Differential Equations

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This document is intended for revision purposes. As a result, it does not contain any exposition. This is based off lectures given by Dr John Taylor in Michaelmas 2020, but the order of content, as well as some of the proofs have been modified after the fact, primarily to provide simpler proofs for theorems. Note that this also contains theorems from examples sheets, as some are useful elsewhere.

Differential Equations is on *Paper 2*.

Contents

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1 Notation

1.1 Little-o and Big-O Notation

Definition (Little-o notation). We say that $f(x) = o(g(x))$ as $x \to x_0$ if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

Definition (Big-O notation). We say that $f(x) = O(g(x))$ as $x \to x_0$ if there exists $M > 0$ and $\delta > 0$ such that for all $0 < |x - x_0| < \delta$,

$$|f(x)| \leq Mg(x)$$

Definition (Big-O notation at infinity). We say that $f(x) = O(g(x))$ as $x \to \infty$ if there exists $M > 0$ and $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$,

$$|f(x)| \leq Mg(x)$$

2 Taylor's Theorem

Definition (Taylor Polynomial). Given $f : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, and that $f^{(n)}$ exists, we define the Taylor Polynomial of degree n about x_0 as

$$P_n(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0)$$

Theorem (Taylor's Theorem). *(Without Loss of Generality set $x_0 = 0$)*

$$f(x) = P_n(x) + E_n$$

where E_n is an error term. We have that $E_n = o(h^n)$. Furthermore, if $f^{(n+1)}$ exists, then $E_n = O(h^{n+1})$.

Theorem (L'Hôpital's Rule). *Let f, g be differentiable at $x = x_0$ and $f(x), g(x) \to 0$ as $x \to x_0$. If $g'(x_0) \neq 0$ and f' and g' are continuous at x_0, then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Proof. From Taylor's Theorem, we have that $f(x) = f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)$, and $g(x) = g(x_0) + (x - x_0)g'(x_0) + o(x - x_0)$. So

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x)}{g(x)} &= \lim_{x \to x_0} \frac{f(x_0) + (x - x_0)f'(x_0) + o(x - x_0)}{g(x_0) + (x - x_0)g'(x_0) + o(x - x_0)} \\
&= \lim_{x \to x_0} \frac{f'(x_0) + \frac{o(x - x_0)}{x - x_0}}{g'(x_0) + \frac{o(x - x_0)}{x - x_0}} \\
&= \frac{f'(x_0)}{g'(x_0)} \\
&= \lim_{x \to x_0} \frac{f'(x)}{g'(x)}
\end{aligned}$$

□

3 Multivariable Calculus

3.1 Multivariate Chain Rule

Proposition (Differential Form of the Multivariate Chain Rule). *For $f(x, y)$, we have that*

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

Proposition (Multivariate Chain Rule). *If $f(x(t), y(t))$, then*

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

3.2 Differentiation of Integrals

Proposition. *For $f(x; c(t))$, we have that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x; c(t))\mathrm{d}x = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial c}\frac{\mathrm{d}c}{\mathrm{d}t}\mathrm{d}x + f(b; c)\frac{\mathrm{d}b}{\mathrm{d}t} - f(a; c)\frac{\mathrm{d}a}{\mathrm{d}t}$$

4 Linear ODEs

4.1 Exponential Function

Definition (Exponential Function)**.** We denote the solution to $\frac{\mathrm{d}f}{\mathrm{d}x} = f$, $f(0) = 1$ as $\exp(x)$.

Definition (Logarithm)**.** We define $\log(x)$ to be the inverse to $\exp(x)$.

Definition (Eigenfunction)**.** For an operator D, an eigenfunction is a function f satisfying

$$Df = \lambda f$$

λ is known as the eigenvalue.

Proposition. $\exp(\lambda x)$ *is the eigenfunction to* $\frac{\mathrm{d}}{\mathrm{d}x}$*, with eigenvalue* λ*.*

4.2 Homogeneous ODEs

Definition (Homogenous ODE)**.** An ODE is homogeneous if all of the terms involve only the dependent variables and its derivatives.

Proposition. *Any linear homogeneous ODE with constant coefficients has solutions of the form* $e^{\lambda x}$*.*

Proposition. *For linear, homogeneous ODEs, any constant multiple of a solution is a solution.*

Proposition. *An n-th degree linear DE has n linearly independent solutions.*

Definition (Characteristic Equation)**.** For a linear homogeneous DE $Ly = 0$, the Characteristic Equation is given by

$$\frac{L(e^{\lambda x})}{e^{\lambda x}} = 0$$

Proposition. *An n-th order ODE requires n initial/boundary conditions.*

Proposition. *The solution(s) to a homogeneous linear constant coefficient ODE can be found using the Characteristic Equation.*

4.3 Forcing

Definition (Inhomogeneous ODE)**.** An ODE is inhomogeneous if there are terms which involve the independent variable, or are constant.

Definition (Forcing)**.** In an ODE $Ly = F$, F is known as the forcing term.

