Dynamics and Relativity

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This document is intended for revision purposes. As a result, it does not contain any exposition. This is based off lectures given by Professor Peter Haynes in Lent 2021, but the order of content, as well as some of the proofs have been modified after the fact, primarily to provide simpler proofs for theorems. Note that this also contains theorems from examples sheets, as some are useful elsewhere.

In this course we will often use the shorthand notation $r = |\mathbf{r}|$ and $\mathbf{r}^2 = \mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2$.

Furthermore, as an applied course, it may be useful to refer to the lecture notes for worked examples.

Dynamics and Relativity is on *Paper 4*.

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1 Newtonian Dynamics

1.1 Defintions

Definition (Particle). A particle is an object with negligible size, mass $m > 0$ and charge q.

Definition (Frame of Reference). In a frame of reference S, there is an origin O and Cartesian axes $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$.

Definition (Position Vector). The position vector $\mathbf{r}$ or $\mathbf{x}$, of a particle is the coordinates of the particle in a frame of reference.

Definition (Velocity). The velocity of a particle is $\mathbf{u} = \dot{\mathbf{r}} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$.

Definition (Momentum). The momentum of a particle is $\mathbf{p} = m\mathbf{u} = m\dot{\mathbf{r}}$.

Definition (Acceleration). The acceleration of a particle is $\mathbf{a} = \dot{\mathbf{u}} = \ddot{\mathbf{r}}$.

1.2 Newton's Laws

Definition (Newton's First Law). For a particle which is not acted on by a force, there exists an inertial frame such that the acceleration of the particle is zero.

Definition (Newton's Second Law). In an inertial frame,

$$\mathsf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

Definition (Newton's Third Law). For every action there is an equal and opposite reaction. That is, the force between two particles are equal and opposite.

1.3 Inertial Frames and Galilean Transformations

Definition (Inertial Frame). In an inertial frame, the acceleration of a particle is zero if and only if the force acting on the particle is zero.

Definition (Boost). For a particle P, let $\mathbf{r}$ be its position vector in a frame S. Let the frame S' be moving with velocity $\mathbf{v}$ relative to S. Then let $\mathbf{r}'$ be the position vector of P in S'. We have that

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t$$

and this is known as a Boost.

Definition (Galilean Transformations). A Galilean transformation is one of the following

- Translation of space - $\mathbf{r}' = \mathbf{r} + \mathbf{r}_0$
- Translation of time - $t' = t_0 + t$
- Rotations and Reflections of Space - $\mathbf{r}' = R\mathbf{r}$ where R is an orthogonal matrix.
- Boosts - $\mathbf{r}' = \mathbf{r} + \mathbf{v}t$

Proposition. *The set of Galilean Transformations generate a group, known as the Galilean group.*

Proposition. *If frames S and S' are related by Galilean transformations, then $\ddot{\mathbf{r}} = \mathbf{0} \iff \ddot{\mathbf{r}}' = \mathbf{0}$. That is, S is inertial if and only if S' is inertial.*

Proposition (Galilean Invariance). *The equations of Physics are invariant under Galilean transformations.*

2 Dimensional Analysis

Definition (Dimension). In Mechanics, we have three basic dimensions, length (L), mass (M) and time (T). We denote the dimension of a quantity X as $[X]$.

Definition (Units). For each dimension, we have a set of units, for example m, kg and s.

Theorem (Bridgman's Theorem). *Suppose the dimensional quantity Y depends on a set of dimensional quantities $X_1, \ldots, X_n$. Let $[Y] = L^\alpha M^\beta T^\gamma$ and $[X_i] = L^{a_i} M^{b_i} T^{c_i}$. Say $Y_1 = C X_1^{p_1} \ldots X_n^{p_n}$, where C is a dimensionless constant.*

If $n \leq 3$, then p_1, p_2, p_3 can be determined exactly.

If $n > 3$, choose dimensionally independent quantities X_1, X_2, X_3 and $n-3$ dimensionless quantities

$$\lambda_i = \frac{X_{i+3}}{X_1^{q_{i1}} X_2^{q_{i2}} X_3^{q_{i3}}}$$

Then $Y = X_1^{p_1} X_2^{p_2} X_3^{p_3} C(\lambda_1, \ldots, \lambda_{n-3})$

3 Forces

3.1 Examples of Forces

Definition (Newtons's Law of Universal Gravitation). For particles of mass m_1, m_2 at $\mathbf{r}_1$, $\mathbf{r}_2$ respectively, the force on particle 1 is

$$\mathbf{F}_1 = -\frac{G m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) = -\mathbf{F}_2$$

Definition (Lorentz Force). For particle with charge q in electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$, the Lorentz force is

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

3.2 1 Dimensional Motion

Definition (Potential). Suppose **F** depends only on x and not $\dot{x}$ or t. We define the potential by

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

Proposition. *$V(x) = -\int^x F(r)\mathrm{d}r$ up the addition of a constant.*

Definition (Kinetic Energy). For a particle with mass m moving with speed $\dot{x}$, we define the kinetic energy

$$T = \frac{1}{2}m\dot{x}^2$$

Proposition. *$E = T + V$ is constant.*

Proof.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}t} = m\dot{x}\ddot{x} + F\dot{x} = 0$$

□

Proposition.

$$\pm\int_{x_0}^{x} \frac{\mathrm{d}x'}{\sqrt{\frac{2}{m}(E - V(x'))}} = t - t_0$$

where $x(t_0) = x_0$.

