 Differentiation of Power Series

Lemma. *If* $\sum_0 a_n z^n$ *has radius of convergence* R*, then so do* $\sum_1 n a_n z^{n-1}$ *and* $\sum_2 n(n-1) a_n z^{n-2}$.

Proof. For $z \in \mathbb{C}$, choose R_0 such that $|z| < R_0 < R$. Then we must have some k such that $\left|a_n R_0^n\right| < k$ for all n. Then

$$\left|n a_n z^{n-1}\right| = \frac{n}{|z|} |a_n R_0^n| \left|\frac{z}{R_0}\right|^n \leq \frac{kn}{|z|} \left|\frac{z}{R_0}\right|^n$$

Using the ratio test, $\sum n \left|\frac{z}{R_0}\right|^n$ converges, since

$$\frac{n+1}{n} \left|\frac{z}{R_0}\right| \to \left|\frac{z}{R_0}\right| < 1$$

Thus $\sum n a_n z^{n-1}$ converges by comparison. On the other hand, if $|z| > R$, then

$$|a_n z^n| \leq |z| \left|n a_n z^{n-1}\right|$$

and $\sum n a_n z^{n-1}$ diverges by comparison. The proof for $\sum n(n-1) a_n z^{n-2}$ follows similarly. □

Lemma. *For* $2 \leq r \leq n$*,* $\binom{n}{r} \leq n(n-1)\binom{n-2}{r-2}$.

Proof.

$$\binom{n}{r} = \frac{n(n-1)}{r(r-1)} \binom{n-2}{r-2} \leq n(n-1) \binom{n-2}{r-2}$$

□

Lemma. $|(z+h)^n - z^n - nhz| \leq n(n-1)(|z| + |h|)^{n-2} |h|^2$

Proof.

$$
\begin{aligned}
|(z+h)^n - z^n - nhz| &= \left|\sum_{r=2}^{n} \binom{n}{r} z^{n-r} h^r\right| \\
&\le \sum_{r=2}^{n} \binom{n}{r} |z|^{n-r}|h|^r \\
&\le n(n-1)\sum_{r=2} \binom{n-2}{r-2} |z|^{n-r}|h|^r \qquad = n(n-1)(|z|+|h|)^{n-2}|h|^2
\end{aligned}
$$

□

Theorem. *Let $f(z) = \sum_0^\infty a_n z^n$, with radius of convergence R. Then f is differentiable for all $|z| < R$, and $f'(z) = \sum_1^\infty n a_n z^{n-1}$.*

Proof. We have already seen that $\sum_1^\infty n a_n z^{n-1}$ converges. Define $f'(z) = \sum_1^\infty n a_n z^{n-1}$. Then we need to show that $\dfrac{f(z+h) - f(z) - hf'(z)}{h} \to 0$ as $h \to 0$.

Let $I = \dfrac{f(z+h) - f(z) - h'f(z)}{h} = \dfrac{1}{h}\sum_{n=0}^\infty a_n((z+h)^n - z^n - nhz^{n-1})$. Then

$$
\begin{aligned}
|I| &= \frac{1}{|h|}\left|\lim_{N\to\infty}\sum_{n=0}^{N} a_n((z+h)^n - z^n - nhz^{n-1})\right| \\
&= \frac{1}{|h|}\lim_{N\to\infty}\left|\sum_{n=0}^{N} a_n((z+h)^n - z^n - nhz^{n-1})\right| \\
&\le \frac{1}{|h|}\sum_{n=0}^{\infty} |a_n|\left|(z+h)^n - z^n - nhz^{n-1}\right| \\
&\le \sum_{n=0}^{\infty} |a_n| n(n-1)(|z|+|h|)^{n-2}|h|
\end{aligned}
$$

□

Now note that $\sum_2^\infty a_n n(n-1)(|z|+|h|)^{n-2} \le \sum_2^\infty a_n n(n-2)(|z|+r)^{n-2} = A_r$, where $|h| < r$ and $|z|+r < R$. Consequently, $|I| \le |h|A_r$, so $I \to 0$ as $h \to 0$.