Proposition. *The general solution to a forced linear inhomogeneous ODE $Ly = F$ is $y = y_c + y_p$, where*

- $Ly_c = 0$ *- Solving the corresponding homogeneous equation, y_c is the complementary function.*
- $Ly_p = F$ *- Finding the particular integral y_p.*

Proof. As L is linear, $L(y_c + y_p) = Ly_c + Ly_p = 0 + F = F$. □

5 Linear First Order ODEs

5.1 Non Constant Coefficients

Consider $y' + p(x)y = f(x)$.

Definition (Integrating Factor)**.** The integrating factor is

$$\mu = \exp\left(\int p \mathrm{d}x\right)$$

Proposition. *The solution to $y' + p(x)y = f(x)$ is given by*

$$\mu(x)y = \int f(x)\mu(x)\mathrm{d}x$$

Proof. Note that $\mu' = p\exp\left(\int p\mathrm{d}x\right) = p\mu$, so $p = \dfrac{\mu'}{\mu}$. So muliplying through by μ, we get that $\mu y' + p\mu y = (\mu y)' = \mu f$. So $\mu y = \int \mu f \mathrm{d}x$. ☐

6 Nonlinear First Order ODEs

General Form is $Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x, y) = 0$

Definition (Separable)**.** The ODE is separable if it can be written in the form $q(y)\mathrm{d}y = p(x)\mathrm{d}x$.

6.1 Exact Equations

Definition (Exact Equation)**.** If for some scalar function $f(x, y)$, we have that

$$\mathrm{d}f = Q\mathrm{d}y + P\mathrm{d}x$$

Then the equation is exact.

Proposition. *If the domain is simply connected (1-connected), then the equation is exact if and only if*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Proof. If $\mathrm{d}f = Q\mathrm{d}y + P\mathrm{d}x$, then $\frac{\partial f}{\partial x} = P(x, y)$ and $\frac{\partial f}{\partial y} = Q(x, y)$. Then we have that $\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$. The reverse implication will not be proven. ☐

Proposition. *The solution to an exact equation* $\mathrm{d}f = Q\mathrm{d}y + P\mathrm{d}x$ *is $f =$ constant.*

7 Graphical Methods

7.1 Graphs of Solutions

In this subsection assume $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$.

Definition (Isocline)**.** An isocline is a curve along which $\frac{\mathrm{d}y}{\mathrm{d}t} = f$ is constant.

We can use isoclines to sketch the solutions to DEs, furthermore as f is single valued, different solution curves do not cross.

7.2 Phase Portrait

In this subsection assume $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$.

A phase portrait plots y on the horizontal axis and $\frac{\mathrm{d}y}{\mathrm{d}t}$ on the vertical axes. Fixed points are roots, and the stability can be determined from the sketch.

7.3 1D phase portrait

In this subsection assume $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$.

A 1D phase portrait marks out the fixed points, as well as the direction a point would move in.

8 Fixed Points

Definition (Equilibrium Point). Equilibrium points are points where $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ for all t.

8.1 Stability

In this subsection assume $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t)$, and that a is a fixed point.

Let $y = a + \varepsilon(t)$ for small ε. Then $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = f(a + \varepsilon, t)$. Expand using a Taylor series, we get

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = f(a + \varepsilon, t) = f(a, t) + \varepsilon \frac{\mathrm{d}f}{\mathrm{d}y}(a, t) + O(\varepsilon^2)$$

As a is a fixed point, $f(a, t) = 0$, and as $\varepsilon \ll 1$, we can neglect terms of $O(\varepsilon^2)$. So

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \varepsilon \frac{\mathrm{d}f}{\mathrm{d}y}(a, t)$$

Definition (Stable Fixed Point). If $\varepsilon \to 0$ as $t \to \infty$, then a is a stable fixed point.

Definition (Unstable Fixed Point). If $\varepsilon \to \pm\infty$ as $t \to \infty$, then a is an unstable fixed point.

Note if $\frac{\partial f}{\partial y}(a, t) = 0$ then we will need higher order terms.

8.2 Autonomous DEs

Definition (Autonomous DE). An Autonomous DE is of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y)$.

In this case,

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = f'(a)\varepsilon$$

and if $f'(a) < 0$, it is stable. If $f'(a) > 0$ it is unstable.

8.3 Discrete Equations

In this section assume $x_{n+1} = f(x_n)$.

Definition (Fixed point). A fixed point satisfies $x_{n+1} = f(x_n) = x_n$.

Let a be a fixed point, $x_n = a + \varepsilon_n$. Then

$$x_{n+1} = f(x_n) = f(a + \varepsilon_n) = f(a) + \varepsilon_n \frac{\mathrm{d}f}{\mathrm{d}x}(a) + O(\varepsilon_n^2) \approx a + \varepsilon_n \frac{\mathrm{d}f}{\mathrm{d}x}(a)$$

Then $\varepsilon_{n+1} = x_{n+1} - a \approx \varepsilon_n \frac{\mathrm{d}f}{\mathrm{d}x}(a)$. Thus if $\left|\frac{\mathrm{d}f}{\mathrm{d}x}(a)\right| < 1$, it is stable. If $\left|\frac{\mathrm{d}f}{\mathrm{d}x}(a)\right| > 1$, it is unstable.