Definition (Equilibrium Points). An equilibrium point x is where $V'(x) = 0$.

Definition (Stable Equilibrium Point). x_0 is a stable equilibrium point if $V''(x_0) > 0$. If $V''(x_0) < 0$ then it is unstable.

Proof. Let x be close to x_0. Then $V(x) = V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + o((x - x_0)^3) \approx V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2$. Differentiating, we get that $V'(x) \approx V''(x_0)(x - x_0)$ and $m\ddot{x} = -V''(x_0)(x - x_0)$. □

3.3 3 Dimensional Motion

Definition (Kinetic Energy). For a particle moving with velocity $\mathbf{u}$, we define the kinetic energy

$$T = \frac{1}{2}m|\mathbf{u}|^2 = \frac{1}{2}m\mathbf{u}\cdot\mathbf{u}$$

Definition (Work). For a particle travelling along a curve C in a vector field F, the work done is

$$W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

Proposition. $\frac{\mathrm{d}T}{\mathrm{d}t} = \mathbf{F}\cdot\mathbf{u}$.

Definition (Conservative Vector Field). A conservative vector field[1] $\mathbf{F}$ is one where $\mathbf{F} = -\boldsymbol{\nabla}V$ for $V : \mathbb{R}^3 \to \mathbb{R}$

Proposition. *If the force is conservative, then $E = T + V$ is conserved.*

Proof.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \boldsymbol{\nabla}v\cdot\dot{\mathbf{r}} = (m\ddot{\mathbf{r}} - F)\cdot\dot{\mathbf{r}} = 0$$

□

[1] See Vector Calculus for more details

Proposition. *For a particle moving along path C from $\mathbf{r}_1$ to $\mathbf{r}_2$, under force field $\mathbf{F} = -\boldsymbol{\nabla} V$, the work done is*

$$V(\mathbf{r}_1) - V(\mathbf{r}_2)$$

Proof.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \boldsymbol{\nabla} V \cdot d\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2)$$

□

Corollary. *If C is closed, then work done is zero.*

3.4 Gravity

Definition (Gravitational Potential Energy)**.** For a particle of mass m, at $\mathbf{r}$ relative to a particle of mass M, the gravitational potential is

$$V(\mathbf{r}) = \frac{-GMm}{|\mathbf{r}|}$$

Proposition. $\mathbf{F} = -\frac{GMm}{|\mathbf{r}|^3}\mathbf{r} = -\boldsymbol{\nabla} V$*, so the gravitational force is conservative.*

Definition (Gravitational Potential)**.** For a particle of mass M, we define its gravitational potential

$$\Phi_g = -\frac{GM}{r}$$

Definition (Gravitational Field)**.** For a particle of mass M, we define its gravitational field

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi_g = -\frac{GM}{|r|^3}\mathbf{r}$$

Proposition. $V(\mathbf{r}) = m\Phi_g(\mathbf{r})$ *and* $F(\mathbf{r}) = m\mathbf{g}$

Proposition. *For masses $M_1, \dots, M_n$ at $\mathbf{r}_1, \dots, \mathbf{r}_n$, we have that*

$$\Phi_g(\mathbf{r}) = -\sum_{i=1}^{n} \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}$$

and

$$\mathbf{g}(\mathbf{r}) = -\sum_{i=1}^{n} \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}(\mathbf{r} - \mathbf{r}_i)$$

Example (1-D Approximation to Gravity)**.** On a planet of radius R, and mass M, we have a particle at radius $R + z$, where $z \ll R$. Then

$$V(R+z) = \frac{-GMm}{R+z} \approx -\frac{GMm}{R} + \frac{GMmz}{R^2} + \dots$$

and if we set $g = \frac{GM}{R^2}$, then $V(R+z) \approxeq \text{constant} + mgz$

Example (Escape Velocity)**.** For a particle with mass m leaving a planet with speed v. Then by conservation of energy, $E = \frac{1}{2}mv^2 - \frac{GMm}{R}$ is constant. Thus if $v \geq \sqrt{\frac{2GMm}{R}}$ the particle will escape. This is known as the escape velocity.

3.5 Electromagnetic Forces

In this subsection, we shall assume that $\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}$. In this case, the force $q\mathbf{E}$ is conservative. Also see Vector Calculus for more information about Electromagnetism.

Definition (Electrostatic Potential). Define the electrostatic potential Φ_e.

Definition (Point Charge). For a point charge Q at the origin, the electrostatic potential is

$$\Phi_e(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0|\mathbf{r}|}$$

Definition (Coulomb Force). For a point charge Q at the origin, and a particle with charge q at $\mathbf{r}$, the coulomb force is

$$F = -q\boldsymbol{\nabla}\Phi_e = \frac{Qq}{4\pi\varepsilon_0|\mathbf{r}^3|}\mathbf{r}$$

3.6 Friction

Definition (Dry Friction). For two objects in contact, we have dry friction

$$|\mathbf{F}| \leq \mu|\mathbf{N}|$$

where μ is the coefficient of friction (which depends on the materials, and whether one is moving relative to the other), and $\mathbf{N}$ is the normal reaction between the two.