5 Special Functions

5.1 Exponential Function

Definition. Define $e : \mathbb{C} \to \mathbb{C}$ by

$$e(x) = \sum_0^\infty \frac{z^n}{n!}$$

Proposition. *e is differentiable, and $e'(x) = e(x)$.*

Proof. Suffices to show that the power series for e has infinite radius of convergence. We have that $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{1}{n+1} \to 0$, so $R = \infty$. Result follows by term by term differentation. □

Lemma. *Let $f : \mathbb{C} \to \mathbb{C}$ be differentiable, and $f'(z) = 0$ for all $z \in \mathbb{C}$. Then $f(z)$ is constant.*

Proof. Consider $g(t) = f(tz) = u(t) + iv(t)$ for a fixed z. By the chain rule, g is differentiable, with $g'(t) = f'(tz) = 0 = u'(t) + iv'(t)$. We must then have that $u'(t) = v'(t) = 0$ for all t. Hence u and v are constant, so $f(z) = f(0)$ for all $z \in \mathbb{C}$. □

Proposition. $e(a + b) = e(a)e(b)$

Proof. Consider $f(z) = e(a + b - z)e(z)$. Then $f'(z) = -e(a + b - z)e(z) + e(a + b - z)e(z) = 0$. So f is constant. Hence $e(a)e(b) = e(a + b)e(0) = e(a + b)$. □

From now on, we consider the restriction $e : \mathbb{R} \to \mathbb{R}$.

Theorem.

(i) *$e : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and $e = e'$.*

(ii) $e(x + y) = e(x)e(y)$

(iii) *$e'(x) > 0$ for all x*

(iv) *e is strictly increasing*

(v) *$e(x) \to \infty$ as $x \to \infty$ and $e(x) \to 0$ as $x \to -\infty$.*

(vi) *$e : \mathbb{R} \to (0, \infty)$ is a bijection.*

Proof. (i) and (ii) follows from the complex case. For (iii), clearly $e(x) > 0$ for all $x \geq 0$. Then as $e(-x) = \dfrac{1}{e(x)}$, we have that $e(-x) > 0$ for $x \leq 0$ as well.

(v) For $x > 0$, $e(x) \geq 1 + x$. So as $x \to \infty$, $e(x) \to \infty$. Furthermore, $e(-x) \leq \dfrac{1}{1 + x} \to 0$.

(vi) Injectivity follows from the fact that it is strictly increasing. Now let $y \in (0, \infty)$ be arbitrary. We can find a, b such that $e(a) < y < e(b)$. Then by the IVT, there exists $x \in (a, b)$ such that $e(x) = y$. □

Proposition. *$e : (\mathbb{R}, +) \to ((0, \infty), \times)$ is a group isomorphism.*

Definition. $l : (0, \infty) \to \mathbb{R}$, $l(x) = e^{-1}(x)$.

Theorem.

(i) *$l : (0, \infty) \to \mathbb{R}$ is a bijection, $l(e(x)) = x$ and $e(l(y)) = y$.*

(ii) *l is differentiable, with $l'(t) = \dfrac{1}{t}$*

(iii) $l(xy) = l(x) + l(y)$

Proof. (i) follows by definition. (ii) follows from the inverse function theorem. (iii) follows from IA Groups, the inverse of an isomorphism is an isomorphism. □

Definition. For $\alpha \in \mathbb{R}, x > 0$

$$r_\alpha(x) = e(\alpha l(x))$$

Theorem. *Suppose $x, y > 0$, $\alpha, \beta \in \mathbb{R}$, then*

(i) $r_\alpha(xy) = r_\alpha(x)r_\alpha(y)$

(ii) $r_{\alpha+\beta}(x) = r_\alpha(x)r_\beta(x)$

(iii) $r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$

(iv) $r_1(x) = x$, $r_0(x) = 1$.