9 Second Order ODEs

9.1 Detuning

In this subsection assume $y'' + by' + cy = f(x)$.

From before, we have seen that this can be solved by finding the complementary function and the particular integral, and that the complementary function can be found by solving the characteristic equation $\lambda^2 + b\lambda + c = 0$, which has roots $\lambda_1, \lambda_2 \in \mathbb{C}$. If $\lambda_1 = \lambda_2$ then we only find one linearly independent solution here, and there should be another.

Consider $y'' + by' + (c - \varepsilon^2)y = 0$, for $\varepsilon \ll 1$. From above, we have that $\lambda^2 + b\lambda + c = (\lambda - \lambda_1)^2$. So the roots to the characteristic equation of our new equation are $\lambda_1 \pm \varepsilon$.

Then $y_c = Ae^{(\lambda_1+\varepsilon)x} + Be^{(\lambda_1-\varepsilon)x} = e^{\lambda_1 x}(Ae^{\varepsilon x} + Be^{-\varepsilon x})$. Expanding $e^{\varepsilon x}$, we get

$$y_c = e^{\lambda_1 x}((A+B) + (A-B)\varepsilon x + O(\varepsilon^2))$$

Suppose further that we have some initial conditions $y_c(0) = C$, $y_c'(0) = D$ and also that we may disregard terms of order ε^2. Using these, we have that $C = A + B = O(1)$, and $D = \lambda_1(A+B) + \varepsilon(A-B)$, then $A - B = \dfrac{D - kC}{\varepsilon} = O(1/\varepsilon)$. Thus we can keep the ε term when taking $\varepsilon \to 0$. Let $\alpha = A + B$, $\beta = \varepsilon(A - B)$, then

$$\lim_{\varepsilon \to 0} y_c = e^{\lambda_1 x}(\alpha + \beta x)$$

Hence if y_1 is a degenerate complementary function, then $y_2 = xy_1$ is a linearly independent complimentary function.

9.2 Reduction of Order

In this section, assume $y'' + p(x)y' + q(x)y = 0$.

Suppose we are given one solution y_1. We can try and find a solution of the form $y_2 = vy_1$. Substituting, we get

$$\begin{aligned}(vy_1'' + 2v'y_1' + v''y_1) + p(x)(vy_1' + v'y_1) + q(x)(vy_1) &= v(y_1'' + py_1' + qy_1) + 2v'y_1' + v''y_1 + pv'y_1 \\ &= 2v'y_1' + v''y_1 + pv'y_1 \\ &= 0\end{aligned}$$

Setting $u = v'$, we find that $u'y_1 + u(2y_1' + py_1) = 0$. This is a separable DE, and once we have found u we can find v, and thus y_2.

10 Phase Space

For an n-th order linear ODE

$$\sum_{k=0}^{n} p_k(x)y^{(k)}(x) = 0$$

The value of $y^{(n)}(x)$ can be determined from $y, \dots, y^{(n-1)}$. We can represent the state of the system as an n-dimensional vector

$$\mathbf{Y}(x) = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

10.1 Linear Independece and Uniqueness of Solutions

Proposition. *Solutions $y_1, \dots, y_n$ to an n-th order ODE are independent if and only if the corresponding solution vectors $\mathrm{Y}_1, \dots, \mathrm{Y}_n$ are independent.*

If $y_1, \dots, y_n$ are solutions to the ODE, then the general solution will be of the form $a_1y_1 + \cdots + a_ny_n$. Suppose we were given initial conditions, which are $y(x_1) = b_1, \dots, y^{(n-1)}(x_n) = b_n$, then we have that $a_1y_1(x_1) + \cdots + a_ny_n(x_1) = b_1, \dots, a_1y_1^{(n-1)}(x_n) + \cdots + a_ny_n^{(n-1)}(x_n) = b_n$. This can be written as a matrix equation

$$\begin{pmatrix} y_1(x_1) & \dots & y_n(x_1) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x_n) & \dots & y_n^{(n-1)}(x_n) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

With this, we can see that the solution is unique if and only if the determinant of the matrix is non-zero.

10.2 Wronskian

Definition (Fundamental Matrix)**.** If the solutions to a DE are $y_1, \dots, y_n$, then the fundamental matrix is

$$\begin{pmatrix} \uparrow & & \uparrow \\ \mathrm{Y}_1 & \dots & \mathrm{Y}_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} y_1 & \dots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix}$$

Definition (Wronskian)**.** The Wronskian, $W(x)$ is the determinant of the fundamental matrix.

$$W(x) = \begin{vmatrix} \uparrow & & \uparrow \\ \mathrm{Y}_1 & \dots & \mathrm{Y}_n \\ \downarrow & & \downarrow \end{vmatrix} = \begin{vmatrix} y_1 & \dots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

If the Wronskian is non-zero for some x, then the two solution vectors are linearly independent. On the other hand, if the two solution vectors are linearly dependent, then the Wronskian is identically zero. Do note the directions of the implications, the reverse implications are not true. For example, if $y_1 = x^2$ and $y_2 = x|x|$, the Wronskian is identically zero, but the two functions are linearly independent.