Definition (Linear Drag). For a solid moving through a fluid, typically a small object through a viscous fluid, the drag experienced is proportional to the velocity.

$$\mathbf{F} = -k\mathbf{u}$$

where k is a constant. For example, in Stoke's Law $k = 6\pi r\eta$, where η is the viscosity of the fluid.

Definition (Quadratic Drag). For a solid moving through a fluid, typically for a large object through a less viscous fluid, the drag experienced is proportional to the velocity squared.

$$\mathbf{F} = -k|\mathbf{u}|\mathbf{u}$$

For example, $k = \rho_{\text{fluid}}C_D R$, where ρ_{fluid} is the density of the fluid, C_D is the drag coefficient and R is the size of the object.

4 Orbits

4.1 Angular Momentum

Definition (Angular Momentum). For a particle with mass m moving under a force $\mathbf{F}$, with position vector $\mathbf{r}$ and velocity vector $\dot{\mathbf{r}}$, we define the angular momentum about the origin as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}$$

Definition (Torque). We define the torgue about the origin as

$$\mathbf{G} = \dot{\mathbf{L}} = m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F}$$

4.2 Central Forces

Definition (Central Force). A central force is a conservative force with potential $V = V(r)$, where $r = |\mathbf{r}|$.

Proposition. *For a central force* F*, we have that*

$$\mathsf{F}(\mathbf{r}) = -\boldsymbol{\nabla} V(r) = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}} = \dfrac{\mathbf{r}}{|\mathbf{r}|}$.

Proposition. *For a central force* F*, the angular momentum is conserved.*

Proof.

$$\dot{\mathsf{L}} = \mathbf{r} \times \mathsf{F} = \mathbf{r} \times \left(-\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}\right) = \mathbf{0}$$

□

Proposition. *The motion under a central force is in a plane through the origin perpendicular to* L.

Proof.

$$\mathsf{L} \cdot \mathbf{r} = (m\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{r} = 0$$

□

4.3 Polars

Recall from Vector Calculus that for polars (r, θ), we have the unit vectors

$$\mathbf{e}_r = \frac{\frac{\partial \mathbf{x}}{\partial r}}{\left|\frac{\partial \mathbf{x}}{\partial r}\right|} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \qquad \mathbf{e}_\theta = \frac{\frac{\partial \mathbf{x}}{\partial \theta}}{\left|\frac{\partial \mathbf{x}}{\partial \theta}\right|} = \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$$

Proposition. $\dfrac{\partial}{\partial t}\mathbf{e}_r = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{e}_\theta$ *and* $\dfrac{\partial}{\partial t}\mathbf{e}_\theta == \dfrac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{e}_r$.

Proposition. $\mathbf{r} = r\mathbf{e}_r$.

Proposition.

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta$$

4.4 Orbits

Proposition. *Under a central force* F, $m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$

Proof. Newton's Second Law - $\mathsf{F} = -\frac{\mathrm{d}V}{\mathrm{d}r}\mathbf{e}_r = m\ddot{\mathbf{r}}$, equating the $\mathbf{e}_\theta$ components yields the result. □

Corollary.

$$\frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0$$

Corollary. $mr^2\dot{\theta}$ *is constant.*

Definition (Angular Momentum per Unit Mass).

$$h = r^2\dot{\theta}$$

Proposition. *Under a central force* $\mathsf{F} = -\boldsymbol{\nabla} V$,

$$m\ddot{r} = -\frac{\mathrm{d}V}{\mathrm{d}r} + \frac{mh^2}{r^3} = \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}t}$$

where $V_{\text{eff}} = V + \dfrac{mh^2}{2r^2}$ *is the effective potential.*

Proposition. *Under a central force* $\mathsf{F} = -\boldsymbol{\nabla} V$,

$$E = T + V = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$$

4.5 Stability of Circular Orbits

For a circular orbit, with radius r_*, we must have that $\frac{\mathrm{d}V_{\text{eff}}(r_*)}{\mathrm{d}r} = V'(r_*) - \frac{mh^2}{r^3} = 0$ as if $r(t) = r_*$ is constant, then $\ddot{r} = 0$.

A circular orbit with radius r_* is stable if $V''_{\text{eff}}(r_*) > 0$. Since in that case we have a local minimum for V at r_*.

4.6 Orbit Equation

The shape of the orbit is determined by $r(t)$ and $\theta(t)$. From $E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) = \text{const}$ we can find $r(t)$ and from $h = r^2\dot{\theta}$ we can find $\theta(t)$. However finding a solution for $r(t)$ from V_{eff} is difficult.

In practice, letting θ be the independent variable ($\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\theta}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{h}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}$) and $u = 1/r$, we get that

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{1}{mh^2u^2}F(1/u)$$

4.7 Kepler Problem

The Kepler problem is the problem of an orbit under a gravitational central force $F = -mku^2$. Substituting into the orbit equation, we get that

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{k}{h^2}$$

The general solution is $u = \frac{k}{h^2} + A\cos(\theta - \theta_0)$, where A and θ_0 are specified from initial conditions. Without loss of generality, let $A \geq 0$. By a rotation, we may also let $\theta_0 = 0$. So $u = \frac{k}{h^2} + A\cos\theta$. Let $l = \frac{h^2}{k}$ and $e = \frac{Ah^2}{k}$. Then the general solution becomes

$$r = \frac{1}{u} = \frac{l}{1 + e\cos\theta}$$

If $e = 0$ we have a circle of radius $r = \frac{h^2}{k}$.