Proof. Follows from definitions. $\square$

Proposition. *For $\alpha \in \mathbb{Q}$, $r_\alpha(x) = x^\alpha$*

Proof. For $n \geq 0, n \in \mathbb{Z}$, $r_n(x) = r_{1+\dots+1}(x) = (r_1(x))^n = x^n$. $r_{-1}(x)r_1(x) = r_0(x) = 1$, so $r_{-1}(x) = x^{-1}$ and $r_{-n}(x) = x^{-n}$. Now for $q \in \mathbb{Z}$, $(r_{1/q}(x))^q = r_q(r_{1/q}(x)) = r_1(x) = x$, so $r_{1/q}(x) = x^{1/q}$. Finally $r_{p/q}(x) = r_p(r_{1/q}(x)) = (x^{1/q})^p = x^{p/q}$. $\square$

Definition (exp, log, Powers)**.** $\exp(x) = e(x)$, $\log(x) = l(x)$, $x^\alpha = r_\alpha(x)$.

Definition (e)**.** $e = \exp(1)$.

Proposition. $(x^\alpha)' = \alpha x^{\alpha-1}$.

Proposition. $(a^x)' = a^x \log a$.

Proposition. *For all k,*

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \infty$$

Proof. From definitions, $\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} > \frac{x^n}{n!}$ for $x > 0$, $n > k$.

Then $\frac{e^x}{x^k} > \frac{x^{n-k}}{n!} \to \infty$. $\square$

5.2 Trigonometric Functions

Definition (cos)**.**

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

Definition (sin)**.**

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

Proposition. *Both power series have radius of convergence $R = \infty$, both are differentiable and* $\cos' = -\sin$, $\sin' = \cos$.

Theorem (Euler's Formula)**.**

$$e^{iz} = \cos z + i \sin z$$

Proof. As the series for e^z is absolutely convergent, we may write

$$e^{iz} = \sum_0^\infty \frac{(iz)^n}{n!} = \sum_0^\infty \frac{(iz)^{2n}}{(2n)!} + \sum_0^\infty \frac{(iz)^{2n+1}}{(2n+1)!}$$

Now $(iz)^{2n} = (-1)^n z^{2n}$ and $(iz)^{2n+1} = i(-1)^n z^{2n+1}$. So

$$e^{iz} = \sum_0^\infty \frac{(-1)^n z^{2n}}{(2n)!} + i \sum_0^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \cos z + i \sin z$$

$\square$

Proposition. $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ *and* $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

Proposition. *There exists a minimum positive real ϖ such that $\sqrt{2} < \frac{\varpi}{2} < \sqrt{3}$, and* $\cos\left(\frac{\varpi}{2}\right) = 0$.

Proof. If $0 < x < 2$, then for $k \in \mathbb{N}$, $2k(2k+1) \geq 4 > x^2$. So $\frac{x^{2k-1}}{(2k-1)!} - \frac{x^{2k+1}}{(2k+1)!} > 0$. Hence

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots > 0$$

So for $0 < x < 2$, $(\cos x)' = -\sin x < 0$, and $\cos x$ is a strictly decreasing function on $(0, 2)$. Now as $(2n+1)(2n+2) > 2$ for $n \in \mathbb{N}$, we have that

$$\cos\left(\sqrt{2}\right) = 0 + \left(\frac{(\sqrt{2})^4}{4!} - \frac{(\sqrt{2})^6}{6!}\right) + \cdots > 0$$

Furthremore, as $(2n+1)(2n+2) > 3$ for $n \in \mathbb{N}$, we have that

$$\cos\left(\sqrt{3}\right) = 1 - \frac{3}{2} + \frac{9}{4!} - \left(\frac{(\sqrt{3})^6}{6!} - \frac{(\sqrt{3})^8}{8!}\right) - \cdots < 1 - \frac{3}{2} + \frac{9}{4!} = -\frac{1}{8} < 0$$

Thus by the IVT, we have some ϖ such that $\cos\left(\frac{\varpi}{2}\right) = 0$ and $\sqrt{2} < \frac{\varpi}{2} < \sqrt{3}$. □

Definition (π)**.** We define π to be the ϖ from above.