11 Second Order ODEs - Continued

11.1 Abel's Identity

In this subsection assume $y'' + py' + qy = 0$.

Theorem (Abel's Identity)**.** *If p and q are continuous on an interval I, then the Wronskian is either identically zero in I, or always non-zero.*

Proof. Let y_1, y_2 be solutions. Then $W(x) = y_1y_2' - y_2y_1'$. In addition, we have that $y_2(y_1'' + py_1' + qy_1) = 0$ and $y_1(y_2'' + py_2' + qy_2) = 0$. Subtracting, we get that

$$y_1y_2'' - y_2y_1'' + p(y_1y_2' - y_2y_1') = y_1y_2'' - y_2y_1'' + pW(x) = 0$$

Now $W'(x) = y_1y_2'' + y_1'y_2' - y_2'y_1' - y_2y_1'' = y_1y_2'' - y_2y_1''$, so what we have is that

$$\frac{\mathrm{d}W}{\mathrm{d}x} + pW = 0$$

Hence

$$W(x) = W(x_0) \exp\left(- \int_{x_0}^{x} p(t) \mathrm{d}t \right)$$

for any $x_0 \in I$. The second term is always positive, as the integral is always finite, by the continuity of p. □

Corollary. *If $p = 0$ then W is constant.*

We can also use Abel's identity to find the second solution to a differetial equation. That is, if we know one solution y_1, then the other solution y_2 satisfies

$$y_1 y_2' - y_2 y_1' = W_0 \exp\left(-\int_{x_0}^{x} p(t)\mathrm{d}t\right)$$

11.2 Equidimensional Equations

Definition (Equidimensional Equation)**.** . A DE is equidimensional if it is unaffected by a multiplicative scaling, say $x \mapsto \kappa x$.

The general form is $ax^2y'' + bxy' + cy = f(x)$, where a, b, c are constants.

To solve equidimensional equations, use $y = x^k$ and solve $ak(k-1) + bk + c = 0$. If there are two distinct roots k_1 and k_2, then the solution is $y = Ax^{k_1} + Bx^{k_2}$.

Note that $z = \log x$ transforms an equidimensional equation into one with constant coefficients, that is,

$$a\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (b-a)\frac{\mathrm{d}y}{\mathrm{d}z} + cy = f(e^z)$$

Solving this as before, we find that the characteristic equation of this is the same as before, and if we have a repeated root k, the solution is $y = Ae^{kz} + Bze^{kz}$, ie. $y = Ax^k + Bx^k \log x$.

11.3 Variation of Parameters

If we know the complementary functions, we can use them to find the particular integrals. Suppose y_1 and y_2 are linearly independent complementary functions, and suppose further that

$$\mathbf{Y}_p = u(x)\mathbf{Y}_1 + v(x)\mathbf{Y}_2$$

Then $y_p = uy_1 + vy_2$ and $y_p' = uy_1' + vy_2'$. We also find that $y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$, so $u'y_1 + v'y_2 = 0$. Differentiating the result for y_p', we find that $y_p'' = uy_1'' + u'y_1' + vy_2'' + v'y_2'$. Hence if $y_p'' + p(x)y_p' + q(x)y_p = f(x)$, using the above we find that $u'y_1' + v'y_2' = f(x)$.

Putting this all together, we get that

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

As y_1 and y_2 are linearly independent, we get that $W(x) \neq 0$. So

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W(x)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

and $u' = \dfrac{-y_2 f(x)}{W(x)}$ and $v' = \dfrac{y_1 f(x)}{W(x)}$. Integrating we can find u and v.

12 Transients and Damping

Consider a particle under a Hooke's Law like force, and a frictional force proportional to velocity. Then we have that

$$my'' = F(t) - ky - Ly'$$

Dividing through by m, and letting $\tau = \sqrt{\dfrac{k}{m}}t$ (and y' will mean $\frac{\mathrm{d}y}{\mathrm{d}\tau}$ from here on out). We get that

$$y'' + 2\kappa y' + y = f(\tau)$$

where $\kappa = \dfrac{L}{2\sqrt{km}}$, $f(\tau) = \dfrac{F\left(\sqrt{\frac{m}{k}}\tau\right)}{m}$.

12.1 Unforced Response

The unforced respose is given by $f = 0$, and is determined entirely by the parameter κ. The characteristic equation is $\lambda^2 + 2\kappa\lambda + 1 = 0$.

If $\kappa < 1$ the system is underdamped, and the resulting motion can be described by

$$y = e^{-\kappa\tau}\left(A\cos\left(\sqrt{1-k^2}\tau\right) + B\sin\left(\sqrt{1-k^2}\tau\right)\right)$$

If $\kappa = 1$, the system is critically damped, and the resulting motion is given by

$$y = e^{-\kappa\tau}(A + B\tau)$$

If $\kappa > 1$, the system is overdamped, and the resulting motion is given by

$$y = Ae^{\lambda_1\tau} + Be^{\lambda_2\tau}$$

Note that in all of these cases, if $L > 0$ then $y \to 0$ as $\tau \to \infty$.