If $0 < e < 1$, we have an ellipse with one foci at the origin.

As $r^2 = x^2 + y^2$, $x = r\cos\theta$, in Cartesians this becomes

$$\frac{(x + ea)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a = \frac{l}{1 - e^2}$ and $b = \frac{l}{\sqrt{1 - e^2}}$.

If $e > 1$, we have a hyperbola with one foci at the origin, asymptotes are $y = \pm\frac{b}{a}(x - ea)$. The perpendicular distance between the incoming trajectory ($r \to \infty$, where the particle is at the asymptote) and the origin is b, and this is known as the impact paramater.

In Cartesians, the orbit is

$$\frac{(x - ea)^2}{a^2} - \frac{y^2}{b^2} = 1$$

If $e = 1$, we have a parabola, $y^2 = l(l - 2x)$.

4.8 Kepler's Laws of Planetary Motion

Proposition (Kepler's First Law). *Orbits of planets are ellipsoidal with the Sun at one of the foci.*

Proposition (Kepler's Second Law). *Line between the Sun and a planet sweeps out equal area in equal time.*

Proof. Area $\approx \frac{1}{2}r^2\delta\theta$, so rate of change of area is $\frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h$ and is constant. $\square$

Proposition (Kepler's Third Law). *If period is P and the semi-major axis is a, then $P^2 \propto a^3$.*

Proof. From the second law, we have that $\pi ab = \frac{h}{2}P$. So $P^2 = \frac{4\pi^2 a^3}{k} \propto a^3$. $\square$

4.9 Rutherford Scattering

Now consider if the central force is repulsive, say $F = \frac{mk}{r^2}$. As before, we have that $u = -\frac{k}{h^2} + A\cos\theta$. If $l = \frac{h^2}{k}$ and $e = Al$, then we have that $r = \frac{l}{e\cos\theta - 1}$. Based on the physical conditions, we must have that $e > 1$.

As before, we have a hyperbola

$$\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a = \frac{l}{e^2-1}$ and $b = \frac{l}{\sqrt{e^2-1}}$. The angle by which the particle is deflected can be calculated from the asymptotes, and it is $\beta = 2\arctan\left(\frac{k}{bv^2}\right)$, where v is in initial velocity at $r \to \infty$.

5 Rotating Frames of Reference

Let S be an inertial frame, S' rotation about z-axis of S, with angular velocity ω. Let the basis vectors for S be $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and for S' be $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ with $\mathbf{e}_3 = \mathbf{e}'_3$. Let $\boldsymbol{\omega} = \omega\mathbf{e}_3 = \mathbf{e}'_3$.

Let $\mathbf{r}$ be the position vector of a particle at rest in S'. Then let $\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S$ be the rate of change of $\mathbf{r}$, as observed in S. Then

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \mathbf{r}$$

Now consider a general vector $\mathbf{a} = \mathbf{a}(t) = \sum_i a_i(t)\mathbf{e}'_i(t)$.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t)\right)_{S'} = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}a'_i(t)\mathbf{e}'_i(t)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t)\right)_S = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}a'_i(t)\mathbf{e}'_i + \sum_i a'_i(t)(\boldsymbol{\omega} \times \mathbf{e}_i) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t)\right)_{S'} + \boldsymbol{\omega} \times \mathbf{a}$$

Applying this twice to the position vector $\mathbf{r}$, we find that

$$\begin{aligned}
\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S &= \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\right]\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\right]\mathbf{r} \\
&= \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\right]\left(\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\mathbf{r}\right) \\
&= \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})
\end{aligned}$$

Hence we have the following

Definition (Newton's Second Law in a Rotating Frame).

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} = \mathbf{F} - 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} - \dot{\boldsymbol{\omega}} \times \mathbf{r} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Definition (Coriolis Force). The Coriolis force is $-2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'}$.

Definition (Euler Force). The Euler Force is $-\dot{\boldsymbol{\omega}} \times \mathbf{r}$.

Definition (Centrifugal Force). The Centrifugal Force is $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$.

Definition (Fictitious Forces). $-2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} - \dot{\boldsymbol{\omega}} \times \mathbf{r} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ are known as ficticious forces. They do not exist in an inertial frame, but we need ficticious forces to explain motion in a rotating frame.

6 Systems of Particles

Definition (Newton's Second Law for a System of Particles). For a system of n particles, we have Newton's Second Law for them. In particular, we are going to separate out the external forces $\mathbf{F}_i^{\text{ext}}$ from the internal forces $\mathbf{F}_{ij}$, which is the force on the ith particle from the jth.