Theorem.

- $\sin\left(\frac{\pi}{2}\right) = 1$.
- $\sin\left(z + \frac{\pi}{2}\right) = \cos z$, $\cos\left(z + \frac{\pi}{2}\right) = -\sin z$.
- $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$.
- $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$.
- $e^{2\pi i} = 1$.

Proof. Follows immediately from angle addition formulae and Euler's Identity. □

5.3 Hyperbolic Functions

Definition (cosh)**.**

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

Definition (sinh)**.**

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

Proposition. $\cosh' = \sinh$, $\sinh' = \cosh$.

Proposition. $\cos z = \cosh(iz), \sin z = -i \sinh(iz)$.

6 Integration

6.1 Definitions

Definition (Dissection)**.** A dissection D of an interval $[a, b]$ is a finite subset of $[a, b]$ containing a and b.

We write $D = \{x_0, \dots, x_n\}$ where $a = x_0 < x_1 < \cdots < x_n = b$.

Definition (Upper Sum)**.** We define the upper sum for a function $f : [a, b] \to \mathbb{R}$ and a dissection D by

$$S(f, D) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x)$$

Definition (Lower Sum)**.** We define the lower sum for a function $f : [a, b] \to \mathbb{R}$ and a dissection D by

$$s(f, D) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x)$$

Proposition. *$s(f, D) \leq S(f, D)$ for all D*

Proof. Clear from definitions. □

Lemma. *If D and D' are dissections of $[a, b]$, with $D \subseteq D'$, then*

$$s(f, D) \leq s(f, D') \leq S(f, D') \leq S(f, D)$$

Proof. First suppose if $D' = D \cup \{y\}$, where $x_{r-1} < y < x_r$. Clearly

$$\sup_{x \in [x_{r-1}, y]} f(x), \sup_{x \in [y, x_r]} \leq \sup_{x \in [x_{r-1}, x_r]} f(x)$$

So

$$(y - x_{r-1}) \sup_{x \in [x_{r-1}, y]} f(x) + (x_r - y) \sup_{x \in [y, x_r]} \leq (x_r - x_{r-1}) \sup_{x \in [x_{r-1}, x_r]} f(x)$$

and $S(f, D) \geq S(f, D')$. A similar argument applies for $s(f, D') \geq s(f, D)$. By induction this works for $|D' \backslash D| = n$. □

Lemma. *If D_1 and D_2 are dissections of $[a, b]$, then*

$$S(f, D_1) \geq S(f, D_1 \cup D_2) \geq s(f, D_1 \cup D_2) \geq s(f, D_1)$$

Proof. Use $D' = D_1 \cup D_2$ in lemma above. □

Definition (Upper Integral)**.** For a function $f : [a, b] \to \mathbb{R}$, we define the upper integral

$$I^*(f) = \inf_D S(f, D)$$

Definition (Lower Integral)**.** For a function $f : [a, b] \to \mathbb{R}$, we define the lower integral

$$I_*(f) = \sup_D s(f, D)$$

Proposition. *For all $f : [a, b] \to \mathbb{R}$, $I_*(f) \leq I^*(f)$.*

Proof. For any dissections D_1, D_2, we have that $s(f, D_1) \leq S(f, D_2)$. Consequently we have that for any D_1, $s(f, D_1) \leq \inf_{D_2} S(f, D_2)$. Then $\sup_{D_1} s(f, D_1) \leq \inf_{D_2} S(f, D_2)$. □

