Vector Calculus

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This document is intended for revision purposes. As a result, it does not contain any exposition. This is based off lectures given by Dr Anthony Ashton in Lent 2021, but the order of content, as well as some of the proofs have been modified after the fact, primarily to provide simpler proofs for theorems. Note that this also contains theorems from examples sheets, as some are useful elsewhere.

Throughout this course a column vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ will represent a vector with respect to the usual Cartesian Axes.

Vector Calculus is on *Paper 3*.

Contents

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1 Curves

Definition (Curve). A curve C in $\mathbb{R}^3$ is the image of a continuous function $\mathbf{x} : [a, b] \to \mathbb{R}^3$, ie

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

Definition (Differentiable Curve). We say a curve C is differentiable if x_1, x_2 and x_3 are differentiable for $t \in [a, b]$

Definition (Regular Curve). We say a curve C is regular if $|\mathbf{x}'(t)| \neq 0$ for all $t \in [a, b]$

Definition (Arc Length Element). Define for a differentiable curve the arc lengthe element

$$\mathrm{d}s = \left|\mathbf{x}'(t)\right| \mathrm{d}t$$

Definition (Length of Curve). Define the length of a curve C as

$$l(C) = \int_a^b \left|\mathbf{x}'(t)\right| \mathrm{d}t = \int_C \mathrm{d}s$$

Definition (Arc Length Function). Define for a differentiable curve the arc length

$$s(t) = \int_a^t \left|\mathbf{x}'(\tau)\right| \mathrm{d}\tau$$

Proposition. *$s(a) = 0$ and $s(b) = l(C)$*

Proposition. *For regular curves, $\frac{\mathrm{d}s}{\mathrm{d}t} > 0$ and*

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}} = \frac{1}{|\mathbf{x}'(t)|} = \frac{1}{\mathbf{x}'(t(s))}$$

Definition (Arc Length Parametrisation). For a regular curve C, we can invert the relationship of s and t, and define $t = t(s)$. Then we can define $\mathbf{r}(s) = \mathbf{x}(t(s))$.

Proposition. $\mathbf{r}'(s) = \dfrac{\mathbf{x}'(t(s))}{|\mathbf{x}'(t(s))|}$

Definition ((Unit) Tangent vector).

$$\mathbf{t}(s) = \mathbf{r}'(s)$$

Definition (Curvature). Define the curvature of a curve at a point as

$$\kappa(s) = \left|\mathbf{r}''(s)\right| = \left|\mathbf{t}'(s)\right|$$

Proposition.

$$\mathbf{t} \cdot \mathbf{t}' = 0$$

Proof. Differentiate $\mathbf{t} \cdot \mathbf{t} = 1$. □

Definition (Principal Normal)**.** Define the principal normal $\mathbf{n}$ by $\mathbf{t}' = \kappa \mathbf{n}$.

Definition (Binormal)**.** Define the binormal $\mathbf{b} = \mathbf{t} \times \mathbf{n}$.

Proposition. $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ *form an orthonormal basis.*

Proposition. $\mathbf{b}' \ \ \mathbf{n}$.

Proof. As $\mathbf{b} \cdot \mathbf{b} = 1$, we have that $\mathbf{b} \cdot \mathbf{b}' = 0$. Furthermore, $\mathbf{t} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{b} = 0$. So

$$0 = (\mathbf{t} \cdot \mathbf{b})' = \mathbf{t}' \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' = \kappa \mathbf{n} \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' = \mathbf{t} \cdot \mathbf{b}'$$

So $\mathbf{t}$ is perpendicular to $\mathbf{b}'$. As $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ form an orthonormal basis, we must have that $\mathbf{b}' \ \ \mathbf{n}$. □

Definition (Torsion)**.** Define the torsion τ by $\mathbf{b}' = -\tau \mathbf{n}$.

Proposition. *The curvature $\kappa(s)$ and torsion $\tau(s)$ determine a curve up to translation/rotation.*

Definition (Radius of Curvature)**.** Define the radius of curvature $R = \dfrac{1}{\kappa(s)}$.

Proposition. *The radius of curvature is the radius of a circle which best fit the curve at a point.*

Proof. Expand the equation for a curve about $s = 0$. Let $\mathbf{t} = \mathbf{t}(0)$ etc.

$$\mathbf{r}(s) = \mathbf{r} + s\mathbf{t} + \frac{1}{2}s^2 \kappa \mathbf{n} + o(s^2)$$

The equation of a circle passing through $\mathbf{r}$ with radius R is

$$\mathbf{x}(\theta) = \mathbf{r} + R(1 - \cos\theta)\mathbf{n} + R\sin\theta\mathbf{t}$$

Expanding for θ small,

$$\mathbf{x}(\theta) = \mathbf{r} + R\theta\mathbf{t} + \frac{1}{2}R\theta^2\mathbf{n} + o(\theta^2)$$

Arc length is $s = R\theta$ on the circle, and we require the second derivatives to be equal, so $s^2\kappa = R\theta^2$ and $R = \dfrac{1}{\kappa}$. □

2 Differentials

Definition (Differential Form)**.** If we have coordinates $(u_1, \ldots, u_n)$ for $\mathbb{R}^n$, then $\mathrm{d}u_i$ are differential forms, and $\{\mathrm{d}u_i\}$ are linearly independent if the u_i are independent.

Definition (Differential)**.** If $f = f(u_1, \ldots, u_n) : \mathbb{R}^n \to \mathbb{R}$, we define

$$\mathrm{d}f = \frac{\partial f}{\partial u_i}\mathrm{d}u_i$$

Definition (Differential (Vectors))**.** If $x : \mathbb{R}^m \to \mathbb{R}^n$, then

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_i}\mathrm{d}u_i$$

Proposition (Multivariate Chain Rule). *Suppose $(u_1, \ldots, u_n)$ are a set of coordinates that depend on $(x_1, \ldots, x_n)$ and $F(u_1, \ldots, u_n) = f(x_1, \ldots, x_n)$, then*

$$\frac{\partial F}{\partial u_i} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial u_i}$$

Proof.

$$\frac{\partial F}{\partial u_i}\mathrm{d}u_i = \mathrm{d}F = \mathrm{d}f = \frac{\partial f}{\partial x_j}\mathrm{d}x_j = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial u_i}\mathrm{d}u_i$$

□

3 Coordinate Systems

Definition (Orthogonal Curvilinear Coordinates). We say that (u, v, w) are a set of orthogonal curvilinear coordinates (OCCs) if the vectors

$\mathbf{e}_u = \dfrac{\frac{\partial \mathbf{x}}{\partial u}}{\left|\frac{\partial \mathbf{x}}{\partial u}\right|}$, $\mathbf{e}_v = \dfrac{\frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial v}\right|}$ and $\mathbf{e}_w = \dfrac{\frac{\partial \mathbf{x}}{\partial w}}{\left|\frac{\partial \mathbf{x}}{\partial w}\right|}$, form a right handed orthonormal basis. That is, $\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$.

Definition (Scale Factor). Define the scale factor $h_u = \left|\frac{\partial \mathbf{x}}{\partial u}\right|$.