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i^{\text{ext}} + \sum_{j=1}^{n} \mathbf{F}_{ij}$$

Proposition. *$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, and $\mathbf{F}_{ii} = \mathbf{0}$.*

Definition (Total External Force). We define the total external force $\mathbf{F}^{\text{ext}}$ as

$$\mathbf{F}^{\text{ext}} = \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}}$$

Definition (Total Mass). For a system of particles, we define the total mass M as

$$M = \sum_{i=1}^{n} m_i$$

Definition (Centre of Mass). For a system of particles, we define the centre of mass $\mathbf{R}$ as

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i$$

Definition (Linear Momentum). For a system of particles, we define the total linear momentum $\mathbf{P}$ as

$$\mathbf{P} = M\dot{\mathbf{R}} = \sum_{i=1}^{n} m_i\dot{\mathbf{r}}_i = \sum_{i=1}^{n} \mathbf{p}_i$$

Proposition. *If $\mathbf{F}^{\text{ext}} = \mathbf{0}$, the $\mathbf{P}$ is constant.*

Proof.

$$\dot{\mathbf{P}} = \sum_{i=1}^{n} m_i\ddot{\mathbf{r}}_i = \mathbf{F}^{\text{ext}} + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{F}_{ij} = \mathbf{F}^{\text{ext}}$$

□

Definition (Angular Momentum). For a system of particles, we define the total angular momentum about the origin as

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i$$

Definition (External Torque). We define the total external torque about the origin as

$$\mathbf{G}^{\text{ext}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}$$

Proposition. *If* $\mathbf{F}_{ij}$ *is parallel to* $\mathbf{r}_i - \mathbf{r}_j$*, then*

$$\dot{\mathbf{L}} = \mathbf{G}^{\text{ext}}$$

Proof.

$$\dot{\mathbf{L}} = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{n} \mathbf{r}_i \times (\mathbf{F}_i^{\text{ext}} + \sum_{j=1}^{n} \mathbf{F}_{ij}) = \mathbf{G}^{\text{ext}} + \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{r}_i \times \mathbf{F}_{ij}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{r}_i \times \mathbf{F}_{ij} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = \mathbf{0}$$

□

Definition (Position Relative to CoM). We can view the positions as being relative to the CoM, that is $\mathbf{r}_i = \mathbf{R} + \mathbf{s}_i$.

Proposition.

$$L = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_{i=1}^{n} m_i \mathbf{s}_i \times \dot{\mathbf{s}}_i$$

Definition (Total KE).

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$$

Proposition.

$$T = \frac{1}{2} M\dot{\mathbf{R}}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\mathbf{s}}_i^2$$

Proof. Expand and note that

$$\sum_{i=1}^{n} \mathbf{s}_i \cdot \mathbf{R} = 0$$

□

6.1 Two Body Problem

This is the special case where we have two particles moving under their mutual gravitational attraction and there are no external forces present.

Let $\mathbf{R}$ be the centre of mass of the two, and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ be the posiiton of the first particle relative to the second. As $\mathbf{F}^{\text{ext}} = \mathbf{0}$, we must have that $\ddot{\mathbf{R}} = \mathbf{0}$. Now consider

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_{12}}{m_1} - \frac{\mathbf{F}_{21}}{m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \mathbf{F}_{12}$$

Equivalently, we have that $\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}$, where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the system. Substituting in Newton's Law of Universal Gravitation, we get that

$$\ddot{\mathbf{r}} = -G(m_1 + m_2)\frac{\mathbf{r}}{|\mathbf{r}|^3}$$

6.2 Rocket Problem

Consider a rocket expelling fuel with velocity u relative to the rocket. We can solve for the motion of the rocket. The momentum at time t of the system is

$$m(t)v(t)$$

and the momentum of the rocket at time $t + \delta t$ is

$$m(t + \delta t)v(t + \delta t)$$

and the momentum of the fuel expelled is

$$(m(t) - m(t + \delta t))(v(t) - u + O(\delta t))$$

The change in momentum between time t and time $t + \delta t$ is

$$\begin{aligned} m(t + \delta t)v(t + \delta t) + &(m(t) - m(t + \delta t))(v(t) - u + O(\delta t)) - m(t)v(t) \\ &= m(t + \delta t)(v(t + \delta t) - v(t)) + u(m(t + \delta t) - m(t)) + O(\delta t) \\ &\approx \left(m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t}\right)\delta t \end{aligned}$$

Thus we have that

$$F^{\text{ext}} = m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t}$$

and this is known as the rocket equation.

7 Rigid Bodies

Definition (Rigid Body). A rigid body is a system of particles where the distance between particles remains constant.

7.1 Moment of Inertia

Recall for a particle rotating about an axis through O, with angular velocity ω, we have that $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$. Recall (from Vectors and Matrices) that $|\boldsymbol{\omega} \times \mathbf{r}| = \omega|\mathbf{r}_\perp|$, where $\mathbf{r}_\perp = \mathbf{r} - \dfrac{\mathbf{r} \cdot \boldsymbol{\omega}}{\omega}\boldsymbol{\omega}$ is the perpendicular part of $\mathbf{r}$. If the particle has mass m, then the kinetic energy is

$$T = \frac{1}{2}mr_\perp^2\omega^2$$

Definition (Moment of Inertia). We define $I = mr_\perp^2$ as the moment of inertia.