Definition (Riemann Integrable)**.** A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if $I^*(f) = I_*(f)$. We write

$$\int_a^b f = I^*(f) = I_*(f)$$

Theorem. *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if given $\varepsilon > 0$, we can always find a dissection D such that $S(f, D) - s(f, D) < \varepsilon$.*

Proof. For every dissection D we have

$$0 \leq I^*(f) - I_*(f) \leq S(f, D) - s(f, D)$$

so one direction of the implication is clear. Conversely, suppose if $I^*(f) = I_*(f)$. Then there are partitions D_1 and D_2 such that $\int_a^b f - \frac{\varepsilon}{2} = I_*(f) - \frac{\varepsilon}{2} < s(f, D_1)$ and $\int_a^b f + \frac{\varepsilon}{2} = I^*(f) + \frac{\varepsilon}{2} > S(f, D_2)$. Then $S(f, D_1 \cup D_2) - s(f, D_1 \cup D_2) \leq S(f, D_2) - s(f, D_1) < \varepsilon$. □

6.2 Elementary Properties of the Integral

Theorem. *If $f : [a, b] \to \mathbb{R}$ is monotonic, then it is integrable.*

Proof. Without loss of generality, suppose f is increasing. Then $\sup_{x \in [x_{j-i}, x_j]} f(x) = f(x_j)$ and $\inf_{x \in [x_{j-1}, x_j]} f(x) = f(x_{j-1})$. Thus for any dissection D,

$$S(f, D) - s(f, D) = \sum_{j=1}^{n} (x_j - x_{j-1})(f(x_j) - f(x_{j-1}))$$

Now let $x_j = a + \frac{(b-a)j}{n}$. Then $S(f, D) - s(f, D) = \frac{b-a}{n}(f(b) - f(a))$. For n large enough, this is less than ε. □

Lemma. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.*

Proof. Suppose not. Then there exists $\varepsilon > 0$, such that for all $\delta > 0$, there exists $x, y \in [a, b]$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

Let $\delta_n = \frac{1}{n}$, then using the above we get x_n and y_n such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon$ for all n. By Bolzano-Weierstrass, we have a convergent subsequence $x_{n_k} \to c$. Now $|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} + c| \to 0$, so $y_{n_k} \to c$ as well. By continuity, $f(x_{n_k}) \to f(c)$ and $f(y_{n_k}) \to f(c)$ as well. Thus $|f(x_{n_k}) - f(y_{n_k})| \to 0$. Contradiction. □

Theorem. *If $f : [a, b] \to \mathbb{R}$ is continuous, then it is integrable.*

Proof. Given $\varepsilon > 0$, from the lemma above, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let $x_j = a + \frac{(b-a)j}{n}$, where n is chosen such that $\frac{b-a}{n} < \delta$. Then for any $x, y \in [x_{j-1}, x_j]$, $|f(x) - f(y)| < \varepsilon$. Then

$$\begin{aligned}
S(f, D) - s(f, D) &= \sum_{j=1}^{n} (x_j - x_{j-1}) \left(\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) \right) \\
&< \sum_{j=1}^{n} \frac{b-a}{n} \varepsilon \\
&= (b-a)\varepsilon
\end{aligned}$$

□

Proposition. *Let f, g be bounded and integrable on $[a, b]$. Then if $f \leq g$ on $[a, b]$, $\int_a^b f \leq \int_a^b g$.*

Proof. If $f \leq g$, then for any dissection D, $\int_a^b f = I^*(f) \leq S(f, D) \leq S(g, D)$. As a result, $\int_a^b f = I^*(f) \leq \inf_D S(g, D) = I^*(g) = \int_a^b g$. □

Proposition. *Let f, g be bounded and integrable on $[a, b]$. Then $f + g$ is integrable, and $\int_a^b f + g = \int_a^b f + \int_a^b g$*