Definition (Line Element). Define the line element

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\mathrm{d}v + \frac{\partial \mathbf{x}}{\partial w}\mathrm{d}w = h_u\mathbf{e}_u\mathrm{d}u + h_v\mathbf{e}_v\mathrm{d}v + h_w\mathbf{e}_w\mathrm{d}w$$

3.1 Cylindrical Polars

Definition (Cylindrical Polars). Define (ρ, ϕ, z) for $0 \le \rho < \infty$, $0 \le \phi < 2\pi$, $-\infty < z < \infty$ by

$$\mathbf{x}(\rho, \phi, z) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ z \end{pmatrix}$$

Proposition. $\mathbf{e}_\rho = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}$, $\mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$ *and* $\mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Proposition. $h_\rho = 1$, $h_\phi = \rho$ *and* $h_z = 1$.

Proposition. $\mathrm{d}\mathbf{x} = \mathbf{e}_\rho\mathrm{d}\rho + \rho\mathbf{e}_\phi\mathrm{d}\phi + \mathbf{e}_z\mathrm{d}z$

Proposition. $\mathbf{x} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$.

3.2 Sphecical Polars

Definition (Spherical Polars). Define (r, θ, ϕ) for $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$ by

$$\mathbf{x}(r, \theta, \phi) = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$$

Proposition. $\mathbf{e}_r = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$, $\mathbf{e}_\theta = \begin{pmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{pmatrix}$ *and* $\mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$

Proposition. $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$.

Proposition. $\mathrm{d}\mathbf{x} = \mathbf{e}_r\mathrm{d}r + r\mathbf{e}_\theta\mathrm{d}\theta + r\sin\theta\mathbf{e}_\phi\mathrm{d}\phi$

Proposition. $\mathbf{x} = r\mathbf{e}_r$

4 Gradient

Definition (Gradient)**.** For $f : \mathbb{R}^3 \to \mathbb{R}$, define the gradient by

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \boldsymbol{\nabla} f(\mathbf{x}) \cdot \mathbf{h} + o(\mathbf{h})$$

as $|\mathbf{h}| \to 0$

Definition (Directional Derivative)**.** For $f : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^3$, the directional derivative of f in the direction of $\mathbf{v}$ is defined by

$$D_{\mathbf{v}} f = \frac{\partial f}{\partial \mathbf{v}} = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{v}) - f(\mathbf{x})}{t}$$

or equivalently,

$$f(\mathbf{x} + t\mathbf{v}) = f(\mathbf{x}) + D_{\mathbf{v}} f(\mathbf{x}) + o(t)$$

Proposition. *$D_{\mathbf{v}} f = \mathbf{v} \cdot \boldsymbol{\nabla} f$.*

Proof. Let $\mathbf{h} = t\mathbf{v}$ in the definition of gradient. □

Proposition. *$\boldsymbol{\nabla} f$ points in the direction of greatest increase of f.*

Proof. By Cauchy-Schwarz, $D_{\mathbf{v}} = \mathbf{v} \cdot \boldsymbol{\nabla} f$ is maximised when $\mathbf{v}$ $\boldsymbol{\nabla} f$. □

Proposition. *If $F(t) = f(\mathbf{x}(t))$, then*

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(f(\mathbf{x}(t))) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \boldsymbol{\nabla} f(\mathbf{x}(t))$$

Proof. Fix t and let $\delta\mathbf{x} = \mathbf{x}(t + \delta t) - \mathbf{x}(t)$. Then

$$\begin{aligned} F(t + \delta t) &= f(\mathbf{x}(t + \delta t)) \\ &= f(\mathbf{x}(t) + \delta\mathbf{x}) \\ &= f(\mathbf{x}(t)) + \delta\mathbf{x} \cdot \boldsymbol{\nabla} f + o(\delta\mathbf{x}) \\ &= f(\mathbf{x}(t)) + \mathbf{x}'(t) \cdot \boldsymbol{\nabla} f \delta t + o(\delta t) \end{aligned}$$

and by the definition of derivative this gives the required result. □

Proposition. *If a surface S is defined implicitly by $S = \{\mathbf{x} : f(\mathbf{x}) = 0\}$. Then $\boldsymbol{\nabla} f$ is normal to the surface.*

Proof. Choose any curve on S. Then $f(\mathbf{x}(t)) = 0$ identically. So $\boldsymbol{\nabla} f \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = 0$ and $\boldsymbol{\nabla} f$ is perpendicular to the tangent of the curve. □

4.1 Calculating the Gradient

Proposition. *In cartesian coordinates,*

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x_i} \mathbf{e}_i$$

Proof.

$$f(\mathbf{x} + \mathbf{h}) = f(x + h_1, y + h_2, z + h_3) = f(\mathbf{x}) + h_1 \frac{\partial f}{\partial x_1} + h_2 \frac{\partial f}{\partial x_2} + h_3 \frac{\partial f}{\partial x_3} + o(\mathbf{h}) = f(\mathbf{x}) + \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} \cdot \mathbf{h} + o(\mathbf{h})$$

□

Proposition. $\nabla f \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$

Proposition. *If (u, v, w) are OCCs, and $f = f(u, v, w)$, then*

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w$$

Proof.

$$\frac{\partial f}{\partial u}\mathrm{d}u + \frac{\partial f}{\partial v}\mathrm{d}v + \frac{\partial f}{\partial w}\mathrm{d}w = \mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{x} = [\nabla f]_u h_u \mathrm{d}u + [\nabla f]_v h_v \mathrm{d}v + [\nabla f]_w h_w \mathrm{d}w$$

Using linear independence we can compare coefficients and the result follows. □

Proposition. *In Cylindrical Polars,*

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z$$

Proposition. *In Spherical Polars,*

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

5 Line Integrals

Definition (Line Integral). For a piecewise smooth curve $C : [a, b] \to \mathbb{R}^3$, and a vector field $\mathbf{F}(\mathbf{x})$, the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\mathrm{d}t$$

Definition (Closed Curve). A curve $C : [a, b] \to \mathbb{R}^3$ is closed if $\mathbf{x}(a) = \mathbf{x}(b)$.

Definition (Circulation). If the curve C is closed, then we write

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

5.1 Exact Differentials and Conservative Fields

Definition (Exact Differential). We say that $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is exact if $\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$ for some scalar function $f : \mathbb{R}^3 \to \mathbb{R}$.

Definition (Conservative Fields). We say that $\mathbf{F}$ is conservative if $\mathbf{F} = \nabla f$ for some scalar function f.

Proposition. $\mathbf{F}$ *is conservative* $\iff$ $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ *is exact.*

Proposition. *If θ is an exact differential form, then $\oint_C \theta = 0$ for any closed curve C.*

Proof. If θ is exact, say $\theta = \nabla f \cdot \mathrm{d}\mathbf{x}$. Then

$$\oint_C \theta = \oint_C \nabla f \cdot \mathrm{d}\mathbf{x} = \int_a^b \nabla f(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}[f(\mathbf{x}(t))]\mathrm{d}t = f(\mathbf{x}(b)) - f(\mathbf{x}(a)) = 0$$

□

Proposition. *If F is conservative, then $\int F \cdot \mathrm{d}\mathbf{x}$ from $\mathbf{a}$ to $\mathbf{b}$ is independent of the path.*

Proof. If C_1 and C_2 are paths from $\mathbf{a}$ to $\mathbf{b}$, then $C_1 - C_2$ is a loop. □

Proposition. *Let (u_1, u_2, u_3) be a set of OCCs. Let $\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \theta_i \mathrm{d}u_i$. Then a necessary condition for θ to be exact is*

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}$$

for all i, j.