Definition (Moment of Inertia for System of Particles). For a system of particles rotating about an axis through the origin and with angular momentum $\boldsymbol{\omega} = \omega\mathbf{n}$, we define the moment of inertia as

$$I = \sum_{i=1}^{n} m_i|\mathbf{n} \times \mathbf{r}_i|^2$$

Then the kinetic energy of the system is $\frac{1}{2}I\omega^2$. Now consider the component of the angular momentum $\mathbf{L}$ parallel to $\mathbf{n}$.

$$\begin{aligned}
\mathbf{L}\cdot\mathbf{n} &= \omega\left(\sum_{i=1}^{n} m_i\mathbf{r}_i \times (\mathbf{n}\times\mathbf{r}_i)\right)\cdot\mathbf{n} \\
&= \omega\left(\sum_{i=1}^{n} m_i\mathbf{n}\cdot(\mathbf{r}_i \times (\mathbf{n}\times\mathbf{r}_i))\right) \\
&= \omega\sum_{i=1}^{n} m_i|\mathbf{n}\times\mathbf{r}_i|^2 \\
&= I\omega
\end{aligned}$$

7.2 Solid Bodies

For a solid body V with density $\rho(\mathbf{r})$, we have the total mass

$$M = \int_V \rho(\mathbf{r})\mathrm{d}V$$

and the center of mass is at

$$\mathbf{R} = \frac{1}{M}\int_V \mathbf{r}\rho(\mathbf{r})\mathrm{d}V$$

The moment of inertia about an axis $\mathbf{n}$ is given by

$$I = \int_V \rho(\mathbf{r})|\mathbf{r}_\perp|^2\mathrm{d}V = \int_V \rho(\mathbf{r})|\mathbf{n}\times\mathbf{r}|\mathrm{d}V$$

See Appendix for a List of Moments of Inertia.

Theorem (Perpendicular Axes Theorem). *For a 2D body we have that*

$$I_z = I_x + I_y$$

where I_z is the moment of inertia through an axis perpendicular to the lamina, and I_x, I_y are the moments of inertia through different, perpendicular axes in the plane of the lamina.

Theorem (Parallel Axes Theorem). *For a rigid body with mass M, moment of inertia I_c in an axis through the centre of mass, then the moment of inertia about a parallel axis distance d away from the centre of mass is*

$$I = I_c + Md^2$$

7.3 Sliding and Rolling

Definition (Slipping Velocity). For an object (cylinder or sphere with radius a) with horizontal velocity v and angular velocity ω (signs taken such that if the object is not slipping then $v = a\omega$), we define the slipping velocity $v_{\text{slip}} = v - a\omega$.

The point of contact is slipping relative to the surface if $v_{\text{slip}} \neq 0$. We have pure slipping motion if $\omega = 0$ and $v_{\text{slip}} \neq 0$. On the other hand, if $v = a\omega$ and $v_{\text{slip}} = 0$, we have pure rolling motion.

8 Special Relativity

8.1 Postulates

In special relativity, there are two postulates:

1. Laws of physics are the same in all inertial frames
2. Speed of light in a vacuum is the same in all inertial frames

Definition (Speed of light)**.** We define the speed of light $c = 299\,792\,458\,\mathrm{m\,s^{-1}}$.

Remark. In this course we will use units such that $c = 299792458$, but in other places it may be the case that the units are chosen such that $c = 1$. In those places, note that the factors of c that appear in this course will not appear.

8.2 Lorentz Transform

For now, let's restrict ourselves to 1 dimensional motion. Suppose the frame S' moves with velocity v in the x direction relative to S. We can consider the motion in the (x, t) plane versus the (x', t') plane. Without loss of generality, let $x = x' = 0$ at $t = t' = 0$.

From postulate 1, constant velocity paths in S will be constant velocity paths in S', so the transformation from one frame to another will be linear. O' moves with velocity v in S, so we have that

$$x' = \gamma_v(x - vt)$$

whereas O moves with velocity $-v$ in S', so we have that

$$x = \gamma_v(x' + vt')$$

Now consider a light ray passing through the origin at $t = t' = 0$. We have that $x = ct$, and by postulate 2, $x' = ct'$. Substituting in, we get that

$$x' = \gamma_v(c - v)t \qquad\qquad x = \gamma_v(c + v)t'$$

Combining these with $x = ct$ and $x' = ct'$, we find the Lorentz factor.

Definition (Lorentz factor)**.**

$$\gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Remark. Note that the Lorentz factor must only depend on speed, and not velocity. This is because there is no "preferred direction" for the Laws of Physics, from Postulate 1.

Proposition.

$$t' = \gamma_v\left(t - \frac{v}{c^2}x\right)$$

Definition (Lorentz Transform)**.** In $1 + 1$ dimensions, the Lorentz transform is

$$\begin{aligned} x' &= \gamma_v(x - vt) \\ t' &= \gamma_v(t - \frac{v}{c^2}x) \end{aligned}$$

The inverse transformation can be found by setting $v \mapsto -v$

8.3 Spacetime Diagrams

Definition (Spacetime diagram)**.** Consider one spatial dimension x, and time t in an inertial frame S. We can plot x on the horizontal axis and ct on the vertical axes.

Definition (Event)**.** A point $P = (x, ct)$ in spacetime is an event.

Definition (World Line)**.** A particle traces out a curve in (x, ct) space. This is known as the world line.

Proposition. *The gradient of the world line of a light ray is* ± 1.

Proposition. *The gradient of a world line of a particle must be always greater than 1.*

The spacetime axes of a different frame can also be plotted, and they will be at an angle θ to the existing axes (and both will be in Quadrant 1).

8.4 Simultaneity and Causality

Definition (Simultaneous)**.** Two events P_1 and P_2 are simultaneous in a frame S if they occur at the same time in S, that is $t_1 = t_2$.

Proposition. *From a frame S' moving with speed $v \neq 0$ relative to S, P_1 and P_2 will not be simultaneous.*

Definition (Light Cone)**.** The light cones of an event P are lines of gradient ± 1 through P.