12.2 Sinusoidal Forcing

Consider $y'' + \mu y' + \omega_0^2 y = \sin(\omega t)$. We would guess that the particular integral is of the form $A\sin(\omega t) + B\cos(\omega t)$. Substituting in, we find that $A = \dfrac{\omega_0^2 - \omega^2}{(\omega_0^2-\omega^2)^2 + \mu^2\omega}$ and $B = \dfrac{-\mu\omega}{(\omega_0^2-\omega^2)^2 + \mu^2\omega}$. Hence $y_p = \dfrac{(\omega_0^2-\omega^2)\sin(\omega t) - \mu\omega\cos(\omega t)}{(\omega_0^2-\omega^2)^2 + \mu^2\omega}$.

If damped, the transient (short term) response is given by the complementary function, whereas the long term behaviour is determined entirely by the particular integral.

12.3 Resonance

Consider if $\omega = \omega_0$ in the above. If $\mu \neq 0$, we find that $y_p \to -\dfrac{\cos(\omega t)}{\mu\omega_0}$ as $\omega \to \omega_0$. Consequently the result is finite amplitude oscillations.

On the other hand, if the system is undamped, that is to say $\mu = 0$, then consider $y'' + \omega_0^2 y = \sin(\omega t)$, where $\omega \neq \omega_0$. We find that $y_p = \dfrac{\sin(\omega t)}{\omega_0^2 - \omega^2}$. Since the original DE is linear, we have that $y_p + Ay_c$ will also satisfy the DE, hence without loss of generality, we can let $y_p = \dfrac{\sin(\omega t)}{\omega_0^2-\omega^2} + A\sin(\omega_0 t)$. Choosing $A = \dfrac{-1}{\omega_0^2-\omega^2}$, we get that $y_p = \dfrac{\sin(\omega t) - \sin(\omega_0 t)}{\omega_0^2 - \omega^2}$. Using trigonometric identities, we get that

$$y_p = \frac{2\cos\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)}{\omega_0^2-\omega^2}$$

Setting $\Delta\omega = \omega - \omega_0$, we get that

$$y_p = \frac{2\cos\left(\left(\omega_0 - \frac{\Delta\omega}{2}\right)t\right)\sin\left(\frac{\Delta\omega}{2}t\right)}{\Delta\omega(\omega+\omega_0)}$$

For $\Delta\omega t \ll 1$, $\sin\left(\dfrac{\Delta\omega t}{2}\right) \approx \dfrac{\Delta\omega t}{2}$. As a result,

$$\lim_{\Delta\omega\to 0} y_p \approx \frac{-t}{2\omega_0}\cos(\omega_0 t)$$

and the amplitude of oscillations grows linearly.

12.4 Dirac Delta Forcing

Consider a family of functions (eg $D(t;\varepsilon) = \frac{1}{\varepsilon\sqrt{\pi}}\exp\left(-\frac{t^2}{\varepsilon^2}\right)$) which satisfy the following:

- $\lim_{\varepsilon\to 0} D(x;\varepsilon) = 0$ for all $x \neq 0$.
- $\int_{-\infty}^{\infty} D(t;\varepsilon)\mathrm{d}t = 1$.

Define the Dirac delta function $\delta(x) = \lim_{\varepsilon\to 0} D(x;\varepsilon)$. Then $\delta(x) = 0$ for all $x \neq 0$ and $\int_{-\infty}^{\infty}\delta(x)\mathrm{d}x = 1$. For continuous functions g, we also have the sampling property,

$$\int_a^b g(x)\delta(x-x_0)\mathrm{d}x = \begin{cases} g(x_0) & \text{if } a < x_0 < b \\ 0 & \text{otherwise} \end{cases}$$

If we have an oscillatory system, which is given a sudden impulse at time $t = 0$, we can represent this by

$$y'' + py' + qy = \delta(t)$$

To solve this, we can solve it for $t < 0$ and $t > 0$ separately. We then impose the jump conditions to connect the two solutions. First of all we require that y is continuous at t. Second, we find that

$$\int_{-\varepsilon}^{\varepsilon} y'' + py' + qy\mathrm{d}t = \int_{-\varepsilon}^{\varepsilon} y''\mathrm{d}t = [y']_{-\varepsilon}^{\varepsilon} = \int_{-\varepsilon}^{\varepsilon}\delta(t)\mathrm{d}t = 1$$

as $\int py'$ and qy both go to zero, since y is continuous. Consequently, we must have a 'jump' of 1 at $t = 0$, ie

$$\lim_{\varepsilon\downarrow 0} y'(\varepsilon) - \lim_{\varepsilon\uparrow 0} y'(\varepsilon) = 1$$

12.5 Heaviside Step Forcing

Define the Heaviside Step function

$$H(x) = \int_{-\infty}^{x}\delta(t)\mathrm{d}t = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$

We can define a system with Heaviside step forcing, $y'' + py' + qy = H(t)$. In this case, the jump conditions are that y and y' are both continuous at $t = 0$.