Proof. If θ is exact, then $\theta = \mathrm{d}f = \frac{\partial f}{\partial u_i}\mathrm{d}u_i$. Then

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial^2 f}{\partial u_j \partial u_i} = \frac{\partial \theta_j}{\partial u_i}$$

□

Definition (Simply Connected, 1-connected)**.** A set $\Omega \subseteq \mathbb{R}^3$ is simply connected, or 1-connected if every loop in Ω can be continuously shrunk to a point while staying in Ω.

Proposition. *The reverse implication is true if the domain of F is simply-connected (or 1-connected).*

6 Area Integrals

Definition (Area Integral)**.** Let $D \subseteq \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$, then the area integral is

$$\int_D f(\mathbf{x})\mathrm{d}A$$

Proposition. *In Cartesians,* $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ *and*

$$\int_D f(\mathbf{x})\mathrm{d}A = \int_Y \left(\int_{X(y)} f(x, y)\mathrm{d}x \right) \mathrm{d}y$$

where $Y = \{y : \exists x, (x, y) \in D\}$ and $X(y) = \{x : (x, y) \in D\}$.

Theorem (Fubini's Theorem)**.** *If $\int_D |f(x, y)|\mathrm{d}A$ is finite, then*

$$\int \left(\int f(x, y)\mathrm{d}x \right) \mathrm{d}y = \int \left(\int f(x, y)\mathrm{d}y \right) \mathrm{d}x = \int_D f(x, y)\mathrm{d}A$$

6.1 Change of Variables

Proposition. *Let $x(u, v)$ and $y(u, v)$ represent a smooth bijection with a smooth inverse, that maps the region D' in (u, v) to the region D in (x, y). Then*

$$\iint_D f(x, y)\mathrm{d}x\mathrm{d}y = \iint D' f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u\mathrm{d}v$$

where $\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \left(\frac{\partial \mathbf{x}}{\partial u} \;\middle|\; \frac{\partial \mathbf{x}}{\partial v} \right)$ *is the Jacobian J. So* $\mathrm{d}x\mathrm{d}y = |J|\mathrm{d}u\mathrm{d}v$.

7 Volume Integral

Definition (Volume Integral)**.** Let $V \subseteq \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$. The volume integral is

$$\int_V f(\mathbf{x})\mathrm{d}V$$

Proposition. *In Cartesians,* $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

7.1 Change of Variables

Proposition. *Let $x(u,v,w)$, $y(u,v,w)$ and $z(u,v,w)$ represent a smooth bijection with a smooth inverse, that maps the volume V' in (u,v,w) to the volume V in (x,y,z). Then*

$$\iiint_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{V'} f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

where $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \det\left(\begin{array}{c|c|c} \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial v} & \frac{\partial \mathbf{x}}{\partial w}\end{array}\right)$ *is the Jacobian J. Thus $\mathrm{d}x\mathrm{d}y\mathrm{d}z = |J|\mathrm{d}u\mathrm{d}v\mathrm{d}w$.*

8 Surface Integral

Recall that a surface can be defined as $S = \{\mathbf{x} : f(\mathbf{x}) = 0\}$, and the normal of the surface is given by $\boldsymbol{\nabla} f$.

Definition (Regular Surface)**.** A surface is regular if $\boldsymbol{\nabla} f \neq 0$ for all $\mathbf{x} \in S$.

Note that in this course, the boundary of a surface S, denoted by ∂S, is empty or piecewise smooth.

Definition (Parametrised Surface)**.** We can also define a surface by a parametrisation, ie

$$S = \{\mathbf{x}(u,v) : (u,v) \in D\}$$

where $D \subseteq \mathbb{R}^2$.

Definition (Regular Parametrisation)**.** A parametrisation (u,v) is regular if for all $x \in S$,

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq \mathbf{0}$$

Definition (Normal)**.** For a regular surface, we can define the normal

$$\mathbf{n} = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|}$$

Definition (Orientation of Boundary)**.** When traversing the boundary, we require that the normal to the surface is to the **left** when viewed from outside the surface.

Definition (Scalar Area Element)**.** For a surface S, we define the scalar area element

$$\mathrm{d}S = \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|\mathrm{d}u\mathrm{d}v$$

Definition (Vector Area Element)**.** For a surface S, we define the vector area element

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\mathrm{d}u\mathrm{d}v = \mathbf{n}\mathrm{d}S$$

Definition (Surface Integral for Scalar Functions)**.** For a surface S and $f : \mathbb{R}^3 \to \mathbb{R}$, we define the surface integral

$$\int_S f\mathrm{d}S = \iint_D f(\mathbf{x}(u,v))\left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|\mathrm{d}u\mathrm{d}v$$

Definition (Surface Integral for Vector Functions)**.** For a surface S and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we define the surface integral

$$\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \int_S \mathbf{F}\cdot\mathbf{n}\mathrm{d}S$$

9 Divergence, Curl and Laplacian

Definition ($\boldsymbol{\nabla}$ in Cartesians)**.** As an operator, we can define $\boldsymbol{\nabla} = \mathbf{e}_i \frac{\partial}{\partial x_i}$

Definition (Divergence)**.** For $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we define the divergence of $\mathbf{F}$ as

$$\mathrm{div}(F) = \boldsymbol{\nabla} \cdot \mathbf{F}$$

Definition (Curl)**.** For $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we define the curl of $\mathbf{F}$ as

$$\mathrm{curl}(F) = \boldsymbol{\nabla} \times \mathbf{F}$$

Definition (Laplacian)**.** For $f : \mathbb{R}^3 \to \mathbb{R}$, we define the Laplacian of f to be

$$\nabla^2 f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f$$

Proposition. *In Cartesians,* $\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$.

Proof.

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \left(\mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot \mathbf{F} = \mathbf{e}_i \cdot \frac{\partial}{\partial x_i} \left(F_j \mathbf{e}_j \right) = (\mathbf{e}_i \cdot \mathbf{e}_j) \frac{\partial F_j}{\partial x_i} = \delta_{ij} \frac{\partial F_j}{\partial x_i} = \frac{\partial F_i}{\partial x_i}$$

□

Proposition. *In Cartesians,* $\boldsymbol{\nabla} \times \mathbf{F} = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} \mathbf{e}_i$

Proof.