Proof. For any dissection, we have that $\sup_{x \in [x_{j-1}, x_j]} (f + g) \leq \sup_{x \in [x_{j-1}, x_j]} f + \sup_{x \in [x_{j-1}, x_j]} g$. So $S(f + g, D) \leq S(f, D) + S(g, D)$. Now choose two arbitary dissections D_1 and D_2. We have that

$$I^*(f + g) \leq S(f + g, D_1 \cup D_2) \leq S(f, D_1 \cup D_2) + S(g, D_1 \cup D_2) \leq S(f, D_1) + S(g, D_2)$$

Fixing D_1 and taking $\inf_{D_2}$, we get that $I^*(f + g) \leq S(f, D_1) + I^*(g, D_2)$. Now taking $\inf_{D_1}$, we get that $I^*(f + g) \leq I^*(f) + I^*(g)$.

Similarly, we get that $I_*(f + g) \geq I_*(f) + I_*(g)$, so $I_*(f + g) = I^*(f + g)$ and $\int_a^b f + g = \int_a^b f + \int_a^b g$. □

Proposition. *Let f be bounded and integrable on $[a, b]$. Then kf is integrable, and $\int_a^b kf = k\int_a^b f$.*

Proof.
$$\sup_{x\in[x_{j-1},x_j]}(kf) = \begin{cases} k\sup_{x\in[x_{j-1},x_j]} f & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ k\inf_{x\in[x_{j-1},x_j]} f & \text{if } k < 0 \end{cases}$$

and the rest follows. □

Proposition. *Let f be bounded and integrable on $[a, b]$. Then $|f|$ is integrable, and $\left|\int_a^b f\right| \leq \int_a^b |f|$.*

Proof. Let $f_+(x) = \max(f(x), 0)$. Then $\sup\limits_{x\in[x_{j-1},x_j]} f_+ - \inf\limits_{x\in[x_{j-1},x_j]} f_+ \leq \sup\limits_{x\in[x_{j-1},x_j]} f - \inf\limits_{x\in[x_{j-1},x_j]} f$. Now given $\varepsilon > 0$, we have a dissection D such that $S(f, D) - s(f, D) < \varepsilon$. Then

$$0 \leq S(f_+, D) - s(f_+, D) \leq S(f, D) - s(f, D) < \varepsilon$$

so f_+ is integrable. $|f| = 2f_+ - f$, so $|f|$ is integrable. Since $-|f| \leq f \leq |f|$, $\left|\int_a^b f\right| \leq \int_a^b |f|$. □

Proposition. *Let f, g be bounded and integrable on $[a, b]$. Then fg is integrable.*

Proof. First assume if $f \geq 0$. Then $\sup\limits_{x\in[x_{j-1},x_j]} f^2 = (\sup\limits_{x\in[x_{j-1},x_j]} f)^2 = M_j^2$, and $\inf\limits_{x\in[x_{j-1},x_j]} f^2 = (\inf\limits_{x\in[x_{j-1},x_j]} f)^2 = m_j^2$.

Then $S(f^2, D) - s(f^2, D) = \sum\limits_{j=1}^n (x_j - x_{j-1})(M_j^2 - m_j^2) = \sum\limits_{j=1}^n (x_j - x_{j-1})(M_j + m_j)(M_j - m_j)$. As f is bounded, we have that $|f(x)| \leq k$. Then $M_j + m_j \leq 2k$. So

$$S(f^2, D) - s(f^2, D) \leq 2k\sum_{j=1}^n (x_j - x_{j-1})(M_j - m_j) = 2k(S(f, D) - s(f, D))$$

So f^2 is integrable. For general f, note that $f^2 = |f|^2$ and use the above. Finally, $fg = \dfrac{(f+g)^2 - (f-g)^2}{4}$ so it is integrable. □

Proposition. *If $a < c < b$, f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and $[c, b]$. Conversely, if f is integrable on $[a, c]$ and $[c, b]$ then it is integrable on $[a, b]$.*

In both cases, $\int\limits_a^b f = \int\limits_a^c f + \int\limits_c^b f$.