Definition (Past Light Cone, Future Light Cone)**.** The parts of the light cone of P which have time $t > t_P$ are known as the future light cone. The parts with $t < t_P$ are known as the past light cone.

Proposition. *If Q is in the past light cone of P, then it will occur before P in all frames. If R is in the future light cone of P, it will occur after P in all frames.*

Proof. Lines of simultaneity have |gradient| less than 1 as they cannot travel faster than light. □

Proposition. *If T is not in either light cone of P, then the ordering of P and T will depend on the observer.*

8.5 Geometry of Spacetime

Definition (Invariant Interval)**.** For two events P, Q with spacetime coordinates (ct_1, x_1), (ct_2, x_2), we define the time separation $\Delta t = t_1 - t_2$ and the space separation $\Delta x = x_1 - x_2$. The invariant interval of P and Q is defined to be

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2$$

In 3 spatial dimensions, we define the invariant interval to be

$$\Delta s^2 = c^2 \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)$$

Proposition. $(\Delta s)^2$ *is invariant under the Lorentz transform. That is, all observers in inertial frames agree on the value.*

Definition (Infitessimal Invariant Interval)**.**

$$\mathrm{d}S^2 = c^2 \mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

Definition (Minkowski Spacetime)**.** Minkowski space time is the combination of 1 time dimension with 3 spatial dimensions into a 4 dimensional manifold. This is often denoted as $\mathbb{R}^{1+3}$.

The inner product in a Minkowski space is given by

$$(t_1, \mathbf{x}_1) \cdot (t_2, \mathbf{x}_2) = c^2 t_1 t_2 - (\mathbf{x}_1 \cdot \mathbf{x}_2)$$

Definition (Time-like separated)**.** Two events are time-like separated if $\Delta s^2 > 0$. There exists a frame such that they occur at the space spatial coordinates but at different times.

Definition (Space-like separated)**.** Two events are space-like separated if $\Delta s^2 < 0$. There exists a frame such that they occur at the same time, but at different spatial coordinates.

Definition (Light-like separated)**.** Two events are light-like separated if $\Delta s^2 = 0$. Each event is in the light cone of the other.

Definition (4-vector)**.** The coordinates of an event in spacetime can be represented as a 4-vector. Where

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

The components of X are $X^0 = ct$, $X^1 = x$, $X^2 = y$, $X^3 = z$

Definition (Inner Product)**.** The inner product in Minkowski space can be represented using a matrix $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ as

$$X \cdot X = X^T \eta X$$

In summation convention, it is $X \cdot X = X^\mu \eta_{\mu\nu} X^\nu$

Definition (Time, Space, Light-like)**.** A 4-vector X is time-like if $X \cdot X > 0$. It is space-like if $X \cdot X < 0$ and it is light-like if $X \cdot X = 0$.

Definition (Lorentz Transformations)**.** The Lorentz transform maps coordinates X in S to coordinates X' in S'. As the transformation is linear (by postulate 1), we can represent this as a 4×4 matrix. In particular, it is the matrices preserving the inner product.

$$X \cdot X = (\Lambda X) \cdot (\Lambda X)$$

or equivalently, $\Lambda^T \eta \Lambda = \eta$.

Definition (Lorentz Group)**.** The set of Λ such that $\Lambda^T \eta \Lambda = \eta$ form as group known as the Lorentz group.

Definition (Proper Lorentz Group)**.** The subgroup of the Lorentz group of matrices with determinant 1 is the proper Lorentz group.

Definition (Restricted Lorentz Group)**.** The subgroup of the Lorentz group that preserves orientation of space and time is known as the Restricted Lorentz Group.

8.6 Rapidity

We shall restrict to the $1 + 1$ dimensional case now. This would correspond to the 2 submatrix of a Lorentz transform.

Let $\Lambda[\beta] = \begin{pmatrix} \gamma_\beta & -\gamma_\beta \beta \\ -\gamma_\beta \beta & \gamma \end{pmatrix}$. This is a boost in the x direction. By calculation, we find that

$$\Lambda[\beta_1]\Lambda[\beta_2] = \Lambda\left[\frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}\right]$$

Let ϕ be such that $\beta = \tanh\phi$. Then the expression above can be written as

$$\Lambda(\phi_1)\Lambda(\phi_2) = \Lambda(\phi_1 + \phi_2)$$

8.7 Relativistic Kinematics

Definition (Proper Time)**.** For a particle at rest in S' with $\mathbf{x}' = 0$, the invariant interval between points on its world line is $\Delta s^2 = c^2(\Delta t')^2$. Define the proper time τ such that $\Delta\tau = \frac{1}{c}\Delta s$.