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\mathbf{e}_j \frac{\partial}{\partial x_j} \right) \times (F_k \mathbf{e}_k) = (\mathbf{e}_j \times \mathbf{e}_k) \frac{\partial F_k}{\partial x_j} = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} \mathbf{e}_i$$

□

Proposition. *In Cartesians,* $\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i}$

Proposition. *For* $f, g : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{F}, \mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$, *we have that*

(i) $\boldsymbol{\nabla}(fg) = (\boldsymbol{\nabla} f) g + f \boldsymbol{\nabla} g$

(ii) $\boldsymbol{\nabla} \cdot (f\mathbf{F}) = (\boldsymbol{\nabla} f) \cdot \mathbf{F} + f \boldsymbol{\nabla} \cdot \mathbf{F}$

(iii) $\boldsymbol{\nabla} \times (f\mathbf{F}) = (\boldsymbol{\nabla} f) \times \mathbf{F} + f \boldsymbol{\nabla} \times \mathbf{F}$

(iv) $\boldsymbol{\nabla}(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\boldsymbol{\nabla} \times \mathbf{G}) + \mathbf{G} \times (\boldsymbol{\nabla} \times F) + (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G} + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F}$

(v) $\boldsymbol{\nabla} \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\boldsymbol{\nabla} \cdot \mathbf{G}) - \mathbf{G}(\boldsymbol{\nabla} \cdot \mathbf{F}) + (\mathbf{G} \cdot \boldsymbol{\nabla})\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\nabla})\mathbf{G}$

(vi) $\boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}) = (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G})$

Proof. Use Cartesians and Suffix Notation. □

Proposition. *For general OCCs,*

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \left(\mathbf{e}_u \frac{1}{h_u} \frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{h_v} \frac{\partial}{\partial v} + \mathbf{e}_w \frac{1}{h_w} \frac{\partial}{\partial w} \right) \cdot (F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{e}_w)$$

Proposition. *For general OCCs,*

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\mathbf{e}_u \frac{1}{h_u} \frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{h_v} \frac{\partial}{\partial v} + \mathbf{e}_w \frac{1}{h_w} \frac{\partial}{\partial w} \right) \times (F_u \mathbf{e}_u + F_v \mathbf{e}_v + F_w \mathbf{e}_w)$$

Definition (Laplacian of Vector Field)**.** For a vector field $\mathbf{F}$, define

$$\nabla^2 \mathbf{F} = (\nabla^2 F_i) \mathbf{e}_i = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{F})$$

9.1 Second Derivatives

Proposition. *For scalar field f and vector field F, we have that $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\mathbf{f} = \mathbf{0}$ and $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathsf{F}) = 0$.*

Proof. Use Cartesians and Suffix Notation. □

Definition (Irrotational)**.** A vector field F is irrotational if $\boldsymbol{\nabla} \times \mathsf{F} = \mathbf{0}$.

Proposition. *Any conservative vector field is irrotational. The reverse implication is true if the domain is 1-connected.*

Definition (Vector Potential)**.** A is a vector potential for F if $\mathsf{F} = \boldsymbol{\nabla} \times \mathsf{A}$.

Definition (Solenoidal)**.** A vector field F is solenoidal if $\boldsymbol{\nabla} \cdot \mathsf{F} = 0$.

Proposition. *If there exists a vector potential for F, then F is solenoidal. The reverse implication is true if the domain is 2-connected. That is, every sphere can be shrunk to a point.*

10 Integral Theorems

10.1 Green's Theorem

Theorem (Green's Theorem)**.** *If $P = P(x, y)$ and $Q = Q(x, y)$ are continuously differentiable functions on A, and ∂A is piecewise continuous, then*

$$\oint_{\partial A} P\mathrm{d}x + Q\mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y$$

10.2 Stoke's Theorem

Theorem (Stoke's Theorem)**.** *If $\mathsf{F} = \mathsf{F}(\mathsf{x})$ is a continuously differentiable vector field, S is an orientable, piecewise regular surface with ∂S piecewise smooth, then*

$$\int_S (\boldsymbol{\nabla} \times F) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathsf{F} \cdot \mathrm{d}\mathsf{x}$$

Remark. Orientable means there is a consistent choice of normal. eg. not a Möbius strip.

Proposition. *If S is an orientable, piecewise regular, closed surface, then*

$$\int_S (\boldsymbol{\nabla} \times \mathsf{F}) \cdot \mathrm{d}\mathbf{S} = 0$$

Proposition. *If F is continuously differentiable and for every loop C, $\oint_C \mathsf{F} \cdot \mathrm{d}\mathsf{x} = 0$, then $\boldsymbol{\nabla} \times \mathsf{F} = \mathbf{0}$.*

Proof. Suppose not. Then there exists a unit vector $\mathbf{k}$ such that $\mathbf{k} \cdot \boldsymbol{\nabla} \times \mathsf{F}(\mathsf{x}_0) > 0$ for some x_0. Let $\varepsilon = \mathbf{k} \cdot \boldsymbol{\nabla} \times \mathsf{F}(\mathsf{x}_0) > 0$. By continuity, there exists $\delta > 0$ such that for $|\mathsf{x} - \mathsf{x}_0| < \delta$, $\mathbf{k} \cdot \boldsymbol{\nabla} \times \mathsf{F}(\mathsf{x}) > \frac{\varepsilon}{2}$.

Now take a loop in the ball $\{\mathsf{x} : |\mathsf{x} - \mathsf{x}_0| < \delta\}$ that lies in the plane with normal $\mathbf{k}$. Then

$$\oint_{\partial S} \mathsf{F} \cdot \mathrm{d}\mathsf{x} = \int_S (\boldsymbol{\nabla} \times \mathsf{F}) \cdot \mathrm{d}\mathbf{S} = \int_S (\boldsymbol{\nabla} \times \mathsf{F}) \cdot \mathbf{k}\mathrm{d}S > \frac{\varepsilon}{2} \int_S \mathrm{d}S > 0$$

Contradiction. □

10.3 Divergence Theorem

Theorem (Divergence Theorem, Gauss' Theorem). *If* $\mathbf{F}$ *is a continuously differentiable vector field, C is a volume with piecewise regular boundary* ∂V*, then*

$$\int_V \nabla \cdot \mathbf{F} \mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

where the normal is chosen to point ***out*** *of* V*.*

Proposition. *If* $\mathbf{F}$ *is continuously differentiable and for every closed surface* S *we have* $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0$*, then* $\nabla \cdot \mathbf{F} = 0$.

Proof. Suppose not. Then for some $\mathbf{x}_0$, $\nabla \cdot \mathbf{F}(\mathbf{x}_0) = \varepsilon > 0$. By continuity, we have a ball where $\nabla \cdot \mathbf{F} > \frac{\varepsilon}{2}$ inside the ball. Choose any volume V inside the ball. Then

$$0 = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \mathrm{d}V > \frac{\varepsilon}{2} \int V \mathrm{d}V > 0$$

Contradiction. □

11 Maxwell's Equations

The following quantities will be used in this section.

- $\mathbf{B}(\mathbf{x}, t)$ - Magnetic Field
- $\mathbf{E}(\mathbf{x}, t)$ - Electric Field
- $\rho(\mathbf{x}, t)$ - Charge Density (per unit volume)
- $\mathbf{J}(\mathbf{x}, t)$ - Current Density (per unit area)
- ε_0 - Permittivity of Free Space
- μ_0 - Permeability of Free Space ($= \frac{1}{\varepsilon_0 c^2}$)

Definition (Maxwell's Equations).

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{3}$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \tag{4}$$

Proposition (Conservation of Charge).

