Differential Equations - Matrix Methods

Shing Tak Lam*

April 22, 2021

This is an extra note on Matrix Methods in DEs, we focus on 2D case here, but the results will generalise.

1 Motivation and Definitions

Problem. Consider the following system of differential equations.

$$\dot{x} = ax + by + f_1(t)$$
$$\dot{y} = cx + dy + f_2(t)$$

where $x = x(t)$, $y = y(t)$ and f_1, f_2 are forcing terms. We would like to be able to find the general solution to this system.

One way to solve this would be to differentiate one equation, and substitute into the other. However this involves a large amount of algebraic manipulation.

Instead, we can represent this system using vectors and matrices.

Definition (Matrix Representation)**.** If we let $\mathsf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathsf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, then the system above can be represented as

$$\dot{\mathsf{X}} = M\mathsf{X} + \mathsf{F}$$

2 Homogeneous Case

First, let's consider the homogeneous case where $\dot{\mathsf{X}} = M\mathsf{X}$. Much like how we defined the characteristic equation of a linear ODE by looking for solutions of the form $e^{\lambda t}$, we will try to find solutions of the form $\mathsf{X} = \mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ is a constant vector. Differentiating and substituting into the original equation, we get that

$$\lambda \mathbf{v} e^{\lambda t} = M\mathbf{v}e^{\lambda t}$$

Dividing through by $e^{\lambda t}$, we get that $M\mathbf{v} = \lambda\mathbf{v}$. This is precisely the definition of an eigenvector/eigenvalue. From this, if we let λ_1, λ_2 be the eigenvalues of M, and $\mathbf{v}_1, \mathbf{v}_2$ be the corresponding eigenvectors, then have two solutions to the system.

$$\mathbf{x}_1 = \mathbf{v}_1 e^{\lambda_1 t} \qquad\qquad \mathbf{x}_2 = \mathbf{v}_2 e^{\lambda_2 t}$$

However, depending on whether the matrix M is real or complex, and whether we want to look for real/complex solutions, there are a few things that we need to be careful for.

*stl45@cam.ac.uk

2.1 Real, Distinct Eigenvalues

If we are looking for real solutions, and the two eigenvalues are real and distinct, then the two resulting solutions will be linearly independent. As a result, we can use the principal of superposition and write the general solution as a linear combination of the two solutions.

$$\mathbf{X} = A\mathbf{x}_1 + B\mathbf{x}_2 = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}$$

2.2 Repeated Eigenvalue

If instead we have that $\lambda_1 = \lambda_2$, then the two solutions from before may not be linearly independent. For this, $\mathbf{x}_1 = \mathbf{v}_1 e^{\lambda_1 t}$ will still be a solution, but we will need to find a second solution.

We will need to look at the eigenspace E_{λ_1}. If we have that $\dim E_{\lambda_1} = 2$, then we have a second, linearly independent eigenvector $\mathbf{v}_2$, and the second solution is

$$\mathbf{x}_2 = \mathbf{v}_2 e^{\lambda_2 t}$$

Writing $\mathbf{X} = A\mathbf{x}_1 + B\mathbf{x}_2$ as before gives us the general solution.

On the other hand, if $\dim E_{\lambda_1} = 1$, then we can't find such a $\mathbf{v}_2$. Let $\mathbf{v}_2$ be a vector satisfying

$$(M - \lambda_1 I)\mathbf{v}_2 = \mathbf{v}_1$$

Then the second solution is

$$\mathbf{x}_2 = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{v}_2)$$

2.3 Complex Eigenvalues

Since the matrix M has real coefficients, if we have complex eigenvalues λ_1 and λ_2, then we must have that $\lambda_1 = \overline{\lambda_2}$. In this case, the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ may also be complex. The following lemma will be useful.

Lemma. *Suppose* $\mathbf{x}$ *is a solution, where* $\mathbf{x} = \mathbf{x}_1 + i\mathbf{x}_2$*. Then* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are solutions.*

Proof.

$$\dot{\mathbf{x}}_1 + i\dot{\mathbf{x}}_2 = \dot{\mathbf{x}} = M\mathbf{x} = M(\mathbf{x}_1 + i\mathbf{x}_2) = M\mathbf{x}_1 + iM\mathbf{x}_2$$

Equating the real and complex parts we get the result required. □

Using this, we have a solution $\mathbf{x} = \mathbf{v}_1 e^{\lambda_1 t}$, and by splitting into the real and complex parts, we can get two linearly independent solutions.

2.4 Complex Eigenvalues - Alternative Method

If we first consider the general system with complex coefficients, and then consider restrictions such that the result is real, we have

$$\mathbf{X} = c\mathbf{x}_1 + d\mathbf{x}_2 = c\mathbf{v}_1 e^{\lambda_1 t} + d\mathbf{v}_2 e^{\lambda_2 t}$$

By some elementary algebra, we find that we must in fact have $\mathbf{v}_1 = \overline{\mathbf{v}_2}$. Letting $\mathbf{v}_1 = \mathbf{u}_1 + i\mathbf{u}_2$, $\lambda_1 = p + qi$ and substituting, we get that

$$\mathbf{X} = e^{pt}((\mathbf{u}_1 \cos qt - \mathbf{u}_2 \sin qt)(c + d) + i(\mathbf{u}_1 \sin qt + \mathbf{u}_2 \cos qt)(c - d))$$

As we want $\mathbf{X}$ to be real, one way is by saying that $c + d$ is real, and $c - d$ is pure imaginary. Without loss of generality, we may assume that $d = \overline{c}$. With this, and noting that $\overline{e^{\lambda_2} t} = e^{\lambda_1 t}$, we get that

$$\mathbf{X} = c\mathbf{v}_1 e^{\lambda_1 t} + \overline{c\mathbf{v}_1 e^{\lambda_1 t}} = \mathrm{Re}(\alpha \mathbf{v}_1 e^{\lambda_1 t})$$

where $\alpha = 2c$, $\alpha \in \mathbb{C}$. This is another method to derive the above result.

3 Particular Integrals

With a inhomogeneous problem, we have a forcing term, and we will need to find a corresponding particular integral. For that, we can use similar methods to how we found particular integrals for a linear ODE.