13 Discrete Equations

The general form of a m-th order linear discrete equation is as such

$$a_m y_{n+m} + \cdots + a_n y_n = f_n$$

Note that the eigenfunction in this case is $y_n = k^n$. We can find solution s just like for Differential Equations using the characteristic equation.

13.1 Second Order Discrete Equation

If the characteristic equation has distinct roots k_1 and k_2, then the complementary function is

$$y_c(n) = Ak_1^n + Bk_2^n$$

On the other hand, if it has a repeated root k, then the complementary function is

$$y_c(n) = Ak^n + Bnk^n$$

The table below has the form of common particular integrals.

Form of f_n	From of $y_p(n)$
k^n, $k \neq k_1, k_2$	Ak^n
k_1^n, k_2^n	$Ank_1^n + Bnk_2^n$
n^p	$c_p n^p + \cdots + c_0$

14 Series Solutions

By considering the solution as a power series, we can use this to find the solution to Differential Equations.

Setting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, or as a convenient trick $y(x) = \sum_{\mathbb{Z}} a_n x^n$, where we define $a_n = 0$ if $n < 0$, we can differentiate term by term, and by substituting into the differential equation and equating coefficients of x we can find a difference equation for the a_ns.

14.1 Method of Frobenius

Given a second order, linear homogeneous ODE $py'' + qy' + r = 0$, the method of Frobenius can be used to find the series solution to it. Suppose we wanted to find a power series expansion about $x = x_0$. We must first classify the point x_0.

If the power series of q/p and r/p converge in some neighbourhood around x_0, then x_0 is an ordinary point. Otherwise, x_0 is a singular point. If the differential equation cen be written as $P(x-x_0)^2y'' + Q(x-x_0)y' + R = 0$, then x_0 is a regular singular point. Otherwise it is an irregular singular point.

Note that $\frac{Q}{P} = (x - x_0)\frac{q}{p}$ and $\frac{R}{P} = (x - x_0)^2\frac{r}{p}$.

If x_0 is an ordinary point, then there are two linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

If x_0 is a regular singular point, then there is at least one solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma}$$

where $\sigma \in \mathbb{R}$, $a_0 \neq 0$. Note for a regular singular point, we substitute the power series into $P(x - x_0)^2y'' + Q(x - x_0)y' + R = 0$, not the original equation. To find the value of σ, consider the coefficient of the lowest degree of x, usually x^σ. Dividing through by $a_0 \neq 0$, we get the indicial equation, which is of the form

$$a\sigma^2 + b\sigma + c = 0$$

Let the roots be σ_1 and σ_2. If $\sigma_1 - \sigma_2 \notin \mathbb{Z}$, then we have two linearly independent solutions.

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma_1} + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\sigma_2}$$

If $\sigma_1 - \sigma_2 \in \mathbb{Z}$, then one of the solutions is of the following form

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma_1}$$

and the other is of the form

$$y_2 = \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\sigma_2} + c y_1 \log(x - x_0)$$

where $c \in \mathbb{R}$, which may or may not be zero. Note that if $\sigma_1 = \sigma 2$, then c must be non-zero.

15 Multivariate Functions

For this section also see Vector Calculus. Here we only consider $f(x, y)$ in Cartesians, but for generalisations to higher dimensions and other othonormal curvilinear coordinates see Vector Calculus.

15.1 Gradient

Definition (Gradient)**.** For $f(x, y)$,

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

Definition (Line element)**.**

$$\mathrm{d}\mathbf{s} = \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$$

Then $\mathrm{d}f = \nabla f \cdot \mathrm{d}s$.

Definition (Directional Derivative)**.** Given a unit vector $\hat{\mathbf{s}}$, the directional derivative $\frac{\mathrm{d}f}{\mathrm{d}s}$ is $\hat{\mathbf{s}} \cdot \nabla f$, and it is the rate of change of f in the direction of $\hat{\mathbf{s}}$.

Note that ∇f points in the direction of greatest increase, and that on the contours of f, $\frac{\mathrm{d}f}{\mathrm{d}s} = 0$, so $\hat{\mathbf{s}} \cdot \nabla f = 0$, which means that $\hat{\mathbf{s}}$ is perpendicular to ∇f.