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Proof. Taking the divergence of (4), we get that

$$0 = \mu_0 \varepsilon_0 \nabla \cdot \left(\frac{\partial \mathbf{E}}{\partial t} \right) + \mu_0 \nabla \cdot \mathbf{J} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) + \mu_0 \nabla \cdot \mathbf{J}$$

Using (1) we get the results required. □

11.1 Integral Formulations

Proposition (Gauss' Law). *For a volume V,*

$$\int_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \mathrm{d}V = \frac{Q}{\varepsilon_0}$$

Proof. Integrate (1) over V and use the Divergence Theorem. □

Proposition (Gauss' Law for Magnetism). *For a volume V,*

$$\int_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

Proof. Integrate (2) over V and use the Divergence Theorem. □

Corollary. *There are no magnetic monopoles.*

Proposition (Maxwell-Faraday Equation, Faraday's Law of Induction). *For a surface S,*

$$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{x} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$

Proof. Integrate (3) and use Stoke's Theorem. □

Proposition (Ampère's Circuital Law). *For a surface S,*

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{x} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$$

Proof. Integrate (4) and use Stoke's Theorem. □

11.2 Electromagentic Waves

Proposition (Maxwell's Equations in Empty Space).

$$\nabla \cdot \mathbf{E} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{3}$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{0} \tag{4}$$

Proposition (Electric Waves in Empty Space).

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mathbf{0}$$

Proof.

$$\nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = c^2 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

□

Proposition (Magnetic Waves in Empty Space).

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = \mathbf{0}$$

Proof.

$$\nabla^2 \mathbf{B} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \left(c^2 \frac{\partial \mathbf{E}}{\partial t}\right) = -c^2 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -c^2 \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{B}}{\partial t}\right) = c^2 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

□

Proposition. *Electromagnetic waves travel at speed c in empty space.*

Proof. See above and the Wave Equation. □

11.3 Electrostatics and Magnetostatics

Proposition (Time Independent Maxwell's Equations).

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{4}$$

Assume further that we're working in a space that is 2-connected. Then let $\mathbf{E} = -\nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$.

Proposition (Time Independent Maxwell's Equations - Reduced).

$$-\nabla^2\phi = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

12 Poisson's and Laplace's Equations

Definition (Poisson's Equation). For $\varphi : \mathbb{R}^3 \to \mathbb{R}$

$$\nabla^2\varphi = F$$

Definition (Laplace's Equation). For $\varphi : \mathbb{R}^3 \to \mathbb{R}$

$$\nabla^2\varphi = 0$$

Definition (Dirichlet Problem).

$$\begin{cases} \nabla^2\varphi = F & \text{in } \Omega \\ \varphi = F & \text{on } \partial\Omega \end{cases}$$

Definition (Neumann Problem).

$$\begin{cases} \nabla^2\varphi = F & \text{in } \Omega \\ \frac{\partial\varphi}{\partial\mathbf{n}} = g & \text{on } \partial\Omega \end{cases}$$

Proposition. *The solution to the Dirichlet Problem is unique. The solution to the Neumann Problem is unique up to +constant.*

Proof. Let φ_1, φ_2 be two solutions, and $\psi = \varphi_1 - \varphi_2$. Then

$$\begin{cases} \nabla^2\psi = 0 & \text{in } \Omega \\ B\psi = 0 & \text{in } \partial\Omega \end{cases}$$

where $B\psi = \psi$ for the Dirichlet problem, and $B\psi = \frac{\partial\psi}{\partial\mathbf{n}}$ for the Neumann problem.

Consider the functional $I[\psi] = \int_\Omega |\nabla\psi|^2 \mathrm{d}V$. Then $I[\psi] \geq 0$, and $I[\psi] = 0$ if and only if $\nabla\psi = 0$ in Ω. Note that

$$I[\psi] = \int_\Omega \nabla\psi \cdot \nabla\psi \mathrm{d}V = \int_\Omega \nabla \cdot (\psi\nabla\psi) - \psi\nabla^2\psi \mathrm{d}V = \int_\Omega \nabla \cdot (\psi\nabla\psi)\mathrm{d}V = \int_{\partial\Omega} \psi\nabla\psi \cdot \mathrm{d}S = \int_{\partial\Omega} \psi\frac{\partial\psi}{\partial\mathbf{n}}\mathrm{d}S$$

Now by the boundary conditions, we have that $\psi\frac{\partial\psi}{\partial\mathbf{n}} = 0$ on ∂S, so $\nabla\psi = \mathbf{0}$ and ψ is constant.

For the Dirichlet problem, we have that $\psi = 0$ on $\partial\Omega$, so by continuity ψ is 0 in $\Omega \cup \partial\Omega$, and the solution is unique.

For the Neumann problem, we have that $\frac{\partial\psi}{\partial\mathbf{n}} = 0$ on $\partial\Omega$, so ψ is constant, and φ_1 and φ_2 differ by a constant. □

12.1 Gauss' Flux Method

Definition (Gauss' Flux Method)**.** Suppose the source term F is spherically symmetric, and the domain $\Omega = \mathbb{R}^3$. Then Poisson's Equation becomes

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi = F(r)$$

Without loss of generality, we may assume that φ is spherically symmetric. Then $\boldsymbol{\nabla}\varphi = \varphi'(r)\mathbf{e}_r$. Integrating over the ball $|\mathbf{x}| < R$ and using the divergence theorem,

$$\int_{|\mathbf{x}|<R} \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\varphi \mathrm{d}V = \int_{|\mathbf{x}|=R} \boldsymbol{\nabla}\varphi \cdot \mathrm{d}\mathbf{S} = \int_{|\mathbf{x}|<R} F(r)\mathrm{d}V$$

Let $Q(R) = \int\limits_{|\mathbf{x}|<R} F\mathrm{d}V$. Then $\int |\mathbf{x}| = R\boldsymbol{\nabla}\varphi \cdot \mathrm{d}\mathbf{S} = Q(R)$. Now on a sphere, $\mathrm{d}\mathbf{S} = \mathbf{e}_r r^2 \sin\theta \mathrm{d}\theta\mathrm{d}\phi$, so on $|\mathbf{x}| = R$, $\boldsymbol{\nabla}\varphi \cdot \mathrm{d}\mathbf{S} = \varphi'(R)R^2 \sin\theta\mathrm{d}\theta\mathrm{d}\phi = \varphi'(R)\mathrm{d}S$. So $Q(R) = \int_{|\mathbf{x}|=R} \varphi'(R)\mathrm{d}S = 4\pi R^2\varphi'(R)$. Finally, we get that

$$\boldsymbol{\nabla}\varphi = \varphi'(r)\mathbf{e}_r = \frac{Q(r)}{4\pi r^2}\mathbf{e}_r$$