Proof. First note that if D_1 is a dissection of $[a, c]$ and D_2 is a dissection of $[c, b]$, then $D_1 \cup D_2$ is a dissection of $[a, b]$. Furthermore, we have that

$$S(f, D_1 \cup D_2) = S(f\restriction_{[a,c]}, D_1) + S(f\restriction_{[c,b]}, D_2)$$

Also, if D is a dissection of $[a, b]$, then

$$S(f, D) \geq S(f, D \cup \{c\}) = S(f\restriction_{[a,c]}, D_1) + S(f\restriction_{[c,b]}, D_2)$$

where $D_1 = (D \cup \{c\}) \cap [a, c]$ and $D_2 = (D \cup \{c\}) \cap [c, b]$.

The first statement implies that $I^*(f) \leq I^*(f\restriction_{[a,c]}) + I^*(f\restriction_{[c,b]})$, while the second implies that $I^*(f) \geq I^*(f\restriction_{[a,c]}) + I^*(f\restriction_{[c,b]})$. So $I^*(f) = I^*(f\restriction_{[a,c]}) + I^*(f\restriction_{[c,b]})$. Similarly $I_*(f) = I_*(f\restriction_{[a,c]}) + I_*(f\restriction_{[c,b]})$. Thus

$$0 \leq I^*(f) - I_*(f) = I^*(f\restriction_{[a,c]}) - I_*(f\restriction_{[a,c]}) + I^*(f\restriction_{[c,b]}) - I_*(f\restriction_{[c,b]})$$

and result follows. □

6.3 Fundamental Theorem of Calculus

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be bounded and integrable. Let $F(x) = \int_a^x f(t)\mathrm{d}t$. Then F is continuous.*

Proof. Say $|f| \le k$ for $x \in [a, b]$. Then

$$|F(x+h) - F(x)| = \left|\int_x^{x+h} f(t)\mathrm{d}t\right| \le \int_x^{x+h} f(t)\mathrm{d}t \le k|h|$$

□

Theorem (Fundamental Theorem of Calculus)**.** *Suppose further that f is continuous at x. Then F is differentiable at x, with $F'(x) = f(x)$.*

Proof. Consider

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| = \frac{1}{|h|}\left|\int_x^{x+h} f(t\mathrm{d}t - hf(x))\right| = \frac{1}{|h|}\left|\int_x^{x+h} f(t) - f(x)\mathrm{d}t\right|$$

If f is continuous, then given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|t - x| < \delta$, then $|f(t) - f(x)| < \varepsilon$. As a result, if we have that $|h| < \delta$, then

$$\frac{1}{|h|}\left|\int_x^{x+h} f(t) - f(x)\mathrm{d}t\right| \le \frac{1}{|h|}\varepsilon|h| = \varepsilon$$

Hence $\dfrac{F(x+h) - F(x)}{h} \to f(x)$. □

Corollary. *Integration is the inverse of differentiation. If $f = g'$ is continuous on $[a, b]$, then for all $x \in [a, b]$*

$$\int_a^x f = g(x) - g(a)$$

Proof. From the FTC, $F - g$ has zero derivative on $[a, b]$. So it must be constant. As $F(a) = 0$, we must have that $F(x) = g(x) - g(a)$. □

Corollary (Integration by Parts)**.** *Suppose f' and g' exists and are continuous on $[a, b]$. Then*

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

Proof. By product rule and FTC. □

Corollary (Integration by Substitution)**.** *Let $g : [\alpha, \beta] \to [a, b]$, $g(\alpha) = a$ and $g(\beta) = b$. In addition, g' exists and is continuous on $[\alpha, \beta]$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then*