15.2 Hessian

The Hessian of a function f is given by

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

15.3 Multivariate Taylor Series

Consider $f : \mathbb{R}^2 \to \mathbb{R}$, then

$$f(\mathbf{r}_0 + \delta\mathbf{r}) = f(\mathbf{r}_0) + (\delta\mathbf{r})\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{(\delta\mathbf{r})^2}{2}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \dots$$

Note that $\delta\mathbf{r}\frac{\mathrm{d}}{\mathrm{d}r} = \delta\mathbf{r} \cdot \nabla$. Considering the quadratic part,

$$\begin{aligned}
(\delta\mathbf{r})^2 \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} &= \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial x}\right) \left(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial x}\right) f \\
&= \delta x^2 f_{xx} + \delta x \delta y f_{xy} + \delta y^2 f_{yy} \\
&= \begin{pmatrix} \delta x & \delta y \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \\
&= \delta\mathbf{r}^T H \delta\mathbf{r}
\end{aligned}$$

So $f(\mathbf{r}_0 + \delta\mathbf{r}) \approx f(\mathbf{r}_0) + \delta\mathbf{r} \cdot \nabla f + \frac{1}{2}\delta\mathbf{r}^T H \delta\mathbf{r}$

15.4 Stationary Points

Note that the stationary points of $f : \mathbb{R}^n \to \mathbb{R}$ are the points where $\boldsymbol{\nabla} f = 0$. Note that the Hessian matrix is symmetric, as partial derivatives are symmetric. As a result, we can diagonalise H with respect to its principal axes. Then

$$\delta\mathbf{r}^T H\delta\mathbf{r} = \delta\mathbf{r}^T \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{pmatrix} \delta\mathbf{r}$$

where the λ_i are the eigenvalues of H. Expanding the quadratic form, we get that $\delta\mathbf{r}^T H\delta\mathbf{r} = \lambda_1\delta r_1^2 + \cdots + \lambda_n\delta r_n^2$, where δr_i is the components of $\delta\mathbf{r}$ with respect to the principal axes of H.

At a minimum point, $\delta\mathbf{r}^T H\delta\mathbf{r} > 0$ for all $\delta\mathbf{r}$, which means that we must have that all of the eigenvalues of H are positive, so H is positive definite.

At a maximum point, $\delta\mathbf{r}^T H\delta\mathbf{r} < 0$ for all $\delta\mathbf{r}$, so H is negative definite.

Otherwise, some of the λ_i are positive and some are negative, and in this case we have a saddle point.

15.5 Signature

For a function $f : \mathbb{R}^n \to \mathbb{R}$, define the sub-Hessian matrices

$$H_i = \begin{pmatrix} f_{x_1x_1} & \dots & f_{x_1x_i} \\ \vdots & \ddots & \vdots \\ f_{x_ix_1} & \dots & f_{x_ix_i} \end{pmatrix}$$

The signature of f is the sequence of signs of the determinants of the Hessians,

$$\text{sign}(|H_1|), \text{sign}(|H_2|), \dots, \text{sign}(|H_n|) = \text{sign}(|H|)$$

Proposition (Sylvester's Criterion). *If the signature is* $1, \dots, 1$, *then it is a maximum point. If the signature is* $-1, 1, -1, \dots, (-1)^n$, *then it is a minimum point.*

15.6 Contours

For a function $f(x, y)$, the contours of f are

- Locally elliptic about minima/maxima
- Locally hyperbolic about saddle points
- Contours only cross at saddle points

To see this, let $\mathbf{x}_0$ be a stationary point, Then f approximately constant about $\mathbf{x}_0$, and $\boldsymbol{\nabla} f \approx 0$. Let $\delta\mathbf{x} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ in terms of the principal axes of H. Then we have that

$$\delta\mathbf{x}^T H\delta\mathbf{x} = \lambda_1\xi^2 + \lambda_2\eta^2 \approx \text{constant}$$

Depending on the signs of λ_1 and λ_2, we get either hyperbolic or elliptic contours.

16 Systems of Linear ODEs

Consider two functions y_1 and y_2 satisfying

$$y_1' = ay_1 + by_2 + f_1$$
$$y_2' = cy_1 + dy_2 + f_2$$

we can write this as

$$\mathbf{Y}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{Y} + \mathbf{F}$$

Any n-th order linear ODE can be written as n first order coupled ODEs, by considering $\mathbf{Y} = \begin{pmatrix} y \\ \vdots \\ y^{(n-1)} \end{pmatrix}$, the differential equation can be written as

$$\mathbf{Y}' = M\mathbf{Y} + F$$

Conversely, any system of n first order coupled ODEs can be written as an n-th order ODE.

16.1 Matrix Methods

To solve a system of linear ODEs,

$$\mathbf{Y}' = M\mathbf{Y} + \mathbf{F}$$

again we use the linearity to find the complementary function $\mathbf{Y}_c$ and the particular integral $\mathbf{Y}_p$. To find Y_c, let $\mathbf{v}$ be a constant vector, then we can try $\mathbf{Y}_c = e^{\lambda t}\mathbf{v}$. Substituting this in, we get $\lambda e^{\lambda t}\mathbf{v} = Me^{\lambda t}\mathbf{v}$ which is equivalent to $\lambda\mathbf{v} = M\mathbf{v}$. Note this means $\mathbf{v}$ are the eigenvectors of M, and λ are the eigenvalues.

To find $\mathbf{Y}_p$, we use the form of $\mathbf{F}$, and either by inspection/guesswork, or variation of parameters.

16.2 Phase Portraits

Returning to $n = 2$, if $\mathbf{Y}_c' = M\mathbf{Y}_c$, then we know that

$$\mathbf{Y}_c = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}$$

The behaviour of the system near the origin will depend on the value of λ_1 and λ_2.