12.2 Superposition Principle

Proposition. *If L is a linear operator, and for $n = 1, \dots,$ $L(\psi_n) = F_n$, then*

$$L\left(\sum_n \psi_n\right) = \sum_n F_n$$

Definition (Superposition Principle)**.** If we have a system $L\psi = F$, where $F = F_1 + \cdots + F_m$, then suffices to solve

$$L(\psi_n) = F_n$$

and

$$L(\psi_1 + \cdots + \psi_n) = F_1 + \cdots + F_n = F$$

12.3 Integral Solutions

Proposition. *Assume $F \to 0$ rapidly as $|\mathbf{x}| \to \infty$. The unique solution to the Dirichlet problem*

$$\begin{cases} \nabla^2\varphi = F & \text{For } x \in \mathbb{R}^3 \\ |\varphi| \to 0 & \text{as } |\mathbf{x}| \to \infty \end{cases}$$

is given by

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}V(\mathbf{y})$$

Proof. This is equivalent to saying

$$\nabla^2\left(-\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\right) = \delta(\mathbf{x})$$

as by differentiating under the integral sign,

$$\begin{aligned} \nabla^2\left(-\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}V(\mathbf{y})\right) &= -\frac{1}{4\pi}\int_{\mathbb{R}^3} F(\mathbf{y})\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}V(\mathbf{y}) \\ &= \int_{\mathbb{R}^3} F(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\mathrm{d}V(\mathbf{y}) \\ &= F(\mathbf{x}) \end{aligned}$$

Now note that for $r \neq 0$,

$$\begin{aligned}
\nabla^2 \left(\frac{1}{r} \right) &= \frac{\partial^2}{\partial x_i \partial x_i} \left(\frac{1}{r} \right) \\
&= \frac{\partial}{\partial x_i} \left(-\frac{x_i}{r^3} \right) \\
&= -\frac{\delta_{ii}}{r^3} + \frac{3 x_i x_i}{r^5} \\
&= 0 = \delta(\mathbf{x})
\end{aligned}$$

If we assume that the divergence theorem holds for delta functions, then for any ball $|\mathbf{x}| < R$,

$$\begin{aligned}
\int_{|\mathbf{x}|<R} \nabla^2 \left(\frac{1}{|\mathbf{x}|} \right) \mathrm{d}V &= \int_{|\mathbf{x}|=R} \boldsymbol{\nabla} \left(\frac{1}{|\mathbf{x}|} \right) \cdot \mathrm{d}\mathbf{S} \\
&= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(\frac{-\mathbf{e}_r}{R^2} \right) \cdot \mathbf{e}_r R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \\
&= -4\pi
\end{aligned}$$

Thus for any $R > 0$,

$$\int_{|\mathbf{x}|<R} \nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \right) \mathrm{d}V = \int_{|\mathbf{x}|<R} \delta(\mathbf{x}) \mathrm{d}V = 1$$

Thus

$$\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \right) = \delta(\mathbf{x})$$

□

12.4 Harmonic Functions

Definition (Harmonic Function). A function φ is harmonic if it satisfies Laplave's Equation, that it

$$\nabla^2 \varphi = 0$$

Proposition (Mean Value Property). *If φ is harmonic on $\Omega \subseteq \mathbb{R}^3$, then*

$$\varphi(\mathbf{a}) = \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x}) \mathrm{d}S$$

For $\mathbf{a} \in \Omega$, r sufficiently small.

Proof. Let

$$\begin{aligned}
F(r) &= \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x}) \mathrm{d}S \\
&= \frac{1}{4\pi r^2} \int_{|\mathbf{x}|=r} \varphi(\mathbf{a}+\mathbf{x}) \mathrm{d}S \\
&= \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \varphi(\mathbf{a} + r\mathbf{e}_r) \sin\theta \mathrm{d}\theta \mathrm{d}\phi
\end{aligned}$$

Then

$$
\begin{aligned}
F'(r) &= \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a}+r\mathbf{e}_r)\sin\theta \mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{4\pi r^2}\int_{\phi=2}^{2\pi}\int_{\theta=2}^{\pi} \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a}+r\mathbf{e}_r)r^2\sin\theta \mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{1}{4\pi r^2}\int_{|\mathbf{x}|=r} \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a}+r\mathbf{e}_r)\mathrm{d}S \\
&= \frac{1}{4\pi r^2}\int_{|\mathbf{x}|=r} \nabla\varphi(\mathbf{a}+\mathbf{x})\cdot \mathrm{d}\mathbf{S} \\
&= \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|=r} \nabla\varphi(\mathbf{x})\cdot \mathrm{d}\mathbf{S} \\
&= \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|<r} \nabla^2\varphi \mathrm{d}V \\
&= 0
\end{aligned}
$$

So $F(r)$ is constant. Letting $r \to 0$ we get the result required. □

Proposition. *If φ is harmonic on some volume $\Omega \subset \mathbb{R}^3$, then φ cannot attain a maximum on any interior point of Ω unless φ is constant.*

Proof. Suppose we have $\mathbf{a} \in \Omega$ such that for all $\mathbf{x} \in \Omega$, $\varphi(\mathbf{x}) \le \varphi(\mathbf{a})$. Then by the Mean Value Property, we can show that φ is constant in a ball centred at $\mathbf{a}$.

$$
\varphi(\mathbf{a}) = \frac{1}{4\pi\varepsilon^2}\int_{|\mathbf{x}-\mathbf{a}|=\varepsilon}\varphi(\mathbf{x})\mathrm{d}S \implies \int_{|\mathbf{x}-\mathbf{a}|=\varepsilon}[\varphi(\mathbf{a})-\varphi(\mathbf{x})]\mathrm{d}S = 0
$$

As the integrand is nonnegative, it must be identically zero. Now choose any other point $\mathbf{y} \in \Omega$. Suppose further that Ω is compact. For any path connecting $\mathbf{a}$ to $\mathbf{y}$, choose balls such that the center of one ball is in the previous ball, and $\mathbf{y}$ is in the last ball. Then by compactness we have a finite subcover. Consequently, we have that $\varphi(\mathbf{a}) = \varphi(\mathbf{y})$, so φ is constant. □

Corollary (Maximum Principle)**.** *If φ is harmonic on Ω, and continuous on the closure of Ω, then for $\mathbf{x} \in \Omega$,*

$$
\varphi(\mathbf{x}) \le \max_{\mathbf{y}\in\partial\Omega}\varphi(\mathbf{y})
$$

13 Cartesian Tensors

Throughout this section, we will only use (right handed) Cartesian coordinates.

Definition (Rank n tensor)**.** An object with components $T_{\underbrace{ij\ldots k}_{n \text{ indices}}}$ that transforms from a basis $\{\mathbf{e}_i\}$ to a different basis $\{\mathbf{e}'_i\}$ according to

$$
T'_{ij\ldots k} = \underbrace{R_{ip}R_{jq}\ldots R_{kr}}_{n\ \ R\text{'s}}T_{pq\ldots r}
$$

is a rank n tensor. Here $R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$ are the components of a rotation matrix, and $R_{ip}R_{jp} = \delta_{ij}$.

Proposition (Scalar)**.** *A scalar is a rank 0 tensor.*

Proposition (Vector)**.** *A vector is a rank 1 tensor.*

Proposition (Linear Map)**.** *A linear map is a rank 2 tensor.*

Proposition. *If $u_i, v_j, \ldots, w_k$ are components of n vectors, then*

$$T_{ij\ldots k} = u_i v_j \ldots w_k$$

is a rank n tensor.

