Differential Equations - Method of Frobenius

Shing Tak Lam*

April 23, 2021

This is a document reviewing the method of Frobenius.

Consider the differential equation[1] which we want to find a series solution for.

$$y'' + py + q = 0$$

1 Ordinary Point

Recall that $x = 0$ is an ordinary point if p and q have convergent Taylor series expansions about $x = 0$. In this case, we have two linearly independent solutions of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

2 Regular Singular Point

If either diverges, then $x = 0$ is a singular point. If $xp(x)$ and $x^2 q(x)$ have convergent Taylor series about $x = 0$, then $x = 0$ is a regular singular point. In this case, we have that

$$p(x) = \frac{1}{x}\sum_{n=0}^{\infty} P_n x^n \qquad q(x) = \frac{1}{x^2}\sum_{n=0}^{\infty} Q_n x^n$$

From Fuch's Theorem, in this case we have a solution of the form

$$y = x^{\sigma}\sum_{n=0}^{\infty} a_n x^n$$

for some $\sigma \in \mathbb{R}$ and $a_0 \neq 0$. To find σ, we substitute this (and the expressions for p and q) into the differential equation and look at the lowest order term. Doing this, we get that

$$a_0\sigma(\sigma - 1)x^{\sigma-2} + a_0 P_0 \sigma x^{\sigma-2} + a_0 Q_0 x^{\sigma-2} = 0$$

As we know that $a_0 \neq 0$, and without loss of generality $x \neq 0$, we must have that

$$\sigma(\sigma - 1) + P_0\sigma + Q_0 = 0$$

This is known as the indicial equation. We can find P_0 and Q_0 quickly from

$$P_0 = \lim_{x\to 0} xp(x) \qquad Q_0 = \lim_{x\to 0} x^2 q(x)$$

*stl45@cam.ac.uk

[1]This is slightly different to the form given in lectures.

Let σ_1, σ_2 be the solutions to the indicial equation. If $\sigma_1 - \sigma_2$ is not an integer, then we have two linearly independent solutions. If $\sigma_1 - \sigma_2$ is an integer, then without loss of generality, $\sigma_1 \geq \sigma_2$ and we have a solution of the form

$$y_1 = x^{\sigma_1} \sum_{n=1}^{\infty} a_n x^n$$

The other solution is of the form

$$y_2 = x^{\sigma_2} \sum_{n=1}^{\infty} b_n x^n + c y_1 \log(x)$$

where c is a real number. If $\sigma_1 = \sigma_2$, then we must have that $c \neq 0$.

3 Finding a Series Solution

To find a series solution, we substitute the series into the differential equation, and by equating the coefficients we get a recurrence relation. By solving this recurrence relation, we can find the coefficients in terms of a_0, which we know is non-zero.

One useful trick is to extend the definition of a_n to $n \in \mathbb{Z}$, where we define $a_n = 0$ for $n < 0$. With this, we can write the series as

$$y = \sum_{n \in \mathbb{Z}} a_n x^n$$

With this, when we differentiate the series, we don't have to worry about shifting the indices. Alternativelty, we can keep the series from starting at $n = 0$ when differentiating.