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(g(t))g'(t)\mathrm{d}t$$

Proof. Set $F(x) = \int_a^x f(t)\mathrm{d}t$ and $h(t) = F(g(t))$. Then

$$\int_\alpha^\beta f(g(t))g'(t)\mathrm{d}t = \int_\alpha^\beta F'(g(t))g'(t)\mathrm{d}t = \int_\alpha^\beta h'(t)\mathrm{d}t = h(\beta) - h(\alpha) = F(b) - F(a) = \int_a^b f(x)\mathrm{d}x$$

□

Theorem (Taylor's Theorem with Integral Remainder). *Let $f^{(n)}$ be continuous for $x \in [0, h]$. Then*

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where $R_n = \dfrac{h^n}{(n-1)!} \int\limits_0^1 (1-t)^{n-1} f^{(n)}(th) \mathrm{d}t$

Proof. Substituting $u = th$, we get that

$$R_n = \frac{1}{(n-1)!} \int_0^h (h-u)^{n-1} f^{(n)}(u) \mathrm{d}u$$

Integrating by parts,

$$R_n = \frac{-h^{n-1} f^{(n-1)(0)}}{(n-1)!} + \frac{1}{(n-2)!} \int_0^h (h-u)^{n-2} f^{(n-1)}(u) \mathrm{d}u = \frac{-h^{n-1} f^{(n-1)(0)}}{(n-1)!} + R_{n-1}$$

Furthermore, $R_1 = f(h) - f(0)$, and we get the result required. □

Theorem (Integral Mean Value Theorem). *Let $f, g : [a, b] \to \mathbb{R}$ be continous. With $g(x) \neq 0$ for all $x \in [a, b]$. Then there exists $c \in (a, b)$ such that*

$$\int_a^b fg = f(c) \int_a^b g$$

Proof. Set $F(x) = \int_a^x fg$ and $G(x) = \int_a^b g$. Applyint the Cauchy Mean Value Theorem, there exists $c \in (a, b)$ such that

$$(F(b) - F(a))G'(c) = F'(c)(G(b) - G(a))$$

ie $(\int_a^b fg)g(c) = f(c)g(c) \int_a^b g$. Dividing through by $g(c) \neq 0$ yields the required result. □

Proposition. *Using $g = 1$ with the Integral form of the reaminder, and the integral mean value theorem, we get Cauchy's Form of the Remainder.*

Proposition. *Using $g = (1-t)^{n-1}$, we get Lagrange's Form of the Remainder.*

6.4 Improper Integrals

Definition (Improper Integral). If $f : [a, \infty) \to \mathbb{R}$, f is bounded and integrable on every $[a, R]$ and as $R \to \infty$ $\int_a^R f \to l < \infty$, then

$$\int_a^\infty f = l$$

Definition (Improper Integral). If $\int_a^\infty f = l_1$ and $\int_{-\infty}^a f = l_2$, then we define

$$\int_{-\infty}^\infty f = l_1 + l_2$$

Theorem (Integral Test). *Let f be a positive decreasing function for $x \geq 1$. Then*

(i) $\int_1^\infty f$ *converges if and only if* $\sum_1^\infty f(n)$ *converges.*

(ii) As $n \to \infty$, $\sum_1^n f(r) - \int_1^n f$ *tends to a limit* l, *with* $0 \leq l \leq f(1)$.

Proof. If $n - 1 \leq x \leq n$, then $f(n-1) \geq f(x) \geq f(n)$, so $f(n-1) \geq \int_{n-1}^n f \geq f(n)$. Adding up, we get that

$$\sum_1^{n-1} f(r) \geq \int_1^n f \geq \sum_2^n f(r)$$

and (i) follows. Now set $\phi(n) = \sum_1^n f(r) - \int_1^n f$. Then $\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^n f \leq 0$. Also, $0 \leq \phi(n) \leq f(1)$, and as $\phi(n)$ is decreasing and bounded below, $\phi(n) \to l$, where $0 \leq l \leq f(1)$. □