Proposition (Kronecker Delta). $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ *is a rank 2 tensor.*

Proof. $\delta_{ij} = \delta'_{ij}$ by definition. Also note that

$$R_{ip}R_{jq}\delta_{pq} = R_{ip}R_{jp} = \delta_{ij}$$

□

Proposition (Levi-Civita Symbol). $\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i\ j\ k) \text{ is an even permutation} \\ -1 & \text{if } (i\ j\ k) \text{ is an odd permutation} \\ 0 & \text{if } (i\ j\ k) \text{ is not a permutation} \end{cases}$ *is a rank 3 tensor.*

Proof. Note that $\varepsilon_{ijk} = \varepsilon'_{ijk}$ by definieion. Also,

$$R_{ip}R_{jq}R_{kr}\varepsilon_{pqr} = \det R \varepsilon_{ijk} = \varepsilon_{ijk}$$

□

Definition (Conductivity Tensor). Suppose that the current produced in a medium $\mathbf{J}$ is proportional to the electric field $\mathbf{E}$. This relationship can be written as $\mathbf{J} = \sigma\mathbf{E}$, or with indices, $J_i = \sigma_{ij}E_j$.

Since $\mathbf{E}$ and $\mathbf{J}$ are vectors,

$$\sigma'_{ij}E'_j = J'_i = R_{ip}J_p = R_{ip}\sigma_{pq}E_q = R_{ip}\sigma_{pq}R_{jq}E'_j$$

As this holds for any E'_j, we must have that

$$\sigma'_{ij} = R_{ip}R_{jq}\sigma_{pq}$$

and σ is a rank 2 tensor known as the electrical conductivity tensor.

Definition (Addition). If $A_{ij\ldots k}$ and $B_{ij\ldots k}$ are rank n tensors, then define

$$(A+B)_{ij\ldots k} = A_{ij\ldots k} + B_{ij\ldots k}$$

which is a rank n tensor.

Definition (Scalar Multiplication). If $A_{ij\ldots k}$ is a rank n tensor, and α is a scalar, define

$$(\alpha A)_{ij\ldots k} = \alpha A_{ij\ldots k}$$

which is a rank n tensor.

Definition (Tensor Product). If $U_{ij\ldots k}$ is a rank m tensor, and $V_{pq\ldots r}$ is a rank n tensor, we define

$$(U \otimes V)_{ij\ldots kpq\ldots r} = U_{ij\ldots kpq\ldots r}$$

which is a rank $m+n$ tensor.

Definition (Contraction). If $T_{ijk\ldots l}$ is a rank n tensor ($n \geq 2$), then we can define the tensor contracting on a pair of indices, say (i, j).

$$T_{iik\ldots l} = \delta_{ij}T_{ijk\ldots l}$$

and this is a rank $n-2$ tensor.

Definition (Symmetric). We say a tensor $T_{ij\dots k}$ is symmetric in (i, j) if

$$T_{ij\dots k} = T_{ji\dots k}$$

Definition (Antisymmetric). We say a tensor $T_{ij\dots k}$ is antisymmetric in (i, j) if

$$T_{ij\dots k} = -T_{ji\dots k}$$

Definition (Totally (Anti)Symmetric). We say a tensor $T_{ij\dots k}$ is totally (anti)symmetric if it is (anti)symmetric for every pair of indices.

Proposition. *δ_{ij} is totally symmetric.*

Proposition. *ε_{ijk} is the only totally antisymmetric rank 3 tensor on $\mathbb{R}^3$ up to multiplication by a constant.*

Proof. For a totally antisymmetric tensor, if two indices are the same then the component must be zero. There are $3! = 6$ non-zero components, and $T_{123} = T_{231} = T_{312}$ and $T_{132} = T_{321} = T_{231} = -T_{123}$. So $T_{ijk} = T_{123}\varepsilon_{ijk}$. □

13.1 Tensor Calculus

Definition (Tensor Field). A tensor field of rank n, $T_{ij\dots k}(\mathbf{x})$ gives a rank n tensor for all $\mathbf{x} in \mathbb{R}^3$.

Proposition. $\frac{\partial}{\partial x_i'} = R_{ij}\frac{\partial}{\partial x_j}$

Proof.

$$\frac{\partial}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'}\frac{\partial}{\partial x_j} = R_{kj}\frac{\partial x_k'}{\partial x_i'}\frac{\partial}{\partial x_j} = R_{kj}\delta_{ik}\frac{\partial}{\partial x_j} = R_{ij}\frac{\partial}{\partial x_j}$$

□

Proposition. *If $T_{i\dots j}(\mathbf{x})$ is a tensor field of rank n, then*

$$\underbrace{\left(\frac{\partial}{\partial x_p}\right)\dots\left(\frac{\partial}{\partial x_q}\right)}_{m\ \text{derivatives}} T_{i\dots j}(\mathbf{x})$$

is a tensor of rank $n + m$.

Proof. Let $A_{p\dots qi\dots j}$ be the expression above. Then

$$\begin{aligned}
A'_{p\dots qi\dots j} &= \left(\frac{\partial}{\partial x_p'}\right)\dots\left(\frac{\partial}{\partial x_q'}\right) T'_{i\dots j}(\mathbf{x}) \\
&= R_{pa}\dots R_{qb}\left(\frac{\partial}{\partial x_a}\right)\dots\left(\frac{\partial}{\partial x_b}\right) R_{ic}\dots R_{jd}T_{c\dots d}(\mathbf{x}) \\
&= R_{pa}\dots R_{qb}R_{ic}\dots R_{jd}\left(\frac{\partial}{\partial x_a}\right)\dots\left(\frac{\partial}{\partial x_b}\right) T_{c\dots d}(\mathbf{x}) \\
&= R_{pa}\dots R_{qb}R_{ic}\dots R_{jd}T_{a\dots bc\dots d}
\end{aligned}$$

□

Proposition. *$\nabla\varphi$ and $\nabla \times \mathbf{v}$ are rank 1 tensor fields, $\nabla \cdot \mathbf{v}$ is a rank 0 tensor field.*

Proposition. *For tensor field $T_{ij\dots k\dots l}(\mathbf{x})$,*

$$\int_V \frac{\partial}{\partial x_k} T_{ij\dots k\dots l}(\mathbf{x})\mathrm{d}V = \int_{\partial V} T_{ij\dots k\dots l}n_k\mathrm{d}S$$

where $\mathbf{n}$ is the outwards normal.

Proof. Let $a, \dots, c$ be constane vectors. Let $v_k = a_i b_j \dots c_l T_{ij\dots k\dots l}$. Applying the divergence theorem,

$$\begin{aligned}\int_V \frac{\partial v_k}{\partial x_k} \mathrm{d}V &= a_i b_j \dots c_l \int_V \frac{\partial}{\partial x_k} T_{ij\dots k\dots l} \mathrm{d}V \\ &= \int_{\partial V} v_k n_k \mathrm{d}S \\ &= a_i b_j \dots c_l \int_{\partial V} T_{ij\dots k\dots l} n_k \mathrm{d}S\end{aligned}$$

As this holds for any $a, \dots, c$, result follows. □

13.2 Rank 2 Tensors

Definition (Symmetric Part). For a tensor T_{ij}, we can define $S_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$

Definition (Antisymmetric Part). For a tensor T_{ij}, we can define $A_{ij} = \frac{1}{2}(T_{ij} - T_{ji})$

Proposition. *Every rank 2 tensor can be decomposed uniquely as*

$$T_{ij} = S_{ij} + \varepsilon_{ijk}\omega_k$$

where $w_k = \frac{1}{2}\varepsilon_{ijk}T_{ij}$ and S_{ij} is symmetric.