- If $\lambda_1, \lambda_2 \in \mathbb{R}$, then we have three cases
  - If $\lambda_1, \lambda_2 > 0$, then we have an unstable node. The solutions move away from the origin.
  - If $\lambda_1, \lambda_2 < 0$, then we have a stable node. The solutions move towards the origin.
  - If $\lambda_1\lambda_2 < 0$, then we have a saddle node. The solutions move towards the origin along the axis with negative eigenvalue, and waya from the origin along the axis with positive eigenvalue.
- If $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = 0$, then we have a centre. The solutions move in concentric ellipses about the origin.
-
  - If $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) > 0$, then we have an unstable spiral. The solutions move in a spiral away from the origin.
  - If $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) < 0$, then we have a stable spiral. The solutions move in a spiral towards the origin.

The direction of rotation for a centre and a spiral can be determined by evaluating the system at a point nead the origin.

16.3 Nonlinear Systems of ODEs

Consider the autonomous system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

The equilibrium points (x_0, y_0) are where $f(x_0, y_0) = g(x_0, y_0) = 0$.

16.4 Stability of Fixed Points

Like before, we can use perturbation analysis to determine the stability of a fixed point. Suppose (x_0, y_0) is a fixed point. Let $(x, y) = (x_0 + \xi, y_0 + \eta)$, then

$$\xi' = f(x_0 + \xi, y_0 + \eta) \approx f(x_0, y_0) + \xi f_x(x_0, y_0) + \eta f_y(x_0, y_0) = \xi f_x(x_0, y_0) + \eta f_y(x_0, y_0)$$
$$\eta' = g(x_0 + \xi, y_0 + \eta) \approx g(x_0, y_0) + \xi g_x(x_0, y_0) + \eta g_y(x_0, y_0) = \xi g_x(x_0, y_0) + \eta g_y(x_0, y_0)$$

Hence

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

This is a homogeneous system of linear ODEs, so the eigenvalues of the matrix will determine the stability of the system.

17 Partial Differential Equations

17.1 First Order Wave Equation

Consider if $y(x, t)$ where

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0$$

Now let $x = x(t)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

Comparing these, we see that if $\frac{\mathrm{d}x}{dt} = -c$, then $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$. Consequently y is constant along lines where $x = x_0 - ct$.

Solving the ODE at $t = 0$, we find that $y(x, 0) = f(x)$. Now note that $y(x, t) = y(x + ct, 0)$, so the general solution is $f(x + ct)$.

17.2 Second Order Wave Equation

Now consider $y(x, t)$ satisfying

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0$$

If we factorise the differential operator[1], then we get that

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) y = 0$$

Clearly the operators commute, so we must have that at least one of $\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0$ and $\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0$ holds.

By comparison to the first order, we see that the respective solutions are of the form $y = f(x + ct)$ and $y = g(x - ct)$. As the equation is linear, $y = f(x + ct) + g(x - ct)$ is a solution.

[1] This works

17.3 Diffusion Equation

Now consider if $c(x, t)$ satisfies

$$\frac{\partial c}{\partial t} = \kappa \frac{\partial^2 c}{\partial x^2}$$

Let $\eta = \dfrac{x^2}{4\kappa t}$, then η is dimensionless. We now look for solutions of the form $y = t^{-\alpha} f(\eta)$. Working out the derivatives, we find that

$$\begin{aligned}
\eta_t &= \frac{-x^2}{4\kappa t^2} = \frac{-\eta}{t} \\
(\eta_x)^2 &= \left(\frac{2x}{4\kappa t}\right)^2 = \frac{4x^2}{16\kappa^2 t^2} = \frac{\eta}{\kappa t} \\
\eta_{xx} &= \frac{2}{4\kappa t} \\
y_t &= -\alpha t^{-\alpha-1} f + t^{-\alpha} f_\eta \eta_t = -\alpha t^{-\alpha-1} f - t^{-\alpha-1} f_\eta \eta \\
y_x &= t^{-\alpha} f_\eta \eta_x \\
y_{xx} &= t^{-\alpha} f_{\eta\eta} \eta_x^2 + t^{-\alpha} f_\eta \eta_{xx} = t^{-\alpha-1} f_{\eta\eta} \kappa^{-1} + \frac{1}{2} t^{-\alpha-1} f_\eta \kappa^{-1}
\end{aligned}$$

Substituting in, we find that

$$-t^{-\alpha-1}(\alpha f - f_\eta \eta) = t^{-\alpha-1}(f_{\eta\eta} + \frac{1}{2} f_\eta)$$

Cancelling $t^{-\alpha-1}$ and rearranging, we find that

$$\eta \frac{\mathrm{d}}{\mathrm{d}\eta}\left(f + f'\right) + \frac{1}{2}(f' + 2\alpha f) = 0$$

Recall that α is arbitrary. Without loss of generality, set $\alpha = \dfrac{1}{2}$ and $F = f + f'$. Then

$$\eta \frac{\mathrm{d}f}{\mathrm{d}\eta} + \frac{1}{2} F = 0$$

We can solve this for F, and using that we can find f.