Proposition. *The proper time is the time expereience in the rest frame of the particle.*

Proposition. *$\Delta\tau$ is invariant under the Lorentz transform.*

Proposition. *For a particle moving with velocity $\mathbf{u}(t)$, we have that*

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{\mathbf{u}}$$

Proof.

$$\mathrm{d}\tau = \frac{\mathrm{d}s}{c} = \frac{1}{c}\sqrt{c^2\mathrm{d}t^2 - |\mathrm{d}\mathbf{x}|^2} = \frac{1}{c}\sqrt{c^2\mathrm{d}t^2 - |\mathbf{u}|^2\mathrm{d}t^2} = \left(1 - \frac{|\mathbf{u}|^2}{c^2}\right)^{1/2}\mathrm{d}t = \frac{\mathrm{d}t}{\gamma_{\mathbf{u}}}$$

□

Definition (4-velocity)**.** For a particle with position 4-vector $X(\tau) = \begin{pmatrix} ct(\tau) \\ \mathbf{x}(\tau) \end{pmatrix}$, the 4-velocity is defined to be

$$U = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \begin{pmatrix} c\dfrac{\mathrm{d}t}{\mathrm{d}\tau} \\ \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \end{pmatrix} = \gamma_{\mathbf{u}}\begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

where $\mathbf{u} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$.

Proposition. *If the 4-position vectors of a particle in S and S' are X and X' respectively, then they are linked by $X' = \Lambda X$. Similarly, $U' = \Lambda U$.*

Proposition. *For any particle, we have that $U \cdot U = c^2$.*

Proof. Transform into the rest frame of the particle. There we have that $U = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}$ and $U \cdot U = c^2$. □

8.8 Relativistic Physics

Definition (Rest Mass)**.** The rest mass m of a particle is the mass of the particle measured in its rest frame.

Definition (4-momentum)**.** The 4-momentum of a particle with mass m and with 4-velocity U is

$$P = mU = m\gamma_{\mathbf{u}}\begin{pmatrix} c \\ \mathbf{u} \end{pmatrix}$$

Definition (Relativistic 3-momentum)**.** The spatial components of the 4-momentum is the relativistic 3 momentum

$$\mathbf{p} = m\gamma_{\mathbf{u}}\mathbf{u}$$

Definition (Relativistic Mass, Apparent Mass)**.** The relativistic mass, or apparent mass of a particle is given by $m\gamma_{\mathbf{u}}$.

Definition (Energy)**.** Define $E = \gamma_{\mathbf{u}}mc^2$. Then $P^0 = E/c$.

Proposition (Mass-Energy Equivalence)**.** *For a stationary particle with rest mass m, we have that $E = mc^2$.*

Proof. For a stationary particle $\gamma_{\mathbf{u}} = 1$. □

Definition (Relativistic Kinetic Energy). The Relativistic Kinetic Energy of a particle is

$$E = (\gamma_{\mathbf{u}} - 1)mc^2 = \frac{1}{2}m\mathbf{u}^2 + \dots$$

Proposition (Energy-Momentum Relation). *For any particle, we have that*

$$E^2 = \mathbf{p}^2c^2 + m^2c^4$$

Proof. P is Lorentz invariant. Transform into the rest frame of the particle, result follows. □

Definition (Massless Particle). A massless particle has zero rest mass, for example a photon.

Remark. Massless particles travel on light-like trajectories, and as a result they do not have a proper time, and we cannot tranform into their rest frame. However they can have a non-zero momentum and energy.

Proposition. *For a massless particle, the 4-momentum is of the form*

$$P = \frac{E}{c}\begin{pmatrix} 1 \\ \mathbf{n} \end{pmatrix}$$

where $\mathbf{n}$ *is a unit vector in the direction of travel.*

Proof. $E = |\mathbf{p}|c$, result follows. □

8.9 Newton's Second Law

Definition (4-Acceleration). For a particle, we define its 4-acceleration by

$$A = \frac{\mathrm{d}U}{\mathrm{d}\tau}$$

Definition (4-Force). For a force $\mathbf{F}$, we define the corresponding 4-force

$$F = \gamma_{\mathbf{u}}\begin{pmatrix} \mathbf{F} \cdot \mathbf{u}/c \\ \mathbf{F} \end{pmatrix}$$

Definition (Newton's Second Law).

$$F = mA = \frac{\mathrm{d}P}{\mathrm{d}\tau}$$

8.10 Applications to Particle Physics

Problems in Particle Physics can be solved using the conservation of 4-momentum. Often it is useful to consider the conservation of 4-momentum in the rest frame of the centre of momentum. That is, the frame where $\sum P = 0$.

Example (Particle Decay). Suppose a particle with mass m_1 with 4-momentum P_1 decays into two particles, with mass m_2,m_3 and 4-momenta P_2, P_3. Then in the centre of momentum frame, equating the 0 components of the 4-momenta, we get that

$$\frac{E_1}{c} = m_1c = \frac{E_2}{c} + \frac{E_3}{c} = \sqrt{\mathbf{p}_2^2 + m_2^2c^2} + \sqrt{\mathbf{p}_3^2 + m_3^2c^2} \geq m_2c + m_3c$$

as a result, decay is only possible if the rest mass of the resulting particles is less than the current particle.

A List of Moments of Inertia

All objects are taken to have mass M, and are uniform unless otherwise stated. The parallel axes and perpendicular axes theorem can be used to calculate the Moments of Inertia of these objects about other axes.

Body	Axis	Moment of Inertia
Ring with radius a	Through centre, perpendicular to ring	Ma^2
Rod with length l	Through one end, perpendicular to rod	$\frac{1}{3}Ml^2$
Disc with radius a	Through centre, perpendicular to disc	$\frac{1}{2}Ma^2$
Disc with radius a	Through centre, in the plane of the disc	$\frac{1}{4}Ma^2$
Sphere with radius a	Through centre	$\frac{2}{5}Ma^2$