Proof. Suffices to show $A_{ij} = \varepsilon_{ijk}\omega_k$.

$$\varepsilon_{ijk}\omega_k = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqk}T_{pq} = \frac{1}{2}(\delta_{ip}\delta_{jq} - \delta_{jp}\delta_{iq})T_{pq} = \frac{1}{2}(T_{ij} - T_{ji}) = A_{ij}$$

If we had two decompositions, taking the symmetric part and the asymmetric part we see that they are the same. □

Definition (Linear Strain Tensor). Suppose each point $\mathbf{x}$ is displaced by a small amount $\mathbf{u}(\mathbf{x})$. Consider two nearby points $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{x}$. Then

$$(\mathbf{x} + \delta\mathbf{x} + \mathbf{u}(\mathbf{x} + \delta\mathbf{x})) - (\mathbf{x} + \mathbf{u}(\mathbf{x})) = \delta\mathbf{x} + \underbrace{[\mathbf{u}(\mathbf{x} + \delta\mathbf{x}) - \mathbf{u}(\mathbf{x})]}_{\text{change in displacement}}$$

Using Taylor's Theorem,

$$u_i(\mathbf{x} + \delta\mathbf{x}) - u_i(\mathbf{x}) = \frac{\partial u_i}{\partial x_j}\delta x_j + o(\delta\mathbf{x})$$

We can decompose $\frac{\partial u_i}{\partial x_j} = e_{ij} + \varepsilon_{ijk}\omega_k$ to get

$$u_i(\mathbf{x} + \delta\mathbf{x}) - u_i(\mathbf{x}) = \underbrace{e_{ij}\delta x_j}_{\text{translational displacement}} + \underbrace{[\delta\mathbf{x} \times \boldsymbol{\omega}]_i}_{\text{rotation}} + o(\delta\mathbf{x})$$

Here, e_{ij} is known as the Linear Strain Tensor.

Definition (Inertia Tensor). For a body V with density $\rho(\mathbf{x})$ and angular velocity $\boldsymbol{\omega}$, the angular momentum about the origin is

$$\mathbf{L} = \int_V \rho(\mathbf{x})\mathbf{x} \times \mathbf{v}\mathrm{d}V = \int_V \rho(\mathbf{x})\mathbf{x} \times (\mathbf{x} \times \boldsymbol{\omega})\mathrm{d}V$$

in components,

$$L_i = \int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \omega_i - x_i x_j \omega_j)\mathrm{d}V = I_{ij}\omega_j$$

where we have defined the Inertia Tensor

$$I_{ij} = \int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)\mathrm{d}V$$

with $\mathcal{V} = \{(x_1, x_2, x_3) : \mathbf{x} = x_i \mathbf{e}_i \in V\}$. The Inertia Tensor is a symmetric rank 2 tensor.

Proposition. *If T_{ij} is symmetric, then there exists a basis $\{\mathbf{e}_i\}$ such that*

$$\left(T_{ij}\right) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

The corresponding coordinate axes are called the principal axes of the tensor.

Proof. Clear from the fact that any real symmetric matrix can be diagonalised by an orthogonal transormation R for which $\det R = 1$. □

13.3 Invariant and Isotropic Tensors

Definition (Isotropic Tensor)**.** We say a tensor is isotropic if it is invariant under changes in Cartesian coordinates. That is

$$T'_{ij\ldots k} = R_{ip}R_{jq}\ldots R_{kr}T_{pq\ldots r} = T_{ij\ldots k}$$

for any rotation R.

Proposition. *Isotropic Tensors in $\mathbb{R}^3$ can be classfied as*

(i) Rank 0 tensors

(ii) There are no non-zero rank 1 isotropic tensors

(iii) $\alpha\delta_{ij}$ for rank 2

(iv) $\beta\varepsilon_{ijk}$ for rank 3

(v) $\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$ for rank 4

(vi) A linear combination of products of ε's and δ's for rank > 4.

Proof. (i) By definition.

(ii) Suppose if v_i are components for an isotropic vector. Take $\left(R_{ij}\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $v_1 = v_2 = 0$. By a different R we can find that $v_3 = 0$.

(iii) Taking rotations by $\frac{\pi}{2}$ about each axis yields the required results.

(iv) - (vi) Same ideas. □

13.4 Multilinear Maps

Proposition. *Given a tensor $T_{i\ldots j}$, $t(\mathbf{a}, \ldots, \mathbf{b}) = T_{i\ldots j} a_i \ldots b_j$ is a multilinear map. Furthermore, given a multilinear map $t(\mathbf{a}, \ldots, \mathbf{b})$, $T_{i\ldots j} = t(\mathbf{e}_i, \ldots, \mathbf{e}_j)$ is a tensor.*

Theorem (Quotient Theorem)**.** *Let $T_{i\ldots jp\ldots q}$ be an array of numbers defined in each coordinate system such that*

$$v_{i\ldots j} = T_{i\ldots jp\ldots q} u_{p\ldots q}$$

is a tensor for each $u_{p\ldots q}$. Then $T_{i\ldots jp\ldots q}$ is a tensor.

Proof. Consider the special case $u_i \ldots j = c_p \ldots d_q$ for vectors $\mathbf{c}, \ldots, \mathbf{d}$. Then $v_{i\ldots j} = T_{i\ldots jp\ldots q} c_p \ldots d_q$ is a tensor, and $v_{i\ldots j} a_i \ldots b_j = T_{i\ldots jp\ldots q} a_i \ldots b_j c_p \ldots d_q$ is a scalar, so it is independent of the choice of basis. Thus we have a multilinear map

$$t(\mathbf{a}, \ldots, \mathbf{b}, \mathbf{c}, \ldots, \mathbf{d}) = T_{i\ldots jp\ldots q} a_i \ldots b_j c_p \ldots d_q$$

By the previous proposition, $T_{i\ldots jp\ldots q}$ is a tensor. □

Definition (Stress Tensor, Stiffness Tensor)**.** Suppose the stress at a point is proportional to the strain. Then let σ_{ij} be the stress tensor. Then we have some c_{ijkl} such that

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

We cannot use the Quotient Theorem as e_{kl} is not arbitrary, as it is symmetric. However, if we assume $c_{ijkl} = c_{ijlk}$ then we can use the quotient theorem. Then c_{ijkl} is known as the Stiffness tensor. Suppose further that the material is isotropic. Then

$$\sigma_{ij} = (\lambda \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}) e_{kl} = \lambda \delta_{ij} e_{kk} + \beta e_{ij} + \gamma e_{ji} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$$

where $2\mu = \beta + \gamma$. This is Hooke's Law for an Isotropic material.
Contracting on (i, j),

$$\sigma_{ii} = (3\lambda + 2\mu) e_{kk} \implies e_{kk} = \frac{\sigma_{ii}}{3\lambda + 2\mu} \implies 2\mu e_{ij} = \sigma_{ij} - \left(\frac{\lambda}{3\lambda + 2\mu} \right) \sigma_{kk} \delta_{